NATIONAL SOCIETY FOR THE STUDY OF EDUCATION

TWENTY-NINTH YEARBOOK

REPORT OF THE SOCIETY'S COMMITTEE ON ARITHMETIC

PUBLIC SCHOOL PUBLISHING COMPANY
BLOOMINGTON, ILLINOIS
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AGENTS

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THE TWENTY-NINTH YEARBOOK

OF THE

NATIONAL SOCIETY FOR THE STUDY OF EDUCATION

REPORT OF THE SOCIETY'S COMMITTEE ON ARITHMETIC

PART I. SOME ASPECTS OF MODERN THOUGHT ON ARITHMETIC
PART II. RESEARCH IN ARITHMETIC

Prepared by the Society's Committee

W. A. Brownell, B. R. Buckingham, G. T. Buswell, C. E. Greene, R. L. West, and F. B. Knight (*Chairman*)

Assisted by the Following Active Members of the Society

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Edited by
GUY MONTROSE WHIPPLE

THIS YEARBOOK WILL BE DISCUSSED AT THE ATLANTIC CITY MEETING OF THE NATIONAL SOCIETY, FEBRUARY 22 AND 25, 1930, 8:00 P.M.

PUBLIC SCHOOL PUBLISHING COMPANY
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EDITOR'S PREFACE

Mindful of the cordial reception accorded the Yearbook on Reading in 1925, the Board of Directors has for several years cherished the idea of issuing eventually a series of yearbooks dealing with the other basic subjects of the elementary school. At the Dallas meeting of the Directors, in February, 1927, it was voted to put two hundred fifty dollars at the disposal of Directors Horn and Judd for the purpose of arranging conferences between them and Professors G. T. Buswell and F. B. Knight to discuss the possibilities of a yearbook on arithmetic, to be published not later than 1930. At the Boston meeting of the Directors in February, 1928, Professor Knight was appointed Chairman of a Yearbook Committee on Arithmetic and was granted a preliminary appropriation of twelve hundred dollars to cover the first year's work on this undertaking. At the Rochester meeting of the Directors in November, 1928, the personnel of the Arithmetic Committee formally presented by Professor Knight was unanimously endorsed, as follows: B. R. Buckingham (Harvard University), G. T. Buswell (Chicago University), J. R. Clark (New York University), C. E. Greene (Denver Public Schools), R. L. West (State Department of Instruction, New Jersey), and F. B. Knight (State University of Iowa), chairman. Following the subsequent resignation of Professor Clark, the Directors approved at their Cleveland meeting in February, 1929, the addition to the Committee of W. A. Brownell (George Peabody College for Teachers). At their Chicago meeting in May, 1929, the Directors placed at the disposal of the Committee for the completion of its work the additional sum of twelve hundred dollars.

In all, the Society has expended through its Committee approximately two thousand dollars in preparing and bringing together in manuscript form the material of this volume. It is a fortunate thing for the members of the Society that its publications have acquired prestige and influence such that men of the professional caliber represented in these pages are willing to devote uncommon amounts of time and energy to these yearbooks with no reward other than the approval of their colleagues. As one of the contributors remarked: "If I were asked to set a price upon what I have done as my share of this book on arithmetic, I would be entirely reasonable if I set the figure at five thousand dollars."

The essential features of the treatment of arithmetic in this Yearbook are so fully set forth in the chapters introducing Part I and Part II and in the summaries of the reviewers embodied in the Appendix that they need not be repeated here. It may be well to mention here, however, that the Committee has eliminated from the Yearbook a considerable amount of material of undoubted value consisting of comparisons between well-known textbooks in arithmetic with respect to the amount of space devoted to various aspects of instruction and drill. This elimination arose from the apprehension of the Committee that such contributions might be construed as propaganda, however carefully the identity of the texts was concealed. Again, the Committee decided for several reasons to refrain from recommending specific details to be observed in the construction of textbooks, workbooks, and drill exercises or in the conduct of classroom teaching. It may be felt that these omissions have reduced the practical usefulness of the Yearbook to the classroom teacher, but they have surely been more than compensated by the wealth of stimulating suggestions that the Yearbook contains for arithmetic supervisors, textbook writers, and professional students of arithmetic in general.

Another feature of the Yearbook worthy of mention is the incorporation along with the report of the Arithmetic Committee of another report made by a Reviewing Committee whose personnel differs from that of the Committee that produced the volume. The plan under which these reviewers operated is explained in the Appendix and in the editor's prefatory note to it.

G. M. Whipple.

PART I

SOME ASPECTS OF MODERN THOUGHT ON ARITHMETIC

CHAPTER I

INTRODUCTION

I. Purposes of the Yearbook

In this *Twenty-Ninth Yearbook* of the Society the attempt has been made to discuss certain crucial problems of arithmetic in such a way as to bring out the theoretical aspects in so far as our present knowledge permits, and also to bring out numerous practical applications which, we trust, will make the discussions useful to supervisors and teachers. Our attempt has been facilitated by certain circumstances among which the following seem outstanding: first, the Arithmetic Committee, in ten full days of conference, secured the manifest advantage of frank exchange of opinion; second, the widespread interest in arithmetic made available for our use a large body of written discussion; third, a reviewing committee received most of the material in galley form in time to prepare a concluding critical chapter, so that this Yearbook contains not only discussions by the committee, but also a deliberate appraisal of these discussions by an independent and competent group of students of arithmetic.

II. Some Limitations

There are certain omissions in this volume which will be recognized immediately by the reader. He will note, for instance, the absence or the scantiness of discussions directed to several important aspects of arithmetic. Thus, problems and problem-solving receive but a partial discussion, because lack of rigorous data permits no thorough and penetrating discussion at present, and in general, the main reason for our neglect of certain issues which stand in obvious need of treatment is the simple fact that there is at present an insufficient amount of consistent scientific findings to furnish a satisfactory basis for such treatments. It seemed better to remain silent when discussion could go little beyond the limits of personal opinion.

A second reason for certain omissions arises, in some instances, from a decision not to proceed beyond general principles to definite and specific suggestions and detailed directions for classroom procedure. Several considerations led to this decision. In some cases sufficiently

close agreement on details could not be reached to present them without, at the same time, presenting a minority opinion. Too frequent expression of minority opinion becomes confusing to the reader. On the whole, the Yearbook is the expression of unanimous agreement of the Committee. While, of course, each member of the Committee has mental reservations about occasional paragraphs in the several chapters (on the score of pertinency as well as probable truth), the Yearbook as a whole represents the judgment of the entire Committee in the interpretation of available data. An exception to this is, of course, the concluding chapter, which is the appraisal of an independent reviewing committee.

Another factor influencing our decision to omit specifics was the desire of members of the Committee to avoid the promotion of their own particular interests in arithmetic, which can be found in print elsewhere. The Committee believes that the Yearbook does not propagandize any particular course of study, textbook, or type of instruction. In leaning backward in its effort to accomplish this purpose, the Committee may have omitted specific suggestions which would prove useful to the supervisor or to the teacher. The student may, however, by using the general points of view presented herein, be enabled to evaluate specific methods found in materials published elsewhere.

III. The Educational Philosophy Underlying the Yearbook

Some readers may feel that this Yearbook is too conservative, that it lacks a bold and daring spirit of progressiveness. There has been a conscious attempt to avoid the urging of any point of view not supported by considerable scientific fact. It has seemed preferable to proceed slowly and on sure ground, to be content with sane and moderate progress, rather than to expound a theory of instruction which, though supported by fine hopes and splendid aspirations, has as yet no basis in objective data. The justification of the contention of those who feel that the spirit of the Yearbook is not progressive will hinge on the definition given to 'progressive.' The changes we have supported are changes which can be made and held, changes based on a sober psychology of learning and of human nature, rather than extravagant changes based on a psychology which in its enthusiasm stubbornly refuses to view all the factors involved. The Committee feels that the treatment of arithmetic in this Yearbook is

as progressive as modern thinking, courses of study, textbook construction, and scientific experimentation give it a right to be, and much more progressive than much of present classroom practice.

The spirit in which this Yearbook was written, while not extreme, assumes the desirability of a liberal school in contrast to the lock-step, teacher-driven school of the '90's. It recognizes the child as the center of interest. The final criterion of all values is considered to be the effect any technique of teaching or any content of instruction has upon the child. The Committee has held in mind, however, that it is the whole child, not a part of him, which is the reality to be kept constantly in mind. A child's present life is but a part of himself, and an educational philosophy based upon the assumption that the present interests, needs, strengths, weaknesses, whims of the child comprise the sole or dominating aspect of the child will in practice render but limited service. The child's future is a part of him. In a sense, that the child will soon become an adult is a reality of childhood which must not be forgotten. Further, it is of importance to realize that the child, whose most precious attribute is 'soon to become an adult,' is destined to live in an actual society, a total environment, which will be, roughly, the United States between the years 1940 and 1980. The actualities of this total environment are not likely to be the ideal ones of which we now dream with fondness. In our more realistic moments we see that the child will be an adult living his life in terms of needs, duties, responsibilities, successes, failures, satisfactions, and monotonies very much like our own.

It is not enough to cast education in terms of children's present interests and desires alone. A child's life at the moment is not a thing in itself, nor is it at all self-sufficient and self-contained. His life is a continuum; his future is more real than his present. He is essentially an organism which is becoming something other than a child. Hence the education of the child must be cast in terms of his becoming an adult as well as in terms of his present status. The child must, of course, learn (and be taught) in ways which utilize the principles of child psychology, but the aims of his education must be influenced by two considerations: (a) his real nature, a potent part of which is his rapid leaving of his present status and his constant becoming an adult, and (b) the demands which life will place upon him *to-morrow*. These demands are not those of our wishing, but

those which will exist in the United States in the next generation, many of which can be predicted with reasonable certainty.

Certain educational philosophers seem to slight this reality of the future. Whether or not these philosophers have this intention, their pronouncements are interpreted to mean that childhood (having its own rights as it surely does) should serve the purposes of childhood only or mostly. The fact that a given unit of material is preparation for adult living is to these persons no sufficient defense for its inclusion in a child's curriculum, even if it be adapted to childhood in accord with good psychology. The aims of the school and the aims of childhood can be cut loose from adult aims and can be considered as quite separate from the aims of the future. Aims, as aims, are thus located in the present rather than in the future. Let the child eat, drink, and be merry, for to-morrow he will—become only an adult. A further present tendency which seems to the Committee dubious philosophy is the attitude of helplessness in predicting the demands which life will place upon the adult of the immediate future, say, 1940 to 1980. This helplessness is often alleged to be a necessary consequence of the rapid changes now occurring in society. Present society is changing in its technicalities, but the fundamentals of competent and useful living are not changing so rapidly that reasonable predictions of demands a generation or two in the future are either impossible or undesirable.

The revolt against the conventional school is a blessing, but it carries with it certain abuses. The Arithmetic Committee feels that a gradual evolution is more desirable than either a revolution or an impatient, extravagant attempt at reform. A temperateness in change may not be quite as thrilling to the philosopher as sweeping radical changes, but it is better sense.

The philosophy of this Yearbook, then, finds aims in the future as well as in the present. It suggests the desirability of preparation for adult living and holds it to be evident that a prediction of the demands of the future is feasible to a reasonable and useful degree of certainty. We should teach, then, those skills, informations, judgments, attitudes, habits, ideals, and ambitions which the child will find adequate and satisfying to the most important part of his whole self; that is, to his future adulthood as well as to his present childhood. How to teach the child can be separated, in discussion, from what to teach—and how to teach is fundamentally more a matter of

psychology based on research and investigation than a matter of philosophy.

IV. The Psychology of Learning Assumed in the Yearbook

Theoretically, the main psychological basis is a behavioristic one, viewing skills and habits as fabrics of connections. This is in contrast, on the one hand, to the older structural psychology which has still to make direct contributions to classroom procedure, and on the other hand, to the more recent *Gestalt* psychology, which, though promising, is not yet ready to function as a basis of elementary education.

The psychological point of view pervading this Yearbook emphasizes the fact that teaching based upon felt needs and interest only is inadequate. Not that felt needs and interest are lacking in vitality and importance, but that neglect of other matters (even if the neglect is only by way of inference) weakens effective teaching and learning to an intolerable degree. Use of all the dynamics of learning, rather than a use of some and a neglect of others, is the position taken. In the older school there was an overconfidence in drill—too often so stupidly administered that it could not possibly effect learning—and a corresponding neglect of interest and of the significance of the work to the worker. In short, there was a failure to capitalize the energy-releasing power of felt needs, together with a very naïve view of the psychological nature of the curriculum. One might suspect that today we are being led by our laudable attention to interest, significance, and the creation or discovery of felt needs to a neglect of other dynamics of human learning. We are in danger of slighting the contributions of properly organized practice; we may allow felt needs to degenerate into whims (often requiring less effort); we may continue (through neglect) to teach subject matter with but a superficial view of its psychological nature. A pedagogy based on a use of all the dynamics of learning in proper proportion is highly preferable to overemphasis upon a single aspect of learning in one generation followed by a corresponding neglect of it in the next generation.

In some quarters it seems the fashion to think that anything that at all resembles conventional practices is necessarily wrong and vicious. There is almost an emotional antipathy to anything that in any way reminds us of the kind of schools we attended as children.

The assumption that children must be unhappy and will develop frightful personalities if exposed to a pedagogy which uses the principles upon which the older schools were based is a sort of inverted old-oaken-bucket delusion. Many aspects of the older school are, after all, distinctly good psychology; they are based on correct theorems relating to human nature. Two instances will illustrate the caution against a lopsided psychology of learning based solely on interest and the felt needs of the child:

(a) A very modern teacher, sure that anything which is conventional is wrong, decided to allow the curriculum of her first-grade class to be guided by the wisdom of her pupils. "Now to-day," she said, "we can do anything we like to do." She suggested several possible projects which were *anything but* customary school work. Silence followed this attempt to discover felt needs. Finally a child summoned up courage and said: "I want to learn to read just as older people do." Should this teacher keep on trying to find something else to do or should she follow her philosophy and assist the youngster in his learning to read with a use of the best techniques of learning at her command?

(b) A third-grade teacher told her class on a certain morning that they could do anything they wished to do—at the same time waving some raffia in her hand and having previously laid in conspicuous places equipment for activities of various kinds. A hand promptly went up, followed by this suggestion: "Let's do some more drill in long division, so that it will not be so hard." What should this teacher do?

The notion that regular school work is or must be essentially intolerable to children is not true. Even skill subjects can be taught, and often are taught, in ways which are even more than tolerable to children. A curriculum laid down before the child enters school can be lived through with zest and enthusiasm. The 'going to school' may, in itself, be an abiding and sufficient project. When an adequate psychology underlies the daily work, many of the criticisms directed against the 'orthodox' school lose their force. Many of the attacks on the 'conventional' school so describe it that the attack is against a straw man.

Much may be said for the following point of view: What to teach should be decided by as wise adults as are available for the task, who will base their decisions as far as possible upon the available body of objective scientific data. How to teach should be based upon a virile psychology of learning which uses all the dynamics of learning—not on the one hand, running to seed, as in the case of the older school, on dead tasks for the tasks' sake; and not, on the other hand, de-

claring a Roman holiday, and becoming concerned only with the factor of interest. The actualities are such that until we are much more skillful than we are, the interest value of many moments in school is not particularly high. But, in spite of this, a total psychology of learning would, on occasion, defend such moments as moments to keep on doing the same thing while improving the interest in it, rather than to avoid doing it altogether. There must be no retreat from the position that felt needs be utilized, that the tasks be made significant, that the sustaining effects of interest be earnestly sought and capitalized. Further, we must continue to seek both increased knowledge of the psychology of the subject matter taught (a field quite neglected by the left wing) and increased skill in the use of such aspects of learning as are suggested by the phrases: drill, continued effort, ability to withstand distraction, persistence though momentarily bored, effort sustained not by rewards at hand but by confidence in values forthcoming in the future, and the intent to master the matter in hand as a permanent possession rather than as a temporary accomplishment.

For the Committee,
F. B. KNIGHT.

GUIDE TO CHAPTER II

CHAPTER II

THE SOCIAL VALUE OF ARITHMETIC

B. R. Buckingham

Introductory—In a Library

One evening not many months ago the members of the Committee under whose auspices this Yearbook has been prepared had the pleasure of visiting the library of Mr. George A. Plimpton of New York City. Mr. Plimpton himself conducted the Committee to the vault which houses his extensive collection of manuscripts and books. Since this was an arithmetic committee, he called special attention to materials in that subject. It was a remarkable exhibit. It impressed one with a sense of the long, hard struggle by which arithmetic has been brought to its present state of development. It gave one respect for the subject, not merely as something to be taught to little boys and girls but as a discipline fit to enlist the powers of mature minds.

Here were vellums as old as the tenth century, themselves transcripts of a line of manuscripts running back to still earlier centuries of the Christian Era and even to the great days of Bagdad and Alexandria. Boethius, the last man whom Cicero might have recognized as a compatriot, was there with his *Arithmetica*. The Venerable Bede, whom Burke called the father of English learning; Isidorus of Seville, whom the Council of Toledo named the "most learned man of the ages"; Paolo Dagomari, with a fourteenth century treatise on commercial arithmetic; Rollandus of Lisbon, who in the early fifteenth century covered all the theoretical arithmetic known and dedicated his work to John of Lancaster, son of Henry IV of England—these and many other writers of ancient and mediaeval times were represented in this rare collection.

The printed books were scarcely less interesting. The Committee saw the first printed arithmetic ever issued, a book more or less com-

mercial in character, published at Triviso in Italy in 1478. Indeed, the activity in Italy in this early period was astonishing. No fewer than two hundred thirteen mathematical works were printed between 1472 and 1500. The Plimpton library likewise has the first arithmetic printed in Germany and the first one printed in England. There is also the arithmetic of Adam Riese (published 1522), which became so famous that "*nach Adam Riese*" is still a common expression in Germany. Scarcely less famous was Robert Recorde, who wrote the first noteworthy arithmetic printed in the English language (published 1542).

Recorde seemed to feel that some explanation was needed for writing an arithmetic in English. His book is in the form of a dialogue between master and student. The student expresses the opinion that the subject has little value. The master replies:

> If number were so vile a thing as you did esteem it, then need it not be used so much in men's communication. Exclude number, and answer to this question, "How many years old are you?"
>
> "Mum," the student replies.
>
> "How many days in a week?" "How many weeks in a year?" "What land hath your father?" "How many men doth he keep?" "How long is it since you came from him to me?"
>
> "Mum," answers the student.
>
> "So that if number want, you answer all by mummes."

Even to-day there are some who think, as the student did in Recorde's book, that arithmetic has little value—at least beyond the ability to compute with small numbers. These men say that the curriculum should be reduced to what they call 'the minimal essentials' and that this residuum should be taught in much less time than is now devoted to the subject. It is the purpose of this chapter to inquire into the social uses of arithmetic for the bearing which such an inquiry may have upon this doctrine.

I. Number Concepts

A sense of the value of numbers such as modern life demands is neither innate nor easily acquired. It is certain that a child is not born with this sense, and among primitive people it is equally certain that adults have not possessed it. Moreover, even among men and

women in modern civilized society the meaning of numbers is often insufficiently understood. Generally speaking, the reason some people can't be induced to keep score at a bridge game is because they can't add a column of figures with facility. They make hard, slow work of it, and experience has taught them to have no confidence in the result. This may not seem to be a deficiency in number ideas so much as a lack of automatic response to number facts. Yet it is more than likely that, both in school and after school days are over, inability to use number combinations with facility is due to a fundamental failure to apprehend the numbers themselves. Merely to drill children in combining meaningless symbols is to develop a misleading automatism which disintegrates at the first period of disuse. Our friend who regularly declines to keep score in a bridge game has a more deep-seated trouble. His knowledge of four, five, and six, for example, lacks the fullness which life in the modern world requires. Observe him when he wants to know how many tricks he has in front of him. He must count them. Moreover, he generally does so by ones and he always has to touch them as he counts. If he has as many as six tricks, he often counts them a second time. Past experience has made him cautious.

The illustration may seem trivial, but the thing illustrated is by no means trivial. The extent to which modern life depends upon mathematics is almost unbelievable, yet no mathematics is possible without number concepts. Without them the very idea of value is defective; the meaning of nature is perverted; the significance of behavior is misapprehended; and the sense of order, sequence, and law is rudimentary. Early prehistoric man had no number ideas beyond two or three, and primitive savages of to-day are said to be similarly handicapped. Correspondingly, the savage has a scale of values which works very badly. He is a prey to superstition and fear. Unable to record or even understand the results of behavior in amounts of this or that, he puts his trust in taboos, amulets, and the howling of meaningless incantations. He sets more value upon a shark's tooth than upon the life of his child. He prepares for battle by fasting and all-night carousal instead of nourishing his body and getting a good night's sleep.

On the other hand, modern life demands a very great range of number ideas. Those who tell us that one may get along with small numbers are mistaken. Look at the morning paper.[1] Under an alluring headline "Does Marriage Wane or Gain?" the reader, if interested, finds a great many numbers. In fact, the argument is *entirely statistical.* To read this one must have a knowledge, appropriate to the purpose, of numbers as large as 5,645. The really exacting character of the article, however, lies not so much in the size of the numbers as in the various meanings given them. The text under the caption "U. S. out to Rescue Wheat" relates that President Hoover wants "100 millions" (otherwise written as "a hundred million dollars") out of the "half billion dollar loan fund" to arrest the slump in the price of wheat. Farther on in the same article, but before it gets off the first page, the numbers "373 millions" and "928 millions" are used, the reference being to bushels of wheat. Another column on the same page is entitled "Strike Peace Parley Called for Monday." It tells the reader that "construction work on $237,000,000 worth of Chicago building projects" has been stopped. Then it gives a table showing the principal buildings involved with the valuation of each. These valuations are all in the millions. But these sums are small compared to the amounts over in the lefthand column where a report is given of the agreement on the German debt. The first three paragraphs fairly bristle with huge numbers. The amount, according to the agreement, is "about $9,000,000,000." This represents "a decrease of $116,000,000,000" from the amount of the original claims in 1919 "which was $125,000,000,000." The reader's interest is supposed to be in no wise dampened by these numbers; for three others follow almost immediately, each in ten or eleven figures. All this, be it remembered, is front page stuff. June 2, 1929, must have been an unfortunate Sunday morning for people who were trained only in small numbers "because large numbers are so seldom used."

Surely one is called upon in these days to have ideas about numbers of all sorts of sizes. Some persons have no adequate concept of numbers as large as a thousand and no useful way at all of thinking about

[1] What follows refers to page 1 of the Chicago *Sunday Tribune* of June 2, 1929.

a million. Such persons are heavily handicapped. On the other hand, highly trained scientists, economists, capitalists, and, in general, men of large affairs are able to react in highly satisfactory ways, especially in the field of their particular interests, to numbers as large as billions or even trillions. Moreover this notion of the size of the numbers about which one ought to know something cuts both ways. It includes *fractions* of units as well as *multiples* of units. Advertisements of fine cars, and even of cars not so fine, bring to one's attention accuracy of measurements to thousandths of an inch. The microscope has revealed, and the public press reports, the existence of life in exceedingly minute forms. The modern adult has personal use for numbers which, merely on the score of size, extend far into subdivisions of one as well as far into aggregations of one.

Without regard to the size of the number concerned, three ways of looking at a number concept will now be considered: first, as to what it does or enables one to do; second, as to the data to which it applies; and third, as to the way in which it is known. The first of these implies the functional idea; the second, the idea of units; and the third, the mathematical idea. Each is merely an aspect of one's set of number concepts. None of these concepts could exist unless it functioned, none unless it referred to units, and none unless it had mathematical meaning. Moreover, any given concept, and hence any system of concepts, is socially valuable according to the number and importance of the ways in which it functions, of the materials in which it is usable, and of the quantitative meanings which it possesses.

1. Concepts Functionally Considered

All concepts are functional. To a greater or lesser degree they serve the purposes of human behavior. While, as has been indicated, it is entirely possible to think of them in other than functional aspects, it is also quite possible to think of these other aspects as serving the purpose of function.

Every normal adult in a civilized environment possesses an extensive equipment of number ideas. In virtue of this fact he behaves in certain ways. The appropriateness of his behavior in the presence of the idea of any particular number varies with the richness of his idea of that number and the consequent versatility with which he can

react to it. One's concepts of small integers are more 'complete' than one's concepts of large integers. Hence reactions to four are more rapid, more accurate, and more varied than reactions to seventy-five or even to fifteen.

For verification of this fact one need do no more than allude to the results of experiments on the apprehension of numbers. It has been found that, generally speaking, an adult can apprehend directly numbers up to seven in the sense that he can tell the number of objects presented together without counting them or grouping them and without the aid of a number pattern. To these small numbers he therefore reacts with maximal certainty and appropriateness. His concepts of these numbers are as nearly perfect as any number concepts which he possesses. He is not so competent, however, in regard to numbers a little larger than seven. He needs the assistance of a pattern, such as the domino arrangement, in order to apprehend these numbers directly. Still larger numbers—say those as large as eighteen or twenty—can only be identified by counting or by the combining of smaller numbers. It seems reasonable therefore to conclude that one's number concepts become progressively less perfect as the numbers increase in size. This means that if we may define a perfect concept as one possessed by a highly endowed individual concerning a small integer, such as four or five, then it is extremely doubtful whether any human being possesses more than a few perfect number concepts. The ways in which concepts may be adequate in one direction and inadequate in another are well illustrated in the case of children.

Some investigators have tested children as to their knowledge of the small numbers and have concluded that this child "knows seven" and that child "knows nine." A child does not know seven in the fullest sense. He only knows seven in the sense used in the test. Throwing down a number of small objects before him we ask him to give us five or eight or three. If he succeeds in giving us five and if by repeating the test we assure ourselves that his success is not accidental, we are then justified in saying that he has a certain functional knowledge of five and that he possesses this knowledge in a particular situation. The functional knowledge is the ability to reproduce the number in question and the particular situation is that in which the reproduction takes place.

Whether a child who is able to reproduce five also has other functional knowledges of five yet remains to be learned. He may or may not be able to identify and name five—answering, for example, such a question as "How many buttons are here?" He may or may not be able to distinguish five from numbers of nearly the same size—succeeding, in the presence of groups of objects, in telling "Which of these is five?" He may or may not be able to match equal numbers. Being shown five cubes, and without being told the fact that there are five, he may be uncertain in the selection of an equal number of pennies from several groups of pennies one of which contains five; or he may fail in making up a group of "that many pennies."

Number ideas in use—which is what is meant when one considers them functionally—may likewise be made to include the speaking of numbers, the reading and writing of them, and any action carried out as dictated by them, such as indicating five by showing all the fingers of one hand or leaving the office in time to catch the 5:18 train.

Accordingly, in any consideration of the social value of arithmetic—and consequently in any educational program based on social value—it is clear that one must take account of the functional nature of number ideas. This is the executive, or dynamic, side of number. It is what one does *numberwise.* It is quantitative reproduction, identification, and discrimination. It is likewise number in language and action. It is *twelve* or *five* or *two hundred fourteen* made manifest. It is the recording of numbers by the use of tallies, letters, digits, and words. It is the communicating of numbers not only by these graphic means but also by means of speech, gesture, and pantomime. It is action under the control of number—such as putting four cups of flour into the mixing bowl as the recipe requires, or taking the third left-hand turn after being so directed, or reading the next fifteen pages according to the teacher's assignment.

Without functional ideas of number the work of the modern world could not go on. Science would be lost for want of appropriate expression. Machinery would go out of use and the civilization of our industrial age would vanish. The mind of man would itself be weakened through the loss of its keenest weapon. Thought would no longer be precise or if precision were ever attained it could not be communicated. Fortuitously or as a matter of rhythm one might put four cups of flour into a mixing bowl, supposing that anything like cups or

flour or bowls existed; but one could not do a variety of such things. One might as a matter of habit take the third turn; but one could certainly not carry out new directions in strange localities. As for reading the next fifteen pages of a book—aside from the fact that there would be no such pages—the requirement could neither be proposed nor undertaken.

2. Concepts in Relation to Units

It is obviously not enough that number concepts should be functionally effective. It is also desirable that concepts be recognized as valid no matter in what medium they may be expressed. Indeed, only by a process of abstraction can one consider the functional character of concepts apart from their application to a variety of uses. If one were competent in the use of number only with reference to a limited application of it, one's competence would still be quite unsatisfactory. The story comes to mind of the feeble-minded boy at the Vineland Training School who carted the coal and whose usual rate of hauling was six tons a day. A car containing thirty-three tons was on the siding. The boy was asked how long it would take him to unload the car. He promptly replied, "Five days and half a day over." No such success, however, in handling numbers would have been possible for him except in connection with carting coal. His competence was in a single line.

When one considers the units in which a number may be expressed, he at once appreciates the vast range of application of number ideas. In prehistoric times man may be supposed to have gradually evolved an abstraction, such as five, from the continued observation of stones, animals, trees, bones, clubs, and the other objects within his experience. It is believed that the earliest distinctions were one, two, many; and later, one, two, three, many. Ideas of the larger numbers were gradually built up from the smaller. Four was two-and-two, or three-and-one, or perhaps one-and-one-and-one-and-one. As these more or less abstract ideas secured recognition, they were seen to apply to a great many conditions and objects. At first it may be supposed that they were used only with reference to objects well within the experience of the individual. Very likely the cave dweller who was entirely capable of applying his number ideas to the objects about him would have been unable to do so with facility if he had encountered the objects

of an environment other than his own. It is quite certain that he would have been confused if he were called upon to apply his number knowledge to units each composed of more than one object—such as pairs or birds or piles of four stones.

Among school children the same kind of difficulty exists. This is particularly true when, as is the case in verbal problems, we require them to deal in images. Problems about oranges, apples, and sticks of candy are easier than problems involving the same numbers of feet or pounds or yards. The latter are still easier than problems involving the same numbers of utterly intangible units such as hours or days or years.

This difference can only be due to the instability of the number ideas. Seven, the teacher impatiently insists, ought to mean seven, whether the problem is dealing with marbles or months, with sheep or 'sibtabs.' If problems involving the same numbers, processes, and phraseology vary in difficulty merely because the units are different, then clearly the number concepts have not reached a sufficiently high degree of abstraction. They are still 'loaded' with objective content. They are not sufficiently disembodied to have unvarying meaning under all circumstances.

Now the circumstances of the modern world to which number ideas must be applied are extremely varied and difficult. An adult who lives successfully in such a world must have highly developed number ideas. He has long ago passed beyond the stage of concrete numbers. Not only are his concepts abstract, but they also exhibit hierarchies of abstraction. For example, he expresses the extent to which two variables are related to each other (the correlation coefficient) by a number, the concrete meaning of which can only be derived by a long series of steps. He devises an intelligence quotient, the units of which have the same remote reference. He measures electric current in kilowatt-hours and the distance of the stars in light-years. He determines the cost of living by means of index numbers. Only a few months ago the world was startled by a new idea in measuring the value of naval vessels through the application of the principle of the index number. If, however, a vessel is rated at 65, it will be by no means easy to answer the question "65 what?" Yet it is certain that unless the answer to this question is available, much misunderstanding will exist. The conclusion is rather obvious. A knowledge of number—merely

the idea of the meaning of *a* number—in terms of its units is by no means secured in the primary grades. It is doubtless true that in this respect our number concepts are being enriched throughout our lives, and the question of the social value of arithmetic cannot ignore the value of the concepts of number which people have at their command.

The meaning of this doctrine for the school is that the curriculum, both as to content and method, should, if social values are recognized, devote more time to the growth of number concepts. In particular it should provide a wide range of material within which the child may exercise his number ideas and number processes. A reduction of arithmetic to the mere units and manipulations found to be employed in common affairs impoverishes the child's thinking. It is said, for example, that the metric system and foreign money should not appear in present-day arithmetic. Yet it is certain that a better conception—a freer, wider, more liberalizing outlook—will be gained concerning the measurement of length, weight, capacity, and value if other than the conventional units are employed. It is not at all necessary that a child should be able to reduce dollars to lire or recite the table of Japanese money. It is very likely unnecessary that he should learn to the point of relatively permanent retention the equivalencies between the common and the C.G.S. systems of measurement. But to allow him to grow up without knowing that there are such things, or to permit him to bump into them for the first time when he has actual and immediate need for them, is to advocate a cheap, threadbare sort of education which may give lip service to the notion of social values, but which has no adequate philosophy of what value is. It is even conceivable that the despised apothecary's weight and the equally despised troy weight should be included in our course of study—not that we insist upon the value, in and of itself, of manipulating scruples and drams, not that we believe everybody should be able to handle grains and pennyweights. Only a few persons are going to have regular need for things of this sort. This is granted. But at the same time it is only fair to point out that the inclusion of just enough of these special units to give the child an idea of their meaning and use is a part of the proper contribution of arithmetic to a liberal education.

One may even go farther. No child is ever likely to have occasion to use such units as cubit, or span, or ell. He is hardly likely to have

use for stone, or hand, or fathom. Yet it is a mistake to cramp the treatment of measurement as it may be offered, at least to bright children, by leaving out of account the measures that have been used in the past—their variation, their anatomical origin, the long processes by which standard units were developed. Suppose that an enriched course for bright children should require them to solve problems involving the furlong or the perch—units definitely ousted from our textbooks by the reductionists. Even then it would not be a complete objection to say that in life they would never have to use such units.

Even if a bowing acquaintance with units of measurement other than those habitually used in the round of common experience were not in itself desirable, they would still be justified, for the pupils who can appreciate them, for the bearing they have upon a better knowledge of the very things which they are certain to have to use. This may be illustrated from another field. It is granted that the first case in percentage is used much more often than the other two cases. But a sufficient justification for the teaching of the second and third cases may be found, not in the fact that these cases will be frequently used, but in the fact that the child who learns them will understand percentage better. It is not too much to say that the chief justification for teaching the third case in percentage is that the first case may be better known. This whole idea of trimming the course of study down to the things which are going to be used directly defeats its own purpose. It means the learning of facts in isolation rather than in rich association; and it means, as usual, that facts so learned are less than half learned.

Even the most narrowly utilitarian training recognizes that you must go beyond the mere treatment of the thing to be learned. In a series of seventy pictures designed to teach employees to sell rayon, considerably less than half of the pictures had anything to do with selling the fabric in question. Most of them dealt with the interesting phases of the manufacture of rayon. First "The wood or cotton is cooked in live steam," second, "The pulpy mass of cellulose is pressed into sheets," and so on through a considerable series until "The man-made fabric" is displayed. Doubtless there are those who would say, "Why all this waste of time about the manufacture of rayon? If you want to teach people to sell rayon, why don't you stick to your topic?" The real answer to this question is that the method employed actually

does teach the selling of rayon more effectively than a so-called 'direct' method would.

To such a pass does a crude philosophy reduce the curriculum that the school is asked to teach and drill upon fragments of topics because other fragments of the same topics have not been found to be employed with a specified percentage of frequency. The course of study is thus impoverished. One is not permitted to teach the number system in any completeness because parts of its "won't be used." One cannot take time to teach the number concepts with richness of meaning and breadth of application because "they are not needed in life." One cannot give a competent treatment of fractions because one is thus carried into parts of the subject which the man in the street fails to use. All this is very distressing to those who believe that the greatest worth in a subject is to be found in a study of it long enough and hard enough and completely enough to obtain from it its inherent educational value. It is believed that a more rounded treatment should be given to the various topics of arithmetic in order that the true relations and inner connections of the subject may be appreciated. It is further believed that a fuller treatment of topics is actually necessary in order that the parts of the topics which are most frequently used may be successfully learned.

3. How Number Ideas Are Known

In the foregoing treatment of number ideas consideration has been given to their use (that is, to the functional question) and to the material with which they deal (that is, to the units of which they are composed). The control which an individual has either of his number ideas in use or of the range to which he can apply them is conditioned by the way in which he knows his numbers. This is the mathematics of number knowledge. If an adult knows a number merely as holding a position in the series one, two, three, four, ..., he has a poorer control of this number idea than he has if, in addition to knowing it in that sense, he also knows it as the sum of two or more smaller numbers. A still more effective control will be added if he can sense the number as a product or perhaps as a combination of a product and a sum. Control in this case means readiness and accuracy in the use of the number for whatever purpose the number may serve. In other words, control rests upon the functional idea.

The question has been debated whether a child should first be taught the small numbers by counting or by measurement. The debate is idle so far as the school is concerned.[2] Preschool life has decided the matter for us. Children typically begin their number experience with counting. There is good evidence for believing that 60 percent of them can count to twenty and that 90 percent of them can count to ten when they enter the first grade. A not inconsiderable number can count to one hundred (see Part II, Chapter IV). On the other hand, children do not of their own accord begin to measure until after they have begun to count. The first vague and unsubstantial ideas of number, therefore, come from counting. Indeed, the counting may precede any number ideas worthy of the name. Nevertheless, counting is the entering wedge. The child may merely learn parrot-fashion a series of words which have a certain rhythm and which he likes to say. He may not know them as numbers at all. Gradually, however, he picks up experience which identifies these empty expressions with the 'how many' of this or that. Thus does the child learn topsy-turvy, as it were, and not at all according to the neatly arranged sequences which adults conceive to be the normal or natural path of progress.

When the words of the sequence one, two, three, four, ..., are understood to be the names of numbers and when each successive number is understood to increase in size, the child has a basis for estimating the magnitude of the numbers which he uses in counting. It does not take him long to realize that seven is more than six or any other number reached in the series before he comes to seven. Thus he develops ideas of more or of less in a quantitative sense. These ideas lack precision, but the child has already advanced far beyond the primitive savage who only knew 'one, two, three, many.'

The child's number ideas, although they may still rest upon counting, become more useful when he realizes that each number in the counting series is *one* more than the number immediately preceding it. This permits real precision in the comparison of adjacent numbers and approximate precision in the comparison of non-adjacent num-

[2] The school here means the American institution for training children which begins with the first grade and takes children of approximately six years of age. Substantially the same statement will hold for the school thought of as beginning with the kindergarten. It may not hold if the school begins with a nursery department.

bers, the precision diminishing as the numbers compared differ more and more from each other.

We may suppose that in his inevitable comparison with non-adjacent numbers, the child comes to need a greater precision than mere counting by ones permits. At any rate, experiment shows that, quite apart from any instruction whatever, children in their normal experience with the things about them early reach an intuitive idea of five as three and two. When this is the case, five becomes more meaningful than it was when it was merely known as occupying a certain position in the number series. A new conception has appeared upon the scene, not to drive out the old one but to supplement and enrich it.

In some such way as this, the child may be supposed to build up a working knowledge of the simpler basic addition facts. At any rate we know that to a much greater degree than the school, by its practice, is assuming, he has these facts at his disposal. According to the investigation reported in Chapter IV of Part II, 89 percent of six-year-old children gave at least one correct answer to verbal problems involving basic addition combinations.

As the child matures and especially as his mental life begins to be affected by formal instruction, he may advance to still richer number concepts. The number eight may not only have, for him, its place in the counting series, but it may also be the combination of various other numbers. It is seven and one, six and two, five and three, four and four, three and five, two and six, and one and seven. It may be apprehended as two fours (which is somewhat different from four and four), as four twos, as two less than ten, and so on. The investigations with the use of number patterns have shown all the conditions just mentioned as well as many others. Thus eight is sensed as two threes and two more, as two fives less two, as three threes less one, and so on. These are all more mature ways of apprehending number, and the result for the individual who can use these superior methods is a quicker, more meaningful, and hence more general application of number ideas. Thus the functionality of which mention was made above is under better control. One reacts more quickly, fully, and competently to number situations. This is surely an important individual asset. It is likewise a clear indication of the social value of the very essential part of arithmetic which has to do with number ideas.

There are still other ways in which numbers may be known—ways in which number ideas or concepts may exist. For example, numbers may be thought of as ratios. In a way this has already been suggested when eight was spoken of as two fours or as four twos. Number as ratio, however, begins with the idea that the number in question is that many times the value of one. Ten, for example, is ten times whatever one is. In fact, it is only when so considered that all numbers become directly comparable. It is only as ten and seven have a common divisor, namely one, that one can be said to be three more than the other, or the other three less than the one. Of course when two numbers are not prime to each other, they have at least one additional common divisor besides one, and an additional basis of comparison with increasing richness of meaning.

The ratio idea is capable of still further extension with important results in the understanding of number. It has already been pointed out that ten is ten times whatever one is. It is also five times whatever two is. In this connection it is entirely possible to think of two as a unit—that is, as a group of two things regarded as a totality. Analysis, however, of this totality discloses its numerical meaning—in other words, its value as a ratio in which one is the second term. Out of this idea a very suggestive notion of the processes of multiplication and division is secured. Children are often taught that multiplication is a short process of addition. It would be better to teach them that it is a short way of adding equal numbers. In the case of the example at the side of the page, however, it is likewise fruitful to apply the idea of ratio and to think of the process as one of finding the number (80) which has the same ratio to the multiplicand (16) that the multiplier (5) has to one. Similarly in division[3] it is a vigorous thought that the process is one of finding the number which sustains the same relation to one that the dividend sustains to the divisor.

$$\begin{array}{r} 16 \\ 5 \\ \hline 80 \end{array}$$

These are no doubt adult ideas, in the sense that they are not to be taught to children at the time they learn the processes of multiplying and dividing. That they should not be taught to brighter children in the upper grades is a pity. That these and other ideas of a broadening and liberalizing tendency should not be known to adults is a painful commentary upon the so-called 'realism' of our age which (quite missing some essential realities) assumes to teach children only the things

[3] Not in partition, where we find one of the equal parts of a number.

they visibly use in getting on in the world. The fact that men's lives would be richer and fuller, that in a fine humanistic sense they would have a better control of the real issues of life, is missed by those who pin their faith to the reported frequency with which business men use square root or farmers use proportion. It is of fundamental importance that men and women in the modern world have number ideas which are true guides to conduct, which are highly abstract and therefore broadly applicable, and which are rich in quantitative significance. These are real acquisitions. To possess such concepts is to be highly endowed in a true and useful sense, even though one may not possess highly trained skill in computing. It is doubtful, however, if those who report their use of arithmetic ever indicate their behavior based upon number ideas independently of computation. These ideas are so much the warp and woof of their intellectual life that they fail to take note of them.

II. Notation

Although it is certain that many of the ancients possessed highly developed number ideas, it is generally conceded that the Hindu-Arabic numerals have made it possible for more people to have better number ideas than was ever the case in ancient times. Even the Roman system, which represented a high degree of development, was clumsy and difficult compared to the system of notation which supplanted it. The story of the way in which numbers are now written is a fascinating one. It ought to be taught to children in school. We must remember that social values include the worthy use of leisure and that the education which ministers to such use of leisure—we call it one kind of 'liberal education'—is of high social importance. The evolution of our number system is an illuminating chapter in human progress. A knowledge of it liberalizes in the sense that it makes the entire field of arithmetic more meaningful by giving it wider horizons.

As a contribution to social utility our system of notation takes high rank. It expresses number in a language which is a help in thinking. It enables one to express large numbers compactly. It permits one to grasp large ideas quickly and accurately, and to manipulate numbers with facility. Accordingly, if we include, as we undoubtedly should, the facilitation of thought and the exchange of ideas as a practical contribution to modern life, then we must admit that our method of writing and saying numbers is of fundamental importance.

Moreover, this system of notation makes possible the fulfillment of modern needs in at least two other respects. In the first place it contributes materially to the management of modern affairs. It is difficult to see how large enterprises, with their complicated systems of records and accounts, could be carried on without the brevity and portability of numerical data which our notation permits. The heavy demands of modern business upon computation of all kinds is generally recognized. The Roman system, and to a still greater degree, the earlier systems of notation, required clumsy and time-consuming manipulation in performing the simplest operations. Even to add a column of figures such as the one at the right was by no means direct when the numbers were expressed as the Romans expressed them. The operation of division was particularly difficult. Much of our scientific work as well as of our commercial accounting would be all but impossible in any of the notations which preceded the general adoption of the Hindu-Arabic system. Think, for example, of the tremendous amount of figuring involved in the collection and the transcription of statistical data. It is true that much of this is done by machinery, but it is questionable whether such machinery could be devised to handle numbers as they were expressed in the older systems.

$$\begin{array}{r} 58 \\ 47 \\ \underline{213} \end{array}$$

In the second place, the present system of notation has contributed heavily to the development of mathematics both in pure and applied fields. For example, our system of logarithms, and the extension of numbers to the right of the decimal point, would be inconvenient if not impracticable without the characteristics of our system of notation. Mechanics and astronomy would be crippled; physics, navigation, and surveying would remain undeveloped.

It is not so much that a decimal system is needed for these purposes. The so-called Hindu-Arabic notation was preceded by many decimal systems. The thing that is unique in our present system is its place-holding characteristic, involving as it does the introduction of a character for zero. The Hindus wrote and read 2055 as "2 sahasra, 5 dasan, 5." It is as if we were obliged to write and say "2 thousands, 5 tens, 5." Since with us the *place* of each digit carries the value of the digit in that place, it is sufficient for us to write the 2 in the thousands' place without giving it a name. Then, in order to provide the empty hundreds' place, we insert 0. Next, instead of having to write 5 tens, we merely write 5 in the tens' place.

Thus it is not difficult to see that society is heavily in debt to arithmetic because of the system of notation which it has developed. This system aids our thinking and makes possible the fulfillment of modern needs in business, in science, and in mathematics.

It has frequently been observed that if we desire to have pupils use their knowledge intelligently, they must learn it intelligently. This is nothing more than to say that we must teach and pupils must learn with attention to *meanings*. If we expect pupils to have a lively sense of the magnitude of numbers and readiness in combining them, we shall secure this advantage in greatest measure, other things being equal, when we teach the numbers with emphasis upon their significance. Every course in arithmetic calls for a reading and writing of number in the Arabic notation, but this seems to be largely an imitative procedure. Not many children are ever given an opportunity to appreciate the significance of the decimal system as we employ it. It is one of the things we "haven't time for." If this notation was achieved only after a long period of experiment on the part of the race, it is fair to suppose that it cannot be appreciated in a few lessons in the third or fourth grade.

In any generous treatment of the meaning of our number system—such a treatment, for example, as bright children may be given—two types of material may well be employed. In the first place, attention may profitably be given to the history of the development of the system of notation as it is now used in most civilized countries. In the second place, the decimal system will be better understood and more fully appreciated if it is shown to be only one of many possible systems. A glance at the theory of numbers is both permissible and desirable for pupils who can understand it. Let them convert decimal numbers into numbers having radices other than ten—the quinary, octary, duodenary, and other systems—and let them perform the reverse operation of converting these non-decimal numbers into the numbers with which they are familiar. This suggestion is made not because any of the children will ever be called upon in their daily round of living to use numbers in these systems, but simply in order that the number system they will use may be made more vivid and real. When a pupil sees that, in the octary system, there are just eight characters, in the nonary system nine, in the senary six, and so on, the fact some day dawns upon him that the ten different characters or digits in the decimal system are part of a larger scheme. Again, the pupil sees

that in the octary system the value of a digit in the second place from the right is obtained by multiplying it by 8, while the value of a digit in the third place is found by multiplying it by 8 times 8, and so on for digits farther to the left. He makes similar discoveries for the nonary system, the senary, and all the others. Having done this, he sooner or later makes the generalization. He sees that the behavior of ten in the decimal system is inevitable, and part of a general law. The broadening effect of this excursion into strange number systems is not unlike the liberation of ideas which follows upon the learning of a foreign language or of any other new mode of expression.

All this may seem a bit far-fetched, but many a schoolroom is humdrum, especially for the more alert children, because it stays upon the level of the easy and the familiar. In our teaching how frequently do we find that when our enthusiasm betrays us into introducing something beyond the experience of the children they suddenly come to life and share our enthusiasm. It is our duty and our privilege to foster among our pupils a broadening of vision as well as a mastery of technique. In a narrow contemplation of the value of arithmetic we are likely to leave out of account the opportunity which the subject affords to communicate large and liberalizing ideas.

III. The Number Facts

Up to this point we have considered number ideas and the present form in which these ideas are expressed—in other words, our notation. We now come to the number facts of arithmetic. The term is here used in a restricted sense. It refers to those identities in arithmetic which a child should know without being obliged to resort to a series of steps. The fact $5 + 9 = 14$ is an identity which we wish all children to know not as a process but, as we say, automatically. Hence, $5 + 9 = 14$ is a number fact.

1. How Many Number Facts Should Be Taught?

Here students of arithmetic differ, depending on the extent to which they are willing to rely upon transfer. All are agreed that the facts involving the adding of two one-place numbers must be known; but how this learning shall be brought about is not a matter of agreement. Some say we should teach forty-five of these number facts, others eighty-one, and still others one hundred. It is clear, of course, that there are one hundred arrangements of two one-place numbers,

and every one is agreed that the sums of these one hundred arrangements should be known by everyone who knows anything about arithmetic. Those who recognize but forty-five facts to be taught reject the nineteen zero facts as inconsequential. Furthermore, they regard $4 + 7 = 11$ and $7 + 4 = 11$ as the same fact, or they believe that after teaching $7 + 4$ one may rely upon transfer to take care of $4 + 7$. This enables them to dispense with the teaching of thirty-six more of the one hundred arrangements. The result is the so-called 'forty-five basic combinations.'

Other students of arithmetic believe that after $4 + 7 = 11$ has been taught, some attention must likewise be given to its reverse. This is the point of view which seems to be gaining acceptance. It leads either to one hundred as the number of basic combinations or to eighty-one, according to whether the nineteen zero combinations are recognized or not.

Still with reference to addition, there are those who believe that certain other identities should be known as facts and that they should, to some extent at least, be specifically provided for. These are the so-called 'higher-decade facts,' the combinations of two-place numbers with one-place numbers. The total number of higher-decade facts whose sums are less than 100 is 765. No one advocates the teaching of all these facts. Some are committed to the teaching of 350 of them, and others to 312. The entire question at issue is the degree to which transfer of learning may be relied upon. No one denies that children should know these facts, both basic and higher-decade. The position that we may provide for this desideratum by teaching forty-five facts indicates the maximal reliance upon transfer. The idea that we should, perhaps in varying degrees, make detailed provision for one hundred basic facts and 350 higher-decade facts, or a total of 450, represents the minimal reliance upon transfer. In other words, the latter view of the case says in effect that the entire number of identities which we ever use in column addition, amounting to a total of 865, may only properly be cared for when we recognize in our teaching 450 of them.

Similar questions arise in the other operations. For example, shall we recognize only ninety division facts, or shall we also recognize the 360 additional facts resulting from division with a remainder? That is, shall we consider 32 divided by 7 as a combination to which the child is supposed to respond automatically, or shall we regard it as a two-step example? If we think of it as a two-step example, how shall

we look upon the second step, namely the subtraction of 28 from 32? Is that to be thought of as a fact or as a process? In other words, have we at once added by this means a whole group of higher-decade *subtraction* facts? Here one is out of the frying-pan and into the fire! Refusal to regard 32 divided by 7 as a fact at once increases the number of subtraction facts; for surely no one would be willing to say that in short division 32 divided by 7 was to be thought of as two steps, and also that 32 minus 28 was to be regarded as a process involving borrowing or equal addition, according to the method of subtraction employed.

The whole question, as far as it relates to the social value of arithmetic, is in reality far more concerned with the facts which should be known than with the facts which should be taught. The facts to be taught become important only as they insure the facts to be known. It is clear that life outside of school as well as in school requires an automatic knowledge of a surprisingly large number of facts. If by 'fact' we mean, as has already been indicated, an identity known automatically without resort to indirection, then it is perfectly clear that in column addition one must have at his disposal as facts such higher-decade combinations as $18 + 7$, $26 + 4$, $32 + 9$, and so on. If a child reaches 18 in his column addition and the next figure is 7, surely no one wishes him to resort to the process of adding 8 and 7 and carrying 1. Similarly, as long as short division is used in dividing by a one-place number, children will be expected in the example at the side of the page to handle 32 divided by 7 and 40 divided by 7 directly; that is, without intermediate steps.

$$\begin{array}{r} \text{45 and 5R} \\ 7\overline{)320\phantom{\text{ and 5R}}} \end{array}$$

The social demand in regard to the automatic control of small numbers is therefore extensive, and the only answer to the question "How many facts should be taught?" is: "Teach each child all the facts he needs and cannot get for himself, and no more." The facts he needs are numerous. The facts to be taught him are doubtless more numerous than the earlier students of arithmetic supposed; but in any event when each child is considered as an individual, the number of facts which should be taught him varies with his ability to profit by transfer. The plain truth is that the great modern social demand for number abilities has necessitated the learning *as facts* of many abbreviations and short cuts which, in a simpler and more leisurely age, would not have been needed at all. Custom now requires that

4×6 shall be learned, although it would be quite possible by roundabout methods to take care of any situation in which this multiplication fact arises by thinking in terms of addition facts—*i.e.*, 6 and 6, 12 and 6, 18 and 6. Again, in a still less exigent day, 6 and 6, as well as 12 and 6, 18 and 6, and all such combinations would not be thought of as facts at all, but merely as situations requiring counting.

The whole question of how many and what facts society requires one to know is only to be answered by considering the way in which number must be used by those who would escape being handicapped. It appears that from this point of view there is a considerable body of number identities which every one must have at his disposal in ready and portable form. Reliance upon the resolution of oft-recurring combinations into still simpler combinations or into counting would indeed diminish the facts to be learned, but would substitute lower for higher order habits—a substitution which the work of the world will not permit. Accordingly, the social value of arithmetic contemplates the learning of many facts for direct use. Moreover, it is idle to find fault with those who point out the facts which life demands. Indeed, those who still stick to the forty-five combinations in each process, do, in reality, expect children to know all the derived facts which they will need. They fail, however, to supply the opportunity to do so except as a matter of transfer.

Those who stand aghast at the huge number of facts which certain investigators declare should be known think of the learning of each of these facts as an isolated act. "What!" they exclaim. "Must we teach one hundred basic facts and two hundred twenty-five higher-decade facts in order to take care of ordinary column addition? And must we teach still more addition facts in order to take care of carrying in multiplication, to say nothing of all these facts in subtraction, multiplication, and division? The thing is preposterous! Children can't learn all these facts, and if they could learn them, they couldn't retain them."

The answer is two-fold. On the one hand, society demands this factual knowledge just as certainly as it demands, in the use of English, the ability to spell thousands of different words. Arithmetic does not fulfill its social uses if one has to make separate examples of $54 + 7$ and $45 + 6$ in multiplying 568 by 9. On the other hand, children who are reasonably well taught do, as a matter of fact, learn and retain these number combinations.

2. How to Teach the Facts So That They Will Be of the Greatest Social Value

a. Teaching the Number Facts with Meaning.—In the first place the number facts should be taught with a full consciousness of their meaning. Hand in hand with the teaching of the facts in the four major operations should go the teaching of the nature of these operations themselves. As the child learns his subtraction facts, he should be getting an intelligent idea of *what subtraction is.* Similarly, as he learns the facts in multiplication and division, he should be getting clear ideas of the meaning of these processes. In order that his arithmetic may be useful to him in the solution of life's problems, he must have a firm grasp of the meaning of its operations.

When a child is taught to say "5 times 7 are 35," he is given a meaningless verbalism. He doesn't know what 'times' means in this connection, and very likely his teacher is not very sure about it either. It is not even clear whether the verb in this expression should be 'is' or 'are.' If, on the other hand, this fact is taught in the form "five 7's are 35" we have expressed the multiplication idea in a form which is readily understood. Five 7's means exactly what it says. It means one 7 and another 7 and still another 7 until we have five 7's. Children have great difficulty with the number facts involving 1. The most common wrong answer to "1×4" is "5." This is doubtless because 1 and 4 have already been associated with 5 in the addition fact "$1 + 4 = 5$." It is not difficult to understand how this confusion comes about when the multiplication combination is read "1 times 4." The point is that this phrase has no meaning. How different is the situation when we teach the combination in the form "one 4 is 4." It is practically impossible for the child to say, "one 4 is 5." Such a statement is evidently absurd.

Even greater confusion is caused by the conventional teaching of the division facts. Children are taught to say "7 is contained in 56 eight times," or "7 goes into 56 eight times," or "56 divided by 7 equals 8." None of these statements has much meaning for a young child. It is much better to say the number of 7's in 56 is 8, or more briefly, "7's in 56 are 8."

All of these more meaningful ways of teaching and saying the number facts will help pupils in solving verbal problems. Many of them do not know what division means, nor how to interpret multiplication, nor how each of these is distinct from addition and subtraction.

The time to teach these basic ideas is during the teaching of the basic facts. For this reason numerous verbal problems should be used from the beginning to illustrate and apply these facts. Part of the reason why children fish around and develop weird rules of procedure is because they have little appreciation of the meaning of the fundamental operations. One child when asked how she could tell whether to add, subtract, multiply, or divide, said: "If there's more than two numbers I adds; but if there's two numbers, I subtracts, unless one of them is a little number. Then I multiples; but if it will go even, I divides." The acuteness which this child exhibited in the formulation of her rule—a rule which really works in a great many cases—would have sufficed to give her a real understanding of the situations requiring the various operations. There are those who believe that the greatest social value of arithmetic arises from the training in thinking involved in solving verbal problems. To the extent that this is true, any teaching of the number facts which results in better problem solving should be adopted.

Allied to teaching with meaning is practice with meaning. We rely far too much upon practice with abstract number. We glibly say that our arithmetic should be life-like, yet it is difficult to make out a case in life for the use of $5 + 6$, or $28 \div 7$, or 9×7 as abstractions. When we have occasion to use these number facts, we do so in problems. Why then should we not practice on the number combinations in problem form? Dewey has well said: "Training by isolated exercises leaves no deposit, leads nowhere; and even the technical skill acquired has little radiating power, or transferable value."[4] Again he says: "In some educational dogmas and practices the very idea of training mind seems to be hopelessly confused with that of a drill which hardly touches *mind* at all—or touches it for the worse—since it is wholly taken up with training skill in external execution . . . Practical skill, modes of effective technique, can be intelligently, non-mechanically *used,* only when intelligence has played a part in their acquisition."

To the writer this can only mean that practice and initial learning should be charged with meaning. Unless this is the case it is entirely possible for a child to know his number facts in abstract form without being able to make any consistent use of them. A child does not *know*

[4] Dewey, John. *How We Think.* Boston: D. C. Heath, 1910.

a fact unless he can use it in a situation requiring it. Merely to be able to *say* "9 times 7 are 63" is insufficient.

Accordingly, we should provide many more verbal problems for practice—practice not only in problem solving but also in using the number facts and processes. Even apart from the value that this kind of practice has for the *use* of the subtraction facts, it may also have a larger effect than we now know upon learning the facts as such. It is recognized in general that things are learned by doing other things requiring their use. It may be, for example, that the performing of a few good '7×9 problems' is worth two or three times as much as an equal number of 'practices' on '$9 \times 7 = 63$' as an abstract fact.

b. Grouping the Number Facts.—On page 31 the topic was set up, "How to teach the facts so that they will be of greatest social value." It was there stated that "in the first place the number facts should be taught with a full consciousness of their meaning."

In the second place, in any consideration of the socially effective teaching of number facts, one must consider how they may be advantageously grouped for learning purposes. Transfer will not take care of all the learning of the derived facts, but it will take care of part of the learning. Moreover, by skillfully grouping the facts for learning purposes, a consciousness of identical elements may be assured and transfer may be facilitated. For example, a knowledge of 7 and 5 does not, in general, insure a knowledge and 17 and 5 or of 27 and 5. This is the common experience of the classroom. Yet this is far from meaning that 17 and 5 and each of the other combinations of the same series must be learned *de novo*. The necessary higher-decade facts should be grouped with the basic facts from which they are derived and which they exemplify and support. Matters may be so arranged that each of the 100 basic facts is seen to have its retinue of higher-decade facts. Thus, each basic fact and its related higher-decade facts constitute a series which may be regarded as a single teaching-and-learning unit. On frequent occasions the pupil may be asked to "tell the whole story about 7 and 5." This will provide for the learning of 17 and 5, 27 and 5, and so on, as far as may be thought desirable, without actually teaching, in the ordinary sense of initial presentation, any of these higher-decade facts. Moreover, it will provide excellent drill on 7 and 5 itself.

Similarly in subtraction certain frequently used examples involving two-place numbers may be directly related to basic facts. Each of

the one hundred subtraction facts gives rise to two series of higher-decade facts which are worth knowing as facts. For example, with 12 — 8 are associated, first, 22 — 8, 32 — 8, and so on; and, second, 22 — 18, 32 — 18, and so on. The case for the social use of higher-decade subtraction, as a series of facts to be learned rather than as processes to be performed, is not so strong as the case for higher-decade addition, yet the case is strong enough. A great many higher-decade subtraction facts must be used *as facts* in short division; and it is also highly convenient in the affairs of life to be able to handle situations involving, say, 27 — 4, or 32 — 26, without having to break the examples up into subordinate subtraction facts. Moreover, if we do not introduce higher-decade subtraction facts, we fail to provide the basic subtraction facts with the fortifying practice which basic addition facts receive, thus perpetuating the greater difficulty of subtraction facts which investigators have invariably found. If, therefore, it is agreed that higher-decade subtraction facts should be taught, they may be taught by combining them with their corresponding basic facts as part of a 'whole story.'

We may likewise take advantage of the principle of grouping when we teach the basic facts themselves. For example, when we teach 5 and 7, we may use the same teaching situation to present 7 and 5, as well as the corresponding subtraction facts 12 less 5 and 12 less 7. There is an economy of effort here which is obvious. Moreover, there is experimental evidence to support this practice. In an investigation carried out at six teaching centers, it was found that the "together" method of teaching the addition and subtraction facts was superior to the "separate" method at five of the centers. Moreover, the conduct of the experiment at the sixth center was such as to cast doubt upon the results at that center.

Inferentially, multiplication and division facts should be grouped and taught together rather than separately, although the writer is not aware of any competent evidence bearing on this question.

The possibilities of grouping have not by any means been exhausted. The 19 zero combinations in addition may usually be taught together; and the same is true with the 10 zero facts in subtraction in which zero is the subtrahend and the 9 others in which the minuend and the subtrahend are the same.

c. Teaching the Number Facts to the Point of Habitual Response.— In the third place, in any consideration of teaching the number facts

so that they will be of the greatest social value, one must recognize the need of reducing the control of these facts to an automatic basis. It is clear that the people who aren't quite sure how many 6 and 9 are, or who hesitate and occasionally go wrong when confronted with the need for multiplying 9 by 7 are at a disadvantage—and it is surprising how many folks are in such a state of uncertainty. Like bad spellers, they fall short of the social standard.

Of course, automatic response in handling the number facts is only to be attained through the operation of the laws of habit formation, and this is no place to enter into such matters. The reader may do well to note, however, that "the social value of arithmetic" is conditioned by the ready command which people have of the subject. Indeed, it is right here that the reductionists make one of their biggest mistakes. They observe that people do not use a certain type of arithmetic without finding out whether they *can* use it or not. Very likely, for example, the reason why these investigators find so little use of the metric system is because people do not know how to use it. This has no bearing upon the real value of a universal decimal system of measures. Much the same observation may be made with reference to the number facts. If the investigators went far enough they would find that the combination $7+9$ occurs in life much less frequently than $2+2$, or $1+4$. Moreover, they would discover something else. They would find that a great many persons, when confronted by $9+7$, never respond to it directly as they do to $2+2$ or $1+4$. This is because they have never learned to respond automatically to $9+7$. They obtain the answer to this combination by using a process. The process for some is "one less than $10+7$." For others it is $(9+9)-2$, and for still others it is $(9+1)+6$ or $(9+3)+4$ or $(7+7)+2$. For these persons $9+7=16$ does not exist as a number fact; and a searching analysis of the number of times it is really so used might lead one to infer that it should not be taught as a fact at all!

The number facts of arithmetic, by their name and nature, are socially valuable. Any principles which would rule out some of them would rightly be called in question as a valid criterion of exclusion. It could properly be said that people failed to use these facts because they did not know how, and that they would use them if they had them at their command. The situations in life may call loudly for the use of a bit of arithmetic, but the human beings who find themselves in these situations may be quite unable to respond with the

needed reactions. It is for this reason that all number facts, once it has been determined what these facts are, should be reduced to the basis of automatic response.

IV. Skill in Processes

Attention has been called in the preceding pages to three broad topics, namely: number ideas, notation, and number facts. Attention will now be directed to the social uses of the number processes. By the term 'processes' is understood those operations which require a combination of number facts. Thus, 6×38 is a process because nobody is expected to learn $6 \times 38 = 228$ as a fact. One performs the process by using the facts $6 \times 8 = 48$, $6 \times 3 = 18$, and $18 + 4 = 22$.

It is clear that if any of the subordinate facts utilized in a given process is incorrectly remembered, the entire process will go wrong—that is, the result will be incorrect. A study of the ways in which errors may be made in each of the processes is exceedingly illuminating and affords the most useful basis for individualizing instruction. Again, the analysis of a process into its unit skills is likewise of great value in the teaching process. Analyses of this sort have been made elsewhere in this Yearbook. No attempt will therefore be made to furnish such material at this point. Its bearing upon the social uses of arithmetic is merely noted.

1. Accuracy and Speed

The two great questions of accuracy and facility in performing the processes have bearing upon social value. Accuracy is of first importance. Speed or facility is decidedly secondary. The speed that is accomplished only at the expense of accuracy is worthless. Men and women in their affairs are not ordinarily interested—at least not profoundly interested—in the speed with which they may compute the value of an investment or the lay-out of a family budget. If the computations lead to a reliable result, it is relatively immaterial whether it takes five minutes or an hour.

A great deal of our current practice in arithmetic involves a harmful emphasis on speed. Our standardized tests have too often created the impression that speed might, in some sense, compensate for the lack of accuracy and stand in the stead of accuracy. When the number of examples or problems attempted falls short of the expectancy, the principal or superintendent is likely to take the attitude

that the children must be spurred to greater speed in trying more examples, even though they get them wrong. Not only is this destructive of usable skill for the affairs of life, but the mental attitude engendered is essentially false. It is the attitude of being content with slip-shod work provided only it is done with speed.

On the other hand, speed when it is not forced—in other words, when it arises naturally in the course of the situation—is of considerable value socially. Mere hurry, the exalting of speed as such, is a besetting sin of our age; but fluency, facility, and the ability to turn out work of high character without dawdling—this is of inestimable value to an individual and to society.

As these ideas apply to the processes of arithmetic, they mean that we shall attempt first to develop accuracy and the ideal of accuracy. Second, we shall seek to secure as high a degree of speed as is consonant with accuracy. In other words, speed will not be speed as such but the speed with which a good job may be performed.

To the end that skill in the processes may be secured—accuracy with its attendant facility—it is important that the processes have a maximum of meaning to the child. Each new process and each new phase of a process should begin in the concrete. In other words, a situation should be shown in which that particular type of computation is required. The movement should then be to the abstract, that is, to practice on the new procedure. But this excursion into the abstract should be broken at intervals by a return to the concrete. Thus the abstract will gain significance. Among other things, it will be apparent when the abstract procedure applies. Too often the social value of arithmetic is lost, not because one may not be able to perform a process, but because one does not know what process to use.

V. The Content of the Curriculum

Throughout the pages of this chapter has run the idea of what is to be taught—in other words, of the subject matter of the curriculum. Yet it is worth while to bring this idea to a focus. What, then, should be taught in the way of number knowledge and the application of number in order that the uses of society may best be served? The position is taken that in answering this question no conflict should arise between the individual and society. In other words, it is held that whatever is of greatest good to the individual is likewise of greatest good to the aggregate of individuals. Whatever makes the

life of the individual fuller and richer does in fact, upon any valid idea of fullness and richness, make him a better member of society. Moreover, whatever accomplishes or contributes to the accomplishment of this result in any widely diffused way is of service to the community of individuals which we call society.

1. Other Criteria Besides Frequency of Use

Social value therefore arises in two ways, and both these ways must be recognized. On the one hand, there is the custom already found to exist in the social organism; on the other hand, there are the practices of selected individuals—practices which although not widely adopted are already of great value to society and would be of still greater value if they were more commonly followed. Accordingly, the criterion of social utility as it is ordinarily determined by a survey of the uses of arithmetic by adults is only a partial criterion. There are other ways of determining value. Some of these are more or less obvious, such as the abilities of pupils, their social status, and the likelihood of their going on to the high school, and so on. These considerations obviously have reference to the individual. They raise sharply the question of whether in arithmetic or in any other school subject it is desirable to have *a* curriculum rather than a series of curricula. The idea of a single curriculum unmodified by individual needs is one of the crude by-products of mass instruction.

There are still other sources of value than those arising from social utility as ordinarily determined. These values arise from an interpretation of life; in other words, they rest upon philosophy. Moreover, they have to do with ideals, attitudes, and appreciations as well as with skills. For example, the position may properly be taken that arithmetic is the only subject in the elementary-school curriculum which affords a basis for quantitative thinking. If it is held that a paramount objective in education is teaching children to think and so to be able to meet new situations, then the study of numbers is the one study which affords both the attitude and the data conducive to quantitative thought. As such it has high value. Moreover, it has this value inherently. It possesses it more or less independently of the number of times people use fractions having large denominators or utilize proportion in their daily affairs. Without quantitative thinking modern life would be impossible. The extent to which this is true may be realized by supposing that number were suddenly to dis-

appear from the writing and printing of man. Under these circumstances "every mill in the whole world would slow down, and every large concern would close until it could replace its accounts, its statistical material, its formulas for work, its measures, its tables, and its computing machinery. Every ship on the seven seas would be stricken with blindness, and would wallow helplessly, awaiting the probable starvation of its human burden. Not a rivet would be driven in a sky-scraper in New York City, because the steel girders would have lost their numbers; Wall Street would close its portals; the engineering world would awaken tomorrow morning to a living death; the mines would shut down; and trade would relapse to the condition of barter as in the days of savagery."[5]

This dismal picture is merely based on the supposition that the printed and written symbols of mathematics should disappear. If the ideas for which these symbols stood were also blotted from the remembrance of man, all the things which Dr. Smith recites in the passage just quoted, together with hundreds of other things which he did not mention, would become permanent. Civilization as we understand the term would be lost. Certainly modern life as we know it could not exist.

If it is true that quantitative thinking means so much to the race, then the curriculum of the school must be formulated so as to transmit to each oncoming generation this fundamental acquired type of thought. Moreover, the body of knowledge which, more than any other, affords the basis for quantitative thought should be estimated as to its inherent value for the high purpose it is to serve rather than for the particular manifestation of certain processes reported to be used a specified number of times by a limited group of adults. In other words, appeal is taken from *usage* to *value*. It is not merely what men use, but what will make their lives fuller and richer. Proportion, it is reported, is seldom used; therefore proportion shall not be taught. Then it will continue not to be used. Its true value may be anything from zero to infinity, but it cannot be taught because it is not used and it cannot be used because it is not taught.

2. All the Usage Cannot be Secured

Even if the frequency with which people use certain processes in arithmetic were a valid criterion, the question of whether this fre-

[5] Smith, David Eugene. "Mathematics in the training for citizenship." *Third Yearbook, National Council of Teachers of Mathematics,* page 13.

quency of use has been ascertained may well be raised. It seems certain that one trouble with selecting content for a curriculum on the basis of reported usage is that no one can secure all, or even much, of the usage. Probably the most pressing problem to which arithmetic may be applied is the problem which for want of a better term may be called that of 'making a living.' This involves the whole question of getting a sufficient income, of spending wisely within one's income, and of investing to provide for old age or disability. Not long ago it was authoritatively stated that the average wage of workers in gainful occupations was twelve hundred dollars per year. Probably the average is now higher, but it is not so high as the cost of a decent living for an average family; for this cannot be estimated at less than two thousand dollars a year. There is a tragic discrepancy here, and it is certain that millions of men and women are tremendously concerned with reducing the discrepancy. This problem of making ends meet is largely quantitative. It involves the use of arithmetic continuously, thoughtfully, and prayerfully. Yet this anxious figuring to see whether the pay-check will last is not reported when the investigator comes around. The setting-up of a family budget, the classification of expenditures in accordance with the budget, the painful balancing of immediate and remote advantages, the tedious relating of costs and values—these are too intimate to be divulged. One of the investigators states that the people from whom he received his reports were not "figuring through complete situations, such as budgeting the family income, determining by careful estimate in advance the desirability of buying a carload of steers and finishing them for market, or determining the relative advantages of renting and buying city property." This is an interesting fact. It probably means that some of these people were indeed failing to think in quantitative terms, but it also means that others, although doing so, were failing to report the fact. Neither condition justifies in the least degree a conclusion regarding the curriculum in arithmetic.

Not only do people fail to report their intimate affairs, but they also fail to report the things which escaped their attention. At luncheon someone says that a certain stock has gone up from $87\frac{3}{4}$ to $91\frac{5}{8}$. At once one tries to understand, and in doing so must subtract, either vaguely or explicitly. Note that the situation involves mixed numbers with fractions whose denominators are unlike. We are told that such conditions are not frequent in life. The subtraction in this case is

casual and there is scarcely a chance that the incident will be remembered or recorded, for the benefit of the investigator, as a use of the subtraction of mixed numbers.

Again, we are told that people seldom compute with large numbers. This morning's paper contained a statement of the surplus stocks of wheat—a matter which has close bearing upon certain major legislation in Congress. The statement is that the present surplus amounts to 130,000,000 bushels, whereas (the news item continues) the surplus a year ago was only 40,000,000 bushels. Upon reading this, what does one do? Clearly the figures are without much meaning unless they are related to each other. If one is to understand the statement at all, one cannot do less than subtract these two numbers. This brings out the relation that the surplus this year is 90,000,000 bushels more than it was last year. A still better understanding of the meaning of the statement will be secured if one also notes that this year's surplus is more than three times that of last year. Here is a case of subtracting and dividing with eight-place numbers, and this, the reductionists say, is so seldom done that such operations should be omitted from the offerings of the school. It is extremely doubtful whether a man who reads this newspaper item and mentally makes the calculations described will remember this as an instance of a use of arithmetic. He is no more likely to report this as a use of the division and subtraction of eight-place numbers than he is to report the use of a complementary infinitive or an adverb of degree in his written or spoken speech.

If it is objected that only a few people would be interested in the item about the surplus stocks of wheat, and that only occasionally would an especially well-endowed reader mentally perform the operations of subtraction and division, and if on the basis of this assertion it is still urged that such operations are too seldom employed to justify teaching them in the schools, then the answer takes another turn. It is proper to reply that the school *should* equip people, first, with the interest in quantitative matters to read such an item, and second, with facility in comprehending and using large numbers to the degree needed in understanding the item. On one or the other of two counts the position of the reductionists is assailable. Either they do not get all the uses or they fail to take account of values independent of frequency of use.

3. Topics of Low Frequency

a. Large Numbers.—What may be taken as the interpretation, so far as the curriculum is concerned, of the fact that a certain investigator found no addition problems involving numbers of more than five places, that in subtraction "the problems had but three or four numbers in the minuend," that in multiplication the vast majority of problems had no more than three-place numbers in the multiplicand, and that in division by far the most common problems had two-place or three-place numbers in either the dividend or the divisor? Shall the conclusion be that numbers of more than five places should be banished from the curriculum? This is by no means justified. Even if teachers were sure that a child would almost never have use for a number of more than five places, they would nevertheless be justified in teaching him numbers of six or seven places in order that he might know five-place numbers better. If he went no farther than five-place numbers, such numbers would always find him exercising his greatest effort. They would represent the limit of his ability. Now the ability of human beings does not stop short at a certain point; it drops gradually. The result would be that the person who was taught numbers of no more than five places would have a vague and uncertain conception of such numbers. Accordingly, in order to assure the handling of five-place numbers, training should be afforded in still larger numbers.

It has already been pointed out that one has use for numbers of six, seven, and eight places, and even for larger numbers. The daily newspaper has lately been carrying items requiring the use of numbers in the hundred-billions. Even if life does not call for the frequent adding of six-place numbers or the subtracting of five-place numbers, nevertheless when these occasions do arise, the 'need' for handling these numbers is just as imperative as though it occurred frequently. Moreover, for the very reason that this need seldom arises, and because it is nevertheless a legitimate need, it is incumbent upon the school to make particular provision for it. Life, by affording practice upon it, will not take care of it; therefore the school must do so.

b. An Opposite Interpretation.—Here arises an interesting comment on the interpretation of 'frequency of use.' An investigator finds that Item A is among the top ten percent in frequency of usage, while Item B is listed at less than one percent. One group will say that Item A should be taught and that Item B should not be taught. It is

not altogether crotchety to say that the inference is precisely the other way around. Item A will be cared for by the school of life; Item B (always supposing that the occasional need is an unescapable one) will have to be vigorously taught in the school just because life does not take care of it.

This reverse type of interpretation goes pretty far, and in its broader aspects is related to one's view of life. A survey commission, for example, finds that in a mill town ninety percent of the children go into the factories. This fact, with the details supporting it, is furnished to two educational experts. One draws from the data that the boys and girls should be taught the things they are going to use in the factory. The other says: "They're going to get enough of that sort of thing anyway. Let's give them something different to make their lives broader and better balanced." The latter expert will doubtless recommend a more humanistic curriculum than the former. Thus, the conclusions of the two experts may be diametrically opposite, though based upon the same unchallenged data.

c. *The Principle of Difficulty.*—This idea easily merges into the idea of difficulty as applied to the curriculum. Doubtless if the fact $2 + 2 = 4$ was never taught in school, it would be known just the same. In fact, two-thirds of the children know it when they enter school. Yet $2 + 2$ and other very easy facts are laboriously taught to every pupil. Moreoever, these easy facts are taught at the beginning of instruction in the number facts, with the inevitable result that they receive more practice than the harder facts. It is possible that if only such addition facts were taught as $7 + 8 = 15$ and $8 + 9 = 17$, time would be saved and the facts would be better known. It is a question whether more of the things which life provides for should not be omitted from the work of the school. Here, then, is another principle applicable to the inclusion or exclusion of material, namely, the *principle of difficulty*. This principle will operate to exclude much that the doctrine of frequency of use would include.

The principle of difficulty will likewise operate to bring into the curriculum some things which would be ruled out upon an uncritical application of the frequency-of-use notion. Consider, for example, the case of common fractions. Here the reductionists have made a rather impressive case. One of them reports that 96 percent of all the fractions which he found to be used were included in this list: $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{1}{5}$, $\frac{2}{5}$, and $\frac{4}{5}$. Another reports that 93.9 percent of all

the fractions which occurred were in the list $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{5}$, and $\frac{1}{8}$. On the other hand, both these investigators and others report the occurrence of fractions with denominators 12, 16, 18, 32, and 64. One report shows not only that the fractional parts of 12 were much used, but also that the addition of all possible fractions with 12 as a denominator was employed.[6] Again, according to a report of department-store transactions involving multiplying, many of the fractions with 36 as a denominator were used.[7] The same report indicates that $\frac{1}{9}$, $\frac{5}{9}$, and $\frac{7}{18}$ were likewise used. Most of these fractions, however, were used only once and, since there were 1079 "uses," a frequency of 1 amounted to about $\frac{1}{10}$ of one percent.

What is the right attitude toward the teaching of multiplication involving fractions of such infrequent occurrence? In the writer's opinion, these fractions should be taught; and if taught at all, they should be taught with sufficient emphasis to accomplish a positive result. The idea almost seems to be that since 96 percent of the fractions in use make up a list of no more than ten, something like 96 percent of teaching emphasis should be placed upon these ten fractions. This is almost as silly as would be a similar interpretation of Ayres's early findings in regard to vocabulary. According to his figures one hundred words constitute 60 percent of the writing people do. Shall we say that 60 percent of the teaching of the schools in spelling and language should be devoted to the one hundred words of most frequent occurrence—to *the, and, but, to, he,* etc? Unless these words have certain instrumental value in approaching other words, the school would in reality be justified in leaving them out of account altogether.

The fact is that frequency of use offers no guide to emphasis in teaching. If we can rely upon its findings, it will tell us how often certain items are used. That is all. From that point on, the curriculum-maker must do his own thinking, and not permit the figures to be a substitute for thinking. Among other things, he must decide whether an item used $\frac{1}{10}$ of one percent or $\frac{1}{100}$ of one percent of the total usage is used often enough to justify teaching it—that is, to justify teaching merely from the point of view of its frequency of use. If the decision on this or any other basis of value is to the effect that the item should be taught as a part of the general curriculum, then the

[6] Mitchell, H. Edwin. "Some social demands on the course of study in arithmetic." *Seventeenth Yearbook of this Society,* Part I, pp. 7-17.

[7] Benz, Harry E. "The Arithmetic of Department-Store Transactions." M.A. thesis, University of Iowa, 1925.

emphasis upon it is by no means indicated by its frequency index. Because of its difficulty or because of its value (determined upon other than a frequency basis) it should perhaps be taught with an emphasis which has little reference to the number of times it will be used. It may be part of a system which, without it, is seriously weakened. For example, in the series of fractions said to constitute 96 percent of the usage, $\frac{1}{5}$ and $\frac{2}{5}$ occur, but $\frac{3}{5}$ and $\frac{4}{5}$ do not. Even if $\frac{3}{5}$ and $\frac{4}{5}$ were *nil* in usage, it would be nonsense to omit them, because they amplify the meaning of fifths—of the idea of five equal parts of which one or more may be considered. One-fifth as *one* of the five equal parts of a unit or of a number and two-fifths as *two* of these equal parts at once suggest to the inquiring mind, "Can't we have *three* of these parts and also *four?*" The reply in substance is: "No, you can't have either three or four of the five equal parts, and you mustn't ask for them. A man who is much wiser than you are has decided that for you. You see, your fathers and mothers failed to tell him that they used three and four of the five parts, and so he knows that you won't use them either."

d. Proportion.—Another topic which has been ruled out of the curriculum because of its alleged lack of use is proportion. Yet proportion has value as a *form of thought* which ought to give it an undisputed place in the offerings of the elementary school. As Osburn has pointed out, "It is fundamental in all our planning for the future when quantitative terms are involved . . . No one can plan budgets, make estimates in advance, determine the relative advantages of certain procedures, or think through complete situations without proportion."[8] Moreover, proportion has what Bagley calls 'preparatory value.' One of the good results of studies of social usage has been the suggestions for additions to the curriculum. This is the positive side of the social utility program, just as elimination is its negative side. Two topics have been suggested for inclusion in the curriculum, namely, similar figures and drawing to scale—the latter exemplified in the making of plans, maps, and graphs. Each of these subjects requires proportion as a prerequisite.

e. The Cases of Percentage.—At the risk of being tedious, one more topic will be mentioned among those which are customarily ruled out of the curriculum on the basis of lack of use; namely, the second and

[8] Osburn, W. J. *Corrective Arithmetic.* Boston: Houghton Mifflin Company, 1929. Vol. II, p. 83.

third cases of percentage. There are two reasons why these cases should be taught. The first is that percentage may be better understood. Even if Case I—the finding of a given percent of a number—were the only case likely to be used, there would still be sufficient reason for some teaching of the other two cases. This first reason is therefore a denial of the sufficiency of frequency of use as a criterion. It is an assertion that other considerations may outweigh lack of usage—in this case the necessary support which the topics in question give to a topic admittedly useful. People in general lack a lively knowledge of the meaning of percentage. If they had a better knowledge of it, they would not, for example, buy on the installment plan. The sellers of goods on the installment plan know their arithmetic; the buyers do not. *Caveat emptor.*

The second reason is that there are many situations in which the second and third cases of percentage are necessary. The second case, it is generally agreed, is used more often in business and social affairs than the third case. Accordingly, a statement of the uses of Case III will be made and the reader will be left to infer that the situations calling for finding what percent one number is of another (Case II) are still more numerous and more common.

Case III of percentage—finding the base when the rate and percentage are given or may be found—is the least common type of percentage; yet it is common enough to require teaching on a purely utilitarian basis. Consider the following situations:

(1) A man estimates that he has completed 70 percent of his job of laying a mile of sewer pipe. He is called upon to figure on a contract for laying another mile. His present experience will enable him to compute the probable cost of various items or the amount of the contract as a whole.

(2) I have finished sixty pages of a manuscript and estimate that I have completed 75 percent of what I have to say. How long is the manuscript likely to be? In general, when one knows the number of units of a task completed and can make an estimate of the proportion completed, the size of the whole task may likewise be judged by using Case III of percentage.

(3) A placard on a counter may say: "Everything on this counter cut 20 percent." On given articles, one may wish to compute the former price.

(4) One sometimes wishes to apply hypotheses in percentage form to actual conditions. For example, the Brown family spends $500 a year for clothing. Family budgets allow about 18 percent of the income for this item. How much income should the Brown family have to justify this expenditure?

(5) Goods are sometimes marked so that a designated discount may be allowed and still leave a given profit. For example, at what price must a book be marked so as to sell at $1.50 after allowing a discount of 25 per cent? In this problem, $1.50 is 75 percent of this marked price. The finding of the marked price is a case of finding the base when the percentage and rate are given; that is, it is Case III.

(6) Part of the objection to Case III centers around problems which are regarded as unacceptable because the answer must be known in making the problem. An example would be: "Sixty percent of my money is $1200. How much money have I?" The objection to this type of problem is not always valid. Even if the answer is known, it may not be known to the person who is interested in solving the problem. Suppose a dealer is known (no matter how) to be accustomed to sell potatoes at an advance of 40 percent on the price he pays the farmer. The farmer ascertains that the dealer is selling potatoes at $2.25 a bushel. The farmer is interested in finding out how much he has a right to expect to receive per bushel. The dealer knows the answer, but it is not the dealer's problem. It is the farmer's problem. Again, the answer may have been known at one time to the person who is interested in the problem, but he may have forgotten it. An editor reported concerning a manuscript, stating among other things that a certain section of it, amounting to 210 pages, was 45 percent of the entire manuscript. Several weeks later the matter came up again when the manuscript was not available, and the editor needed to know the length of it. He had forgotten this fact, but was able to compute it by using Case III.

(7) Finally, the American Hardware Association was not deterred by the ban against Case III from advocating that gain be computed as a percent of the selling price rather than as a percent of the cost. Business men quite generally accepted this principle without regard to the feelings of the reductionists. A bedroom suite cost a furniture dealer $47. At what price should he sell it in order to take care of estimated overhead and profit? The older way was for him to apply a certain percent (say 60) to the cost price ($47), thus employing Case I. But according to the way which has lately become popular, he will prefer to use, as his price-marking formula, a different percent (say 45) and to apply it to the unknown selling price as the base. The difference between the cost price and the selling price is to be 45 percent *of the latter*. Therefore the cost price itself must be 55 percent of the selling price. In technical terms, the percentage and rate are known and the base is to be found. The only way, therefore, that the furniture dealer may solve his problem is by using Case III.

This last example illustrates very neatly one inadequacy of the idea of social utility based upon frequency of usage. In 1922 almost no uses of Case III were discovered in business, and in 1925 the elimination of Case III was still recommended. Two or three years later a business procedure came into use which required Case III, and in a few more years this procedure may be all but universal. The method of scoring frequency of use makes no provision for change.

4. When Should Regular Number Work Begin?

This is one of the half dozen most important questions in administering the curriculum. It is also profoundly important in considering the problem of social values. Recently students of arithmetic have made it clear that the school is hurrying children too rapidly over the fundamental parts of the subject, particularly the development of number concepts, the appreciation of the number system, and the knowledge of number facts. The drift of opinion is toward spending more time on these matters. Skill in mere computation has not only been regarded as the test of success in teaching arithmetic, but this type of skill has also been emphasized too soon.

Now, unless school people are willing to go much farther than inertia has yet permitted them to go in postponing difficult topics, such as long division, to later grades, the only way in which a well-sustained treatment of the true fundamentals of arithmetic may be secured is by starting them early.

It is usual to beg the question by asking, "When should *formal* arithmetic begin?" A very acceptable reply to this question might be that it should never begin, any more than formal reading or formal history should begin. The literature on education is liberally sprinkled with warnings against formal instruction. Evidently the term as used in this connection needs definition.

Such a definition is provided by the writers of the chapter on arithmetic in the *Third Yearbook of the Department of Superintendence of the National Education Association*. On page 37 the following statement is made: "The term 'formal arithmetic' as here defined means formal drill work and a definite attempt to teach for automatic reproduction the number combinations in the fundamental processes." One may find fault with this definition from a number of points of view, but the purport of it is clear enough, both as to what it says and as to what it leaves unsaid. In the same chapter the following statement is also made: "Since the purpose of drill is to fix by repetition something previously comprehended, it is evident that drill should follow, not precede, adequate experience and understanding."

The writers of the arithmetic chapter in the *Third Yearbook* then present certain evidence as to the unimportance of teaching number in Grades I and II. They first cite the study of Dr. Joseph S. Taylor, who reported that at the end of the second school year children who had received no arithmetic work in the first grade except counting

were at an advantage over those who had done two years of formal work. The inference is made that "the totals appear to favor omitting formal number work from the first year and giving the time to reading." An informal statement from Professor Stone (presumably Professor John C. Stone) is likewise reported to the effect that pupils who began formal work in the third grade were indistinguishable from others in the same city who had begun number work in the first year. Finally a reference is made to H. L. Smith's Bloomington survey. The purport of this reference is "that children who had had no formal number work in grades one and two were not handicapped at the end of the third year in competition with children who had had number work regularly during the first and second years."

One is therefore prepared to find in the *Fourth Yearbook* of the same organization a pronouncement on this question. A portion of it reads as follows: "Arithmetic should be taught when a child has need for it and when he has the proper background of experience to profit by it. While the available data relative to the question, "When shall formal arithmetic begin?" are incomplete, they point to the desirability of more careful attention to reading ability and the entire omission of formal number work, except for counting, from the first grade. . . . The inference is that with this ability [*i.e.*, reading ability and an experience basis for number work] developed, children who do not begin formal arithmetic until some time in the second or at the beginning of the third grade will at the end of the third grade not be handicapped in competition with children who have had formal number work regularly during the first and second grades."

It will be observed in the foregoing quotations that two points of view are stressed: first, that arithmetic should be taught when the child has a proper background to profit by it; a view which is further shown to mean that drill should follow, not precede, adequate experience and understanding. The second point of view is that arithmetic —shall we say formal arithmetic?—should not be taught in the first and second grades because time spent in doing so is wasted. The support for this point of view is found in the studies of which Superintendent Taylor's may be taken as a type.

There is good reason to believe that when a normal child enters grade one he already has "the proper background of experience to profit" by arithmetic teaching. In other words, properly selected and motivated drill, if given him at that time, will in fact "follow, not

precede, adequate experience and understanding.'' The number abilities which children already possess when they enter grade one at six years of age are much greater than the school assumes. Evidence to this effect in scattered form has been available for a number of years—mostly, however, in foreign reports. In Chapter IV of Part II of this Yearbook a new investigation of the subject is reported, confirming the former investigations and supported by supplementary data. These children, as has already been said, can count, 90 percent of them to ten, 60 percent of them to twenty, and not a few to one hundred. They can do this either by rote or with objects. They have usable concepts of numbers as far as ten, in the sense that they can make up a group of objects of a specified number or identify the number of objects in a group and name the number. If the one hundred basic addition combinations are arranged from easy to hard and selections from the first eighty-eight of these combinations are taken, half the children can solve half the combinations. Quite evidently they must have had ''an experience basis for number'' before coming to school, and from every tenable point of view they are ready to do additional learning based on an experience which is far from inadequate.

The results of studies like those of Dr. Taylor do not mean what they seem to mean. When children who have been taught number for two years do not at the end of the third year surpass those who began number in the third year, there may or may not be an indictment of the teaching and learning of number in the first two years. There is certainly, however, an indictment of the *third-grade* teaching and learning. Presumably it will be admitted that at the beginning of the third grade the children who had been taught number work during the first two years knew more about it than did the children who had not been taught number work. Why should this superiority vanish? The answer is not an appeal to mythical experiences; for the children who have been learning number during the first two years have certainly been having number experience. They have been building up their concepts of large numbers through combining smaller numbers. They have been gaining a notion of the marvelous structure of our number system. Even the most wooden first-grade and second-grade teaching can scarcely avoid a by-product of this sort. The real reason why the superiority of the number-taught children vanishes is that the third-grade teacher has third-grade standards to maintain and allows

these children to mark time while she brings along as rapidly as she can those who are just beginning number. In other words, she sets out to make them all alike by the end of the third grade, and when they turn out to be alike, great wonderment is expressed.

There are hundreds of situations of similar character. Some of them are experimental. The permanent effects of any administrative device or teaching method are often found to be small, if not *nil,* in comparison with a rival procedure. The children are caught in the lockstep, or, to vary the figure, they become the victims of the leveling effect of the school. It is always so fatally easy to level downward, to permit bright children and those who have had exceptional advantages to drop back. This is why children who have attended kindergarten are found to be little different, in the second or third grade, from those who have not. This is why the boy who has had high-school chemistry comes out at the end of his first college year of chemistry even with the boy who never studied chemistry before he went to college. It is why the professor of American history finds no difference between those who have studied history in the high school and those who have not. If the children who enter the third grade with two years of number training, even of a mediocre sort, do not continue to surpass those who have had no number training—assuming that other conditions are equal—then they have simply failed to receive an equally stimulating experience in the third grade. They have been educationally 'gypped.'

The evidence is now sufficient, it is believed, to justify a decision to teach regular—perhaps formal is not the word—number work in the first grade. The decision rests upon two types of information: First, information to the effect that we need a longer time to develop rich number ideas and a just sense of the use of number; and second, information to the effect that children upon entering grade one have already begun the subject and are ready to go on with the subject. They have had many number experiences outside of school and their readiness for further experience is abundantly evident from the things they do when left to their own devices.[9] Their teaching should begin *where they are* and proceed gradually—not hurriedly—toward the building up of skills and attitudes which will make their child life and their adult life more abundant.

[9] Smith, Nila B. "An investigation of the uses of arithmetic in the out-of-school life of first-grade children." *Elementary School Journal,* 24: April, 1924, 621-626.

VI. Forms of Thought

1. Generalization

Arithmetic supplies something that no other elementary-school subject even attempts. It provides the widest type of generalization and affords the organization of experience. Every part of the dictum $5 + 4 = 9$ is an abstraction, and the dictum itself is an abstraction. Five is a characteristic the same yesterday, to-day, and forever. It is unaffected by our moods; it is as true at the poles as at the equator; it neither expands nor contracts with the heat, and suffers no diminution through barometric pressure. Four and nine have the same generality of meaning, and the statement that the sum of five and four is nine is a general truth, because no matter what material one has in mind it is always true for that material.

Thus number introduces the pupil to a breadth of view which nothing else can. It is not found in nature; it is man-made. Nature exists in apparent confusion; man introduces order and system through the invention of number. By means of number applied to units of measurement he describes with precision and reproduces his ideas with exactness. In numbers the physician prescribes for the sick with the certainty that his prescription will be accurately compounded. By numbers as applied to time the great railroad and steamship companies systematize the transportation of goods and people. Houses and streets are located by numbers, and even ships at sea are accurately placed by a system so general in its application and so non-existent in nature as to represent one of the great triumphs of man. While, therefore, society has tended to break up into minutely specialized groups, one of its cohesive forces is a common number system. By means of it values may be computed and transmitted; and each part of this elaborate organism recognizes its relationship to the other parts and to the whole.

The control, therefore, of quantitative thinking is the major result of the learning of mathematics, and no part of mathematics is more potent for this purpose than the mathematics of the elementary school. If a child knows the number system very well indeed, he will never have to learn as an abstract rule the fact that three-fourths of a number is three times as much as one-fourth of it, nor the fact that dividing by one-third is numerically the same as multiplying by three, nor the further fact that dividing by two-thirds will

be half as much as dividing by one-third. He will not misplace the decimal point in dividing because his lively sense of the meaning of numbers will make any but the correct location of the point an absurdity. He will be able to estimate in advance the approximate size of his results, and he will be furnished with a critical instrument to apply to his results after he has obtained them.

2. The Relation between Numbers

Arithmetic not only affords means of generalization through the mere apprehension of number; it also builds up abstractions through relating numbers to each other. One of the fundamental ways of thinking to which arithmetic contributes is *ratio*. Number itself as ratio has already been referred to in this chapter. The reference here is rather to a form of thought. If you furnish two children with a quantity of beads and direct one child to lay down three every time the other lays down two, the resulting groups of beads at any point in the progress of the experiment is in the ratio of three to two. From such a simple origin the meaning of ratio may be developed to include the relationship between two sides of similar triangles—an application which will not be reached until years later. Meanwhile an instrument of thought will be provided which even on the basis of frequency of use has high value. What, for example, is the meaning of fifty-fifty, of the statement that the vote was three to one, of the virtues of the naval formula 5 : 5 : 3? What does the history book mean when it speaks of the free and unlimited coinage of silver at a ratio of sixteen to one? What does the sporting page mean when it says that the odds are seven to two that the Athletics will win the pennant?

From the idea of ratio to the idea of equality of ratios is but a step in generalization. The value of proportion for the practical uses of life has already been mentioned. Its value is stressed in this connection because of its potency as an instrument of thought. The writer has had frequent occasion to observe that men and women trained abroad or in Canada have secured a grasp of proportion which children in this country—especially since the reductionists came upon the scene—do not acquire. Proportion starts in experience and supplies one item of a new experience. One is called upon to deduce from the old experience what the completion of the new experience will be. Suppose that in a year of overproduction a farmer raises 1600 bushels of wheat on 60 acres of land. The following year he wishes to diversify

his farm, reducing his wheat crop to about 1000 bushels. How many acres should he plant? The given experience is represented by the yield of 1600 bushels on 60 acres. The term of the new experience which is supplied is the 1000 bushels. The farmer is required to deduce from his experience what the second term of the new situation will be. It does not matter in the least so far as the form of thought is concerned whether such a problem is solved by the customary type of proportion or by some 'rule of thumb.' The essential thinking is the same.

3. Functional Thinking

Although the idea of variables is customarily thought of as belonging in secondary mathematics, it does in reality receive some treatment even in the intermediate grades. Moreover, the relationship of one variable to another may be apprehended and will give rise to an important type of thinking. The value of a fraction with constant denominator varies with the size of its numerator. Similarly, the value of a fraction with constant numerator varies inversely as the size of its denominator. In multiplication the product increases with the size of either the multiplier or the multiplicand, provided the other factor is unchanged. The interest on a sum of money for a given time increases with the rate. The distance the automobile travels in a given time is a function of its rate per hour. The area of a rectangle varies with either dimension when the other is unchanged. The length of a bar in a bar-graph varies with the size of the number it represents.

Even a course of study in arithmetic which is utterly unprogressive introduces, although it may not fully capitalize, the notion of variation and of function. Arithmetic would be made socially more valuable if these inherent facts of variation and function were made explicit.

4. Ideas of Type

As a child learns about number and applies number to his environment, he develops unconsciously an attitude toward different classes of objects and ideas. He calls a certain man short because he has a notion of the typical man. If the child's experience had been among the Hottentots, he would have a different concept of the 'man of average height.' The same man whom he now calls short he would then call tall. Similarly, the child has, just as an adult has, notions of how many examples ought to be given in a lesson, how much an

average day's work is, what a fair price is for a baseball, how much time it ought to take to dress in the morning, and how many miles is a reasonable day's run with an automobile. The whole world, so far as one has any extended experience of it, is a series of amounts which are reasonable or to-be-expected or usual. Thinking is greatly facilitated by these ideas of type. It could be facilitated still more if pupils were made conscious of some of the interesting distinctions which the idea of type permits.

5. Ideas of Variation

Ideas of type, however, are rather crude unless they are supplemented by ideas of variation. The child sees a man on the street and calls him short. This has been explained as due to a knowledge that this man is shorter than the typical man. That, however, is not the whole story. The man would not be called short if in the child's experience men ranged from one foot tall to eight feet tall. This shortish fellow, who perhaps is a couple of inches below the average, would then be thought of, not as deviating from type, but as being substantially at type. The reaction of children and of adults to their environment is profoundly colored by their ideas of variation taken in connection with their ideas of type; and the thinking of the race is conditioned by these two notions. Tables are thirty inches high and the variation about this height is very small. Accordingly, a table will be called high even if it differs by a little from the type for tables. One might be quite distressed to find oneself writing at a desk a couple of inches higher than 'usual.' The seats of chairs are pretty well standardized at eighteen inches, and so nicely adjusted are we to this fact that hastily dropping into a chair short by an inch may be attended with surprising results. A variation of half a dozen pounds from one's accustomed weight is a topic for grave concern. On the other hand, a variation of an equal number of pounds in a load of coal or the weight of an elephant would be negligible. An inch on a tarpon is immaterial; on a trout it is something to talk about.

Of course there are many other ways of thinking, and any catalogue which can be made here must of necessity be merely illustrative. Ideas of precision, of relationship, of balance, of orderly procedure; ideas of cause and effect, of the general and the specific, of analysis and synthesis—these and a host of others are facilitated if they are not indeed conditioned by number. Logic itself is mathematical,

and the sciences as they develop tend more and more to become mathematical.

VII. Problem-Solving

The aim of education, no matter how conceived or how phrased, includes the fitting of children for life. There are some who are so insistent upon the present life of the child as a basis for school procedure that they appear to ignore the inescapable fact that as a social institution the school has been established primarily for the purpose of the induction of immature members of the community into community life as adults. The touchstone which the man on the street applies to education is whether it will enable a person to get on better; and his idea is basically correct. The fact that utilization of the child's present life happens to be in harmony with the aim of preparation for adult life is an incident in the learning process. Rightly does the school draw upon the present environment, the present capacities, and the present interests of the child because this assures a motive. But no one in his senses will argue for an education whose ultimate aim is to prepare children to be children.

The theater upon which children of today will be called upon to play adult parts will be a theater materially different from that in which their fathers and mothers are now actors. No argument is needed to establish the fact that the world is rapidly changing. Accordingly, the life to which education seeks to fit children is a life which in many of its important aspects is unknown. Even to the extent to which it can be known—even in regard to the aspects of life which are of slow change—no one knows what particular station an individual child will assume as an adult, nor which of the myriads of life's problems he will be called upon to solve.

The result is that in an attempt to fit children for adult life no complete reliance may be placed upon the setting up of habits or patterns of reaction to meet predictable conditions. It follows therefore that education must give the organism itself flexibility. The child must be endowed with an ability to make his own adjustments, and thus to meet unforeseen circumstances. Habits are essential for economy of action. They should be taught and learned. In arithmetic, for example, the number facts should be reduced to a habit basis. But confronted by unfamiliar situations one's habits no longer meet the need; one's number facts are insufficient.

All this means that the educated person is a thinking person and that a fundamental problem of the school is to teach children to think. This may amount to the teaching of an attitude rather than a technique. The product may even be loosely referred to as a habit, but the meaning is clear. In the training of boys and girls for successful living, teachers must have prominently in mind the inculcation of an ability to think and a willingness to take thought where more stereotyped modes of behavior fail to serve the purpose. As an attitude this means an open-minded preference for conclusions that are properly grounded, a tendency to distinguish between tested beliefs on the one hand and mere assertions, guesses, or opinions on the other hand. As an ability it means the power to use methods of reasoning appropriate to the problems of life. Without this attitude and that ability a man, no matter what he knows, is not educated. Since one is not born with these characteristics, and since the casual circumstances of environment are not enough to compel their acquisition, it is not too much to say that the most distinctive office of education is to cultivate them.

Human beings cannot be taught to think unless they are placed in situations requiring thought. In the social environment these situations are called problems, and arithmetic affords exercises which are likewise called problems. There is, however, very material distinction between the problems of arithmetic and the problems of life. In arithmetic the problems do not 'arise.' They are stated by one person to another, the latter being remote from the scene of action. Accordingly, the problems of arithmetic, even at their best, fall far short of equivalence with the problems of life, while at their worst they have no reference whatever to the problems of life.

Nevertheless, the problems of arithmetic may be made exceedingly valuable. The difference between half a loaf and no bread may be the difference between life and death. The real task is to make the descriptive problems in arithmetic go as far as possible in affording lifelike opportunities for thought.

In the first place, attention should be given to the situations within which the problem arises. Except with very little children, it is likely that the problem "How many are five horses and seven horses?" is just as abstract as "$7 + 5 = ?$." It is everywhere agreed that problems in arithmetic should refer to real life. This is all very well if we understand the term 'real' in a truly psychological sense. Excellent problems for little children may be quite fanciful. The

imaginative life of such a child is very real. In any event, the details embodied in the description of a problematic situation must be considerable, because only through richness of detail can the description really be faithful. Details should be set forth (1) for interest, (2) for vividness, and (3) for distractions.

(1) Details which make for interest insure that to some degree the child will identify himself with the problem. He will appreciate the critical situation in which affairs are, and will want to know what the answer is. Having sensed the crises and desiring to reach a solution, he will be led to form a hypothesis as to his procedure. After examining the hypothesis in the light of the conditions, he will reach a decision as to its validity. Still with his interest aroused he will proceed to the verification by solving the problem. It will be apparent from what has just been said that the interesting details must not be mere sugar coating. They should be natural to a situation which is itself inherently interesting.

(2) Details should likewise appear for the sake of vividness. The problem must make an appeal. The situation must be understood, and if it is understood, the problem is half solved—solved not because the writer of the problem did the child's thinking for him, but because he placed at his disposal clear and unambiguous data.

(3) The problems of life contain extraneous facts. Part of the difficulty in solving such problems is to avoid distraction and select the appropriate data. If, therefore, the problems of arithmetic are to simulate the problems of life, they should contain details such as naturally arise in a life situation but which have nothing to do with the solution of the problem. When the problem writer presents nothing but the numbers and the relationships which the child will have to use, he detracts both from the reality and from the problematic character of the situation he describes.

Few people are writing problems in such detail as has just been suggested. Only in part is this due to their failure to realize that they should write such problems. The real difficulty is that the school people are not demanding problems of this sort and are unwilling to pay for them. At present teachers are willing to accept a book which has relatively few problems because the problems themselves are of small value. But if the right kind of problems were presented, the textbook would have to be much larger, not only because each problem would be longer, but because teachers would want more of them in

the book. Such a book in the first place would have to be sold at a higher price, thus raising a definite obstacle to its sale. Such a book, in the second place, would become the target of competitive criticism —'fads,' 'frills,' 'impractical ideas,' and so on—with the result that the school people would forsake the book they were already beginning to admire. It is only slowly that a new idea can be embodied in textbook form. Nevertheless, the superintendents, principals, and supervisors who bemoan the fact that teachers do not teach children to solve problems have the situation in their own hands. It will be time enough to find fault with the teachers when they have permitted them to have problems worth teaching.

It goes without saying that problems must be expressed in language which children can understand, that the computations should be within their powers, and that the complexity—the number of steps—involved should be within the maturity of their reasoning processes.

With all their shortcomings, however, the problems available to the school are important—far too important to be omitted or slighted. Problem for problem, they do indeed lack the educative value of the perplexities, doubts, and crises of life. But they are the only substitute the school has or can have in any abundance, and to drop them from our teaching would be a confession of defeat. They afford practice in thinking situations through and these situations are no more artificial than the problems which are hailed with delight in the teaching of geography and history. Teachers formulate problems about Braddock's Campaign, the Panama Canal, and the Constitutional Convention. Are not these as remote as any problems in arithmetic? Yet we praise the result in one case and are dubious about it in the other.

The problems of arithmetic are the most significant part of the subject. In solving problems the pupil gets all the benefit that he can get from doing abstract examples, plus a benefit which is peculiar to the problems—namely, a training in thinking; and training in thinking is, as has been shown, a major purpose of the school. If the question of the social value of arithmetic is raised, the subject simply cannot be discussed without reference to problems and problem-solving.

In still other ways verbal problems are important in rendering arithmetic socially valuable. In the first place, they form a significant point of departure for the processes which will subsequently have to be practiced in abstract form. If carrying in multiplying is to be

taught, it is well to show a situation which calls for this procedure. This affords a better occasion for it and gives it a livelier meaning.

In the second place, such problems, after a period of drill in the process, call for the process in use. The multiplication *idea* is fortified when, in an interesting situation, the pupil finds that it applies. The process almost takes on life when it is found to work. Moreover, this use of a process is an insurance against the failure of the process itself. The pupil may do well with it as long as he can concentrate on it. When, however, he is obliged to pay attention to a number of other conditions, his skill in the process may break down. Thus it is desirable to give him an opportunity to use the process under a variety of conditions.

In the third place, the problem parts of arithmetic present the subject in its functional aspect. This is in line with the best thought as applied in other subjects as well as in arithmetic. In drawing, for example, children at the outset are encouraged to draw things instead of merely making lines and curves. In music they play pieces from the beginning. In language they first express themselves and later study matters of form. In life one scarcely ever works an abstract example; when one performs a number process, it is almost always part of a problem. The school would do well to pattern more closely after life in this respect.

VIII. Attitudes

While children are learning arithmetic they are developing an attitude toward quantitative matters. As they grow in number ideas, they not only clarify and extend these ideas but they also take a position favorable or otherwise toward them. As they become familiar with the number system they assume a mental and emotional disposition towards it. Besides their knowledge of number facts and processes they acquire a certain warmth of feeling toward them.

1. The Importance of Attitude

In any estimate of the social value of arithmetic the character of the attitudes developed in the learning of arithmetic can scarcely be over-emphasized. Probably no product of instruction is more important than this or is more usually disregarded. The failure to take sympathetic account of the child as he faces this wonderful, yet highly artificial, world of number is responsible for most of the failures in

arithmetic in school and for the fact that, when freed from the school, children drop their arithmetic as completely as possible. On the other hand, when because of native talent or wise teaching the child has learned to assume a sympathetic attitude toward quantitative matters, he will not only enjoy arithmetic as a child but will use it as an adult.

2. Rationalizing the Processes

One reason why grown people do not use arithmetic to a greater degree is because they did not learn it intelligently. Too many of them learned it as a bag of tricks. You do this and then that, and if you didn't make a mistake, your answer is correct. A normal child who cannot understand carrying in addition is probably not ready to learn it. A doctrine has flourished long enough which holds in effect that telling a child what steps to take is enough; that the getting of the right answer is a sufficient justification to him for the process. This is an admission of failure. It has been found that many children cannot understand the rationale of the subtraction of whole numbers or of multiplication by a two-place number; so it is decided that they don't need to understand these things, anyway. The correct inference is that they have not learned the full meaning of numbers as expressed in our decimal notation. Their prerequisites are at fault —not the prerequisites of computation but the prerequisites of number sense. Let the child in the early grades be taught number ideas along with his computations. Let him get a feel for number, a sense of what he can do with it. Then he will recognize the meaning and value of what he is doing. The number system will be so fully appreciated that the steps in a new process need not long remain mysteries to him. The result of such teaching will be the development of a wholesome attitude which, after all, is what is most likely to carry over into adult life.

3. The Procedure versus the Answer

In considering this matter of attitudes, one is justified in alluding to the difference between the attitude of independent thinking and the attitude of getting the answer. It is undesirable that the effort of the child should be directed toward a manipulation of figures that will produce the answer rather than toward a procedure which will meet the conditions of the problem. There is a certain lack of mental integrity when a pupil exercises his ingenuity in working backwards from the answer, and then exhibits his work as a way of *getting* the

answer. The results may indeed look right, but the attitude is wrong.

Although it is true that in any problem in life the correct solution includes obtaining the right answer, nevertheless it is possible to stress this attitude too much in the solution of the problems arising in school. The child who has done honest thinking of his own is entitled to praise, even though his answer is faulty. When both process and answer are correct, it is the rightness of the process which should be regarded as important. Indeed, it is often unnecessary that any answer should be obtained. The solution of many problems may be indicated rather than carried out.

No catalog of methods or of management of subject matter as they effect attitudes can be attempted in these pages. After all, we touch here the spirit of the school. If the school sets for itself the task of drilling boys and girls in mere computation, it is likely to miss the soul of the subject and to engender hostile attitudes. But if it can teach the child, along with the computation and all the rest of the items in the course of study, something fine and wholesome which is not in the course at all, namely an intimate feeling for number—if it can do that, it will succeed genuinely. Its work will have the highest value for the individuals who have passed through it, and so for society. The man who has acquired such an attitude can instantly feel the application of number to a problem which confronts him. To him number has become something more than subject matter. It has become a way of looking at life.

GUIDE TO CHAPTER III

CHAPTER III
THE ARITHMETIC CURRICULUM

R. L. West, Charles E. Greene, and W. A. Brownell

Introduction

The *curriculum* may be defined as the totality of subject matter, activities, and experiences which constitute a pupil's school life. A *course of study* is the material, usually in pamphlet form, which sets forth for the teacher such items as the objectives and content of a given subject, and the activities and books to be used to accomplish desired results. The present tendency to treat all these items in a course of study, in contrast to the earlier tendency to include only a brief outline of subject matter, is a recognition of the interrelationship between different phases of the educational process.

The curriculum which a pupil will actually experience depends, on the one hand, on theories of sociology and, on the other, on principles of methodology. The knowledge which the pupil acquires is selected according to theories of the relative value of all the possible items in the subject. The activities of the pupil depend upon the teacher's point of view and the methods which he practices. As a matter of fact, all findings in any phase of education must ultimately affect the curriculum if any application of such findings is to function.

The reader will find, therefore, that nearly every chapter in this Yearbook presents points of view and material which have a definite bearing on the curriculum in arithmetic. In the Introduction there is a statement of principles accepted by the Committee. Nearly all of these statements involve matters which affect the curriculum. The value of arithmetic in modern life with its implication for selection of content is set forth in Chapter II. Some of the practices of methodology which affect very largely the specific activities of the pupil are found in Chapter IV. How tests of various kinds may be used to determine the pupil's progress in the curriculum appears in Chapter V. In Chapter VI there are frequent references to the curriculum problems met by student teachers. Many of the studies in Part II also concern the curriculum.

Because of the number of places in the Yearbook where material concerning the curriculum may be found, it has not seemed necessary

in this chapter to treat the curriculum exhaustively. Instead, certain special topics have been selected to illustrate the changing theories and practices in the arithmetic curriculum as evidenced particularly by published studies and by courses of study. A discussion then follows of several topics which appear to need more adequate treatment in the future. Finally, in view of the widespread activity recently in the construction of courses of study, there is included a section dealing with the problem of course-of-study construction.

I. Changing Aims and Purposes in the Arithmetic Curriculum as Evidenced in Selected Courses of Study[1]

1. Introduction

What knowledge is of most worth? What provisions can be made whereby the things counted desirable may be most economically and effectively learned? The history of education is replete with the many and varied answers as well as the relative emphases given to these age-old, yet ever present, questions. These answers and emphases may be traced as they are disclosed in the changing purposes which direct the education program, the growth of and change in curriculum content, the introduction from time to time of 'special methods' movements, to mention but a few aspects of the educational program.

When applied to the field of mathematics in the elementary school, what do the varied answers and shifting emphases suggest as to present trends and tendencies, and indicate as to future directions of movement? The purpose of this section is to report major changes in curriculum goals in the field of elementary mathematics as evidenced in selected courses of study dating from 1850 to the present.

2. Nature and Use of Courses of Study

Before considering the purposes of education in relation to the questions of 'what' and 'how' to teach as found in courses of study, it is significant for proper evaluation to note something of the nature

[1] This material is based upon an analysis of from eight to twenty courses of study for each decade from 1850 to 1900, for each five-year interval from 1900 to 1920, and for the period 1921-1928, inclusive. The courses from 1850 to 1920 were selected on the basis of geographic representation, size of community, and typicalness. For the period from 1921 to 1928, only those courses judged as outstanding were used. A total of 151 courses was analyzed. The Committee acknowledges the assistance given it by H. B. Bruner and Florence B. Stratemeyer, of Teachers College, Columbia University, in the preparation of Sections I and IV of this chapter.

of the course of study as a document and the extent and type of use to which it has been put. In general, the course of study, as such, did not exist at the beginning of the period under discussion (1850), being either a small section of the annual report of the superintendent of schools and the 'school visitors' or interwoven in the context of this report. Not until about 1890 did it become general practice to issue a separate course-of-study bulletin, and not until 1910 did separate courses in the several subject fields appear. Table I indicates the average and range in number of pages devoted to arithmetic in a selection of courses from 1850 to 1928. The figures indicate that the

TABLE I.—AVERAGE NUMBER OF PAGES AND THE RANGE IN PAGES DEVOTED TO ARITHMETIC IN A SELECTION OF SIXTY-ONE COURSES

Year	1850	1870	1890	1910	1920	1925*	1928*
No. of Courses........	7	10	15	8	9	6	6
Average No. of Pages ..	.9	1.5	3.9	27.5	35.1	206	231
Range in Pages........	0.5–2	0.5–3	0.5–11	2.5–78	8–98	83–376	114–322

*These figures are based on the actual courses of study alone, no consideration having been given to the large amount of supplementary materials, such as manuals for testing, research monographs, etc., which quite often accompany courses issued subsequent to 1921.

course of study in the early decades was a textbook course with sparsity of detail, whereas in recent years voluminous suggestions have been included.

Although the course of study in arithmetic in 1850-1860 was, in the main, limited to an indication of the required pages in the textbook as compared with the 200-300 pages not uncommon in modern arithmetic courses, the phrase "binding upon all teachers and officers of the schools,"[2] doubtless indicates that the older course more exactly described classroom practice than does the more modern course which suggests a wide variety of materials with which "teachers are urged to experiment."

While some of the early courses of study indicated a suggestive and experimental attitude, a few even in the new century maintained the more rigorous doctrine of strict adherence to prescribed materials of instruction. The following excerpts suggest the struggle between the two points of view in administration:

[2] Wilmington, Delaware—*City of Wilmington, Reports Concerning the Public Schools, 1881-1882,* p. 40.

1870

It is evident that a definite prescribed course of study is essential to the harmonious and successful working of every system of schools.[3]

The negative character of the provisions in our programme, respecting instruction in arithmetic, would seem to leave the teachers free to handle this branch according to their individual judgment. If this were the case, the teaching and its results would reflect the views and the abilities of the teachers. But the very absence of directions on the programme tends to crush out all independence and originality in teaching arithmetic. For the textbook is the programme, and the examinations are naturally based upon it.[4]

1880

The course of study ought to be binding upon all teachers and officers of the schools. The Board of Education only should make changes in it. Changes thus made should at once be accepted as the law of the schools. With the present diversity of opinion as to what should be taught, there is no safe course but the one just stated. Teachers are, of course, at liberty to use matter not in the textbooks to illustrate the subjects, and should do so.[5]

A course of study for the primary grades can be but poorly outlined. Certain work in reading and numbers can be marked out; but the many means for the development of the perceptive faculties could not be stated in a volume.[6]

1890

The course of study should be explicitly stated, and strictly adhered to by teachers.[7]

1900

The following course of study is merely indicative and represents the average results to be attained in the schools; but it must be borne in mind that classes in different localities and even in the same school differ in ability; hence, the amount of work completed by any class within a given time cannot be regarded as a standard for all classes. The quality of work

[3] Jacksonville, Illinois—*Sixth Annual Report of the Superintendent of Public Schools, to the Board of Education of the City of Jacksonville, Illinois, 1873*, p. 15.

[4] District of Columbia—*Department of Education (U. S.) Special Report of the Commissioner of Education on the Condition and Improvement of Public Schools in the District of Columbia, Submitted to the Senate June, 1868, and to the House with Additions, June, 1870*, p. 475.

[5] Wilmington, Delaware—*City of Wilmington, Reports Concerning the Public Schools, 1881-1882*, p. 40.

[6] Seattle, Washington Territory—*First Annual Report of the City Superintendent with the Rules and Regulations and Course of Study for the Public Schools of Seattle, Washington Territory, 1884-1885*, p. 17.

[7] Springfield, Illinois—*Thirty-First Annual Report of the Public Schools of the City of Springfield, Illinois, for the year ending August 31, 1889*, p. 10.

done, however, should be uniform and there should be uniformity in the degree of progress made in the various branches. No effort should be made to fix a given time in which a certain work should be completed merely for the purpose of doing so much work in that time.[8]

1925

The course provides that all work in the first and second grades should be informal and that books shall be in the hands of the teachers only. In the third grade, formal arithmetic is begun with texts in the hands of the pupils. Teachers should follow the text very closely in order that the scheme may function to the best advantage. It may be advisable to supplement with mimeographed sheets of problems or with original problems taken from the child's experience; but one should not deviate from the general plan of the text. . . . . Every teacher of arithmetic in the Los Angeles Public Schools should make an intensive study of the *New Method in Arithmetic* by Thorndike, a copy of which will be placed in every school. This is essential.[9]

1928

Each teacher will be expected to study and try out the methods and materials included in the course and at the end of each semester report her general opinion, her criticisms, and her suggestions for changes.[10]

Although subject matter, forms, activities, devices, and other suggestions have been provided, it is not intended to stifle initiative on the part of the individual teacher. While it is essential that a great part of the work herein presented should be given as presented, much room is left for added suggestions, methods, and devices. Care must be taken, however, that such added methods are in accordance with good authority and a recognized method of learning.[11]

3. Purposes and Functions: Changing Concepts of the Goals of Education in the Field of Arithmetic

The changing point of view and emphasis thus suggested have been in part contingent upon the changing conception of the desired goals of education. The history of education shows that aims and purposes, conditioned in turn by social and economic factors, are the factors which direct and govern the answers made to the problems both of the

[8] Louisville, Kentucky—*Louisville Public Schools—Course of Study, 1889*, p. 7f.

[9] Los Angeles, California—*Course of Study, Grades One to Six, 1924-1925*, p. 340.

[10] Winston-Salem, North Carolina—*Arithmetic Course of Study, Grades One to Seven, 1927*, p. 2.

[11] Cicero, Illinois—*Arithmetic Course of Study, Grades One to Eight*, p. 1.

content of instruction and the method of presenting it. Early in educational history the sources and purposes of education and the curriculum shifted in the main from church to state; from the few constituting the ruling class to the masses; from the needs of the professional and leisure classes to the industrial, commercial, and agricultural workers; and during the relatively short span of seventy-five years covered in this study, the change from formal and disciplinary emphasis to stress upon work-a-day interests and activities and upon the demands of social usage is apparent.

During this short period, as Professor David E. Smith has pointed out,[12] the aim has been (1) utilitarian, (2) cultural, (3) traditional, (4) a mere show of knowledge, (5) a quickener of the wit, (6) the basis of a remunerative trade, (7) enjoyment or pleasure, or (8) the development of self-activity and independence on the part of the pupil. The teaching of arithmetic for the purpose of making pupils keen and quick-witted, leading to the use of puzzle problems and textbooks consisting primarily of "difficult and impractical problems designed to catch the unwary pupil," was early displaced by the doctrine of mental discipline. This doctrine, with its lack of regard for the child's natural interests and its attendant theory of transfer of training which considered the study of arithmetic as a means of developing the general power of reasoning, the memory, or the general habit of accuracy, was the most dominant of any of the goals of the seventy-five year period. Although educational leaders pointed to the fallacy of the doctrine and modern psychology and experimental studies have shown that our abilities tend to be specialized, its influence was at work long after it was held untenable as a doctrine—is, in fact, exhibited to-day. This suggests one phase of the dualism of formal discipline on the one hand and the demands of social usage on the other, a dualism which has continued up to the present time in concurrent and parallel practices with respect to the purpose and content of instruction as well as teaching method. This is illustrated in the following quotations from courses of study published in 1914 and 1921:

> The ultimate purpose in all teaching, aside from information, is to develop an interest in life; a keen comprehension of one's environment; a desire to think independently and the power to arrive at correct deductions when certain facts are given.[13]

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[12] Smith, D. E. *The Teaching of Elementary Mathematics*, Ch. I.

[13] East St. Louis, Illinois—*Detailed Course of Study for Grade Classes in Arithmetic*, 1914, p. 41.

And yet in the same course we find:

> Emphasize the necessity for making neat, legible figures. The arrangement of work in tablet or on blackboard is indicative of certain proclivities which will have much influence in later years.[14]

And in another course we read:

> Although we no longer teach arithmetic primarily for its disciplinary value mentally, it would be a mistake to say that arithmetic has no value as discipline. In fact, the overthrow of the dogma of formal discipline has been responsible for a great deal of slipshod teaching in mathematics. The sanest point of view which a teacher may hold in regard to this whole problem may best be stated by emphasizing the moral side rather than the mental. There is no doubt that our work in arithmetic as well as that in all other subjects must teach proper habits of thinking. The boy or girl must learn from the arithmetic lesson the habits of self-reliance, honesty, concentration, and industry.[15]

Of the one hundred nineteen courses examined for the period prior to 1921, but thirty-six even indicated their aims or objectives, and only nine of these specifically labelled the statement as an aim or purpose. In contrast, of the twelve courses examined for the years 1921-1928, eleven contain general objectives for arithmetic and ten have specific objectives for each grade.

The change in the point of view, although so gradual that the two tendencies exist concurrently, may be seen from the following quotations illustrative of the changing conception of the function of the teaching of arithmetic from the acquisition of skill and knowledge only (power) toward 'growth' and the development of the whole child.

1850

> Arithmetic and geography are prominent studies in this department, and the work of mental training should be prosecuted with vigor.[16]

1860

> In arithmetic, our schools stand unrivaled, and the power of mind acquired by the proper study of this subject might, if rightly directed, accomplish all that could be desired of all other subjects.[17]

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[14] *Ibid.*, p. 8.

[15] Akron, Ohio—*Course of Study in Arithmetic, Elementary Grades,* 1921, pp. 3f.

[16] New Haven, Connecticut—*Report of the Superintendent of Common Schools to the General Assembly, May Session, 1858.*

[17] Cincinnati, Ohio—Superintendent's Report—33rd Annual Report, 1862, p. 22.

1880

At least in this subject 'practical utility' and 'mental discipline' are not at variance; neither need be sacrificed to the other.[18]

1890

Mathematics occupies an important place in every curriculum of studies for its (a) practical, and (b) disciplinary values. For the great mass of people its practical value is confined chiefly to arithmetic; its disciplinary value, on the other hand, pervades the entire realm of the purely practical, and increases in intensity as it approaches the generalized domain of concrete geometry and elementary algebra.[19]

Arithmetic—What rank the study of Arithmetic holds with the other school studies, as a means of mental development, is a question upon which there is much difference of opinion. I am inclined to believe that it does not rank very high. In this, as in other branches, however, the amount of mental discipline acquired depends greatly upon the methods used in teaching. If the pupil is required to commit definitions and rules to be used simply as aids in performing mechanical operations for obtaining results, the various steps of which are not understood, then the reasoning faculties are but little, if any, improved.[20]

1905

The teacher should bear in mind that her work is to develop the child rather than to teach subjects.[21]

1910

This introduces a third purpose of mathematics, to aid in developing the poetic spirit and minister to pure enjoyment. Perhaps it seems more likely to do the opposite, but that is partly because we have not permitted the free play and gambol of the mind around mathematical facts.[22]

1915

The time was when difficult abstract problems in arithmetic were defended solely on the ground of disciplinary value. More modern thought and methods do not undervalue the disciplinary feature of arithmetic, but

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[18] Boston, Massachusetts—*School Document No. 17, Suggestions Accompanying the Course of Study for Grammar and Primary Schools.* 1878, p. 17.

[19] Saginaw, Michigan—*Saginaw East Side, Public Schools—Manual of Instruction.* 1896-1897, p. 38.

[20] Pittsburgh, Pennsylvania—*Grade and Course of Study for the Pittsburgh Public Schools.* Edition of 1893, p. 6.

[21] Los Angeles, California—*Courses of Study for the Public Schools of the City of Los Angeles for the Year of 1906-7*, p. 7.

[22] Augusta, Maine—*Course of Study in Mathematics for the Elementary Schools.* 1909, p. 9.

they do recognize the fact that a subject may be practical and still have disciplinary value; that, in truth, the more closely a subject is related to one's interests, the more intense will be the application and the greater will be its value as a mental training.[23]

1920

Arithmetic has grown out of social needs and continues to assist man to make better social adjustments, for which reason it is justly entitled to a place in the curriculum. Arithmetic was found by the earliest peoples to be necessary and so was developed by them to meet real needs. It is necessary in the social life today, and, so far as we may see into the future, it always will be necessary.[24]

1926

One of the tests of the educational value of a subject is whether or not it affords means for developing industry and enterprise. The subject of arithmetic affords an outstanding opportunity for the cultivation, not alone of industry and enterprise, but of honesty, initiative, and many other traits quite as desirable. Every teacher should utilize the subject matter of this course of study toward the development of these desirable traits and should lose no opportunity to impress upon the pupils the value of qualities thus developed in determining success in every other phase of school work.[25]

1928

Arithmetic has its place in the curriculum as the indispensable tool by which all the quantitative relationships in ordinary living can be interpreted and satisfactorily handled. The individual must constantly come in contact with and adjust himself to these relationships. Without arithmetic it would be impossible to attain full command of the basic arts and skills; certain knowledge, habits, and appreciations vital to worthy home life would be impossible of attainment; the individual would be unable to choose intelligently or serve efficiently in his vocational life; and the quality of his citizenship is most certainly determined by his ability to interpret quantitative relationships and values.[26]

II. Changing Time Allotments

The changes in the aims and purposes of the curriculum in arithmetic as outlined in the preceding section have been accompanied by changes in the time allotted to this subject.

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[23] Joplin, Missouri—*Course of Study and Teachers' Manual for the Elementary Schools.* 1915, p. 103.

[24] Indiana State—*Manual with Courses of Study for the Elementary Schools of Indiana.* 1919, p. 36.

[25] Denver, Colorado—*Course of Study Monograph Number Four, Arithmetic—Grades One to Six.* 1926, p. 13.

[26] Kansas City, Missouri—*Course of Study in Arithmetic, Grades One to Six.* 1928, p. 12.

1. Desirability of Optimal Time Allotments

Theoretically it should be possible to set up optimal time allotments for arithmetic in the various grades. Obviously, however, such time allotments are dependent on such items as the objectives of arithmetic, the relative value of arithmetic in the total curriculum, the efficiency of teaching, and evidence from careful studies to determine the relation of accomplishment to pupils' capacities. Much time must elapse before any of these items are determined with any degree of certainty. Meanwhile time allotments must be prescribed and schools must operate.

2. Value of Analyzing Practice

There is considerable value in determining actual practice in the problem of setting up a time allotment. Whereas present practice is often provided by tradition or by imitation, it does represent, to an extent, what experience in the classroom has demonstrated to be necessary. The following is a list of the most important investigations concerning time allotment which have guided educational practice since 1900:

(1) Payne, Bruce R. *Public Elementary School Curricula.* Silver, Burdett and Company, 1905.
(2) Holmes, Henry W. "Time distributions by subjects and grades in representative cities." *Fourteenth Yearbook* of this Society, (1915) Part I, Ch. II, pp. 21-27.
(3) Ayer, Fred C. *Time Allotments in the Elementary School.* United States Bureau of Education, City School Leaflet No. 19, February, 1925. (Also in the *Second Yearbook of the Department of Superintendence,* February, 1924, pp. 139-172.)
(4) Mann, Carleton H. *How Schools Use Their Time,* Teachers College, Columbia University, Contributions to Education, No. 333, 1928.

The Mann investigation, made in 1926, published in 1928, has the merit of examining the practices in both large and small cities. By the use of annual reports for six cities in 1886, Payne's study in 1904, and Holmes's study in 1914, Mann presents some interesting tables showing the comparative average amounts of time given to elementary subjects. This comparison for arithmetic[27] is shown in Table II.

There is a considerable decline in the time given in 1914 compared with 1904 and 1866, especially in the lower grades. Between 1914 and

[27] Mann, C. H. *Op. cit.*, pp. 19-23.

TABLE II.—AVERAGE AMOUNTS OF TIME IN MINUTES PER WEEK AND PERCENTAGES OF TIME IN EACH GRADE ALLOTTED TO ARITHMETIC

Grade	I		II		III		IV		V		VI		Total I-VI	
	Aver.	Percent	Aver.	Percent	Aver.	Percent	Aver.	Percent	Aver.	Percent	Aver.	Percent	Aver.	Percent
1866 (6 Cities)...	205	13.7	249	16.5	234	15.3	273	17.2	307	18.9	307	18.9	1575	16.8
1904 (6 Cities)...	161	12.2	204	14.3	248	16.6	261	17.2	266	17.0	270	17.4	1410	15.9
1914 (50 Cities)...	71	5.6	149	11.2	203	14.4	231	15.4	223	14.9	226	15.0	1103	12.0
1926 (444 Cities)..	80	5.9	146	10.2	196	12.8	211	13.3	215	13.3	215	13.3	1063	11.6

1926 not much change occurred, although every grade except Grade I is slightly less in 1926. Mann reports that in 1845 Boston gave one-third of the total time to arithmetic, so that evidently great changes in the curriculum took place between that date and 1866.

For forty-two cities that reported that they taught arithmetic in all of the first six grades, the averages as found by Ayer were as follows:[28]

Grade	I	II	III	IV	V	VI	Total
Minutes Per Week	96	143	193	206	211	211	1060

An unpublished study of fifty-two cities of New Jersey[29] made by a committee of the New Jersey Teachers Association in 1925, shows an average total of 1157 minutes per week for arithmetic for the first six grades distributed as follows:

Grade	I	II	III	IV	V	VI
Minutes	97	165	216	231	228	220

In comparative rank of the percentage of time given to all subjects, Mann finds that arithmetic stood second in all of the four years named.[30]

Another interesting item is the recommended time allotment as found by Mann in fifteen states. (See Table III).

TABLE III.—AVERAGE AMOUNTS OF TIME IN MINUTES PER WEEK AND PERCENTAGES OF TIME IN EACH GRADE RECOMMENDED FOR THE ELEMENTARY SUBJECTS BY FIFTEEN STATES[31]

Grade	I		II		III		IV	
	Aver.	Percent	Aver.	Percent	Aver.	Percent	Aver.	Percent
Arithmetic........ ...	67	4.9	105	7.5	179	11.4	186	11.6

Grade	V		VI		Total	
	Aver.	Percent	Aver.	Percent	Aver.	Percent
Arithmetic........	198	12.0	198	12.0	933	10.1

[28] Ayer, F. C. "Time allotments in the elementary school." *Second Yearbook, Dept. Supt., N.E.A.*, 1924, p. 140.

[29] On file in the office of the State Commissioner of Education, Trenton, N. J.

[30] Mann, C. H. *Op. cit.*, p. 26.

[31] *Ibid.*, p. 38.

Recommended time equals 933 minutes, or 10.1 percent of the total available, whereas 444 cities report 1063 minutes, or 11.6 percent. Evidently, course-of-study makers believe that the time given to arithmetic should be somewhat less than is actual practice at present.

As is usual in all such investigations, Mann found very great variations in practice between individual school systems. He says:[32]

> All cities report arithmetic as one of the subjects of their elementary school curriculum, but some schools devote almost 4½ times as much time to it as others. The total number of minutes per week given to arithmetic, in all grades ranges from a minimum of 405 minutes to a maximum of 1,797 minutes. Twenty-five percent of the schools devote less than 916 minutes per week to this subject, while 25 percent give more than 1,245 minutes per week. The medial amount of time spent on arithmetic in the first six years of school is 1,076 minutes per week. The degree of variability as indicated by the interquartile range is greatest in communities of less than 2,500 population, while the closest agreement as shown by this same measure is in communities of 30,000 to 100,000.

It would be very profitable if a careful testing program in arithmetic could be carried out to determine whether cities giving such large amounts of time are profiting thereby or whether those giving small amounts are able nevertheless to secure satisfactory results. Only by such investigations, with careful study of all the variables involved, can we hope to define conditions under which specified results ought to be expected from certain time allotments.

3. Suggested Time Allotments

From the studies analyzed, the following schedule of time allotment is suggested in Table IV. The median time reported by Mann is added for comparison.

TABLE IV.—SUGGESTED MINIMUM AND MAXIMUM TIME ALLOTMENTS IN MINUTES PER DAY AND PER WEEK FOR ARITHMETIC (GRADES I–VI) COMPARED WITH MEDIAN TIME IN 444 CITIES REPORTED BY MANN

Grade	I	II	III	IV	V	VI	Total
Per Day	15 to 20	25 to 30	35 to 40	40 to 45	40 to 45	40 to 45	195 to 225
Per Week	75 to 100	125 to 150	175 to 200	200 to 225	200 to 225	200 to 225	975 to 1125
Mann (Per Week)	80	146	196	211	215	215	1063

[32] *Ibid.*, p. 76.

In interpreting these recommendations it should be noted that the time given means the total time devoted by a pupil in a day or a week to all forms of arithmetic activities, either in individual or in class work.

School systems that are spending much less or much more time than these amounts should examine their achievements carefully to determine whether more time is needed to produce satisfactory results or whether excess time is being accompanied by wasteful or inefficient teaching.

III. Changing Content of Instruction with Particular Attention to Eliminations and Additions

Paralleling the changes in purpose and function and the shifting emphasis in time, corresponding changes involving eliminations, additions, and variations in emphasis have developed. A comparison of present-day courses of study or of textbooks with those of only a few years ago shows that many topics and types of problems formerly in general use have been eliminated and that others have been added.

1. Doctrines Producing Elimination of Material

Certain educational theories have been operating to produce certain eliminations. These have developed from general movements and changes in educational thought. Among the important educational ideas which have influenced makers of courses of study and of textbooks to eliminate material are the following:

a. Changes in the Doctrine of Transfer.—Arithmetic has had difficulty in emerging from the fog of so-called 'mental discipline' which surrounded it for so long. The importance of this doctrine in the changes which have taken place in the general aims of arithmetic has already been mentioned in Chapter II and earlier in this chapter. Even to-day there is plenty of evidence that those who were trained in the type of arithmetic employed forty or fifty years ago believe that they did receive some form of general mental, or disciplinary, value from the solution of difficult and complicated puzzle problems. Mark Sullivan, in his *Our Times,* quotes former Governor James M. Cox of Ohio as declaring that "the tests of rapidity in addition, multiplication, and division gave spur to the mind and were a good form of mental gymnastics." Former Justice John H. Clarke wrote that he

had found the capacity derived from mental arithmetic very valuable in the hurry of trials of cases.[33]

Modern psychology has brought into question the idea that the mind can be effectively trained for use in practical life situations by means of mental gymnastics on material that is difficult and puzzling, yet unconnected with such situations. The former faith in ready and general transfer from one field to another has been rather thoroughly dispelled.

On the other hand, certain opponents of the old psychology have undoubtedly gone too far in denying the possibility of transfer. That some general result occurs, even in the performance of an apparently absurd mathematical problem, is probably true. That certain habits and points of view can be acquired in such work which will carry over to general application is also not only possible but probable.

It is not the function of this section to discuss at length the theories of transfer,[34] but to point out that the attack on the old faith has resulted in the elimination of many types of arithmetic problems which could not be defended except for such disciplinary ends.

Most courses of study to-day advocate problems which will exemplify the applications of arithmetic to actual life situations which are now, or will be, of value to the pupil. This idea is, of course, tied up with the conception of social utility which plays a very large part at the present time in the determining of material to be included in any course of study.

b. The Doctrine of Social Utility.—This doctrine sets up the criterion of social usefulness as the chief basis for the selection of subject matter for a given course. It has found expression in numerous studies to discover what knowledge and skills people actually use. These studies have been for the purpose of investigating processes and problems used by parents, salesgirls, bank clerks, and others. Such investigations have accomplished much good in showing the kind of work actually performed by people in various phases of life. Naturally they also showed that much material taught in school was not needed in life and thus they aided in eliminating content.

There is serious danger, however, as has been pointed out in Chapter II, of carrying the idea of social utility too far. Many educators talk quite glibly about the 'scientific construction of the curriculum.'

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[33] Sullivan, Mark. *Our Times*, Vol. I, pp. 120-121. Charles Scribner's Sons.

[34] An extended discussion of these theories may be found in the *Twenty-Seventh Yearbook* of this Society, Part II, 1928, Ch. XIII.

What they frequently mean is the selection of curriculum content by means of so-called 'scientific' investigations. To them an investigation is considered 'scientific' if the data collected are treated correctly from a statistical standpoint. But frequencies, medians, and correlations do not determine what should actually go into a curriculum. The facts thus collected and presented are interesting and valuable, but they present only a part of the whole picture. Suppose it were proved, for example, that a very large percent of the problems encountered by the ordinary citizen require only the addition, subtraction, multiplication, and division of whole numbers. This would not prove that arithmetic should be confined to these processes, because it may be that arithmetic has many other social values besides its use in the actual computation involved in the day's affairs. Scientific studies can give the raw facts of specific conditions as they exist; they aid in the problem of setting up objectives, but they must be considered along with other facts and with the outcomes desired.

The doctrine of social utility is often interpreted to mean that that only is useful which is shown to be actually in use. In reality, utility should be broadly enough envisaged to include the value of a subject to mankind in all of its phases. Arithmetic, for example, has informational value. It functions in the concepts that a citizen has of his world. It forms a basis for the orientation of a person to many fundamental conceptions of modern society. It is necessary to examine carefully, therefore, the topics advocated for elimination in the light of *all* the values which arithmetic can offer. For example, many have recommended the elimination of longitude and time. Yet in the operation of the radio and in traveling, a considerable part of the population needs to have at least a reasonable understanding of differences in time. Although this might be thought to be a geographical concept, there are mathematical elements involved, so that this topic can be easily considered as an application of aspects of number to practical life situations.

2. Topics Commonly Recommended for Elimination

It is interesting to note that as long ago as 1887, Francis A. Walker[35] persuaded the Boston School Committee to eliminate such topics as mensuration of unusual surfaces and solids, compound proportion, compound interest, equation of payments, exchange, metric system, and compound partnership.

In 1895, the Committee of Fifteen recommended that obsolete and impractical parts of arithmetic be eliminated.

Walter A. Jessup[35] in the *Fourteenth Yearbook* of this Society recommended the following eliminations from the elementary course of study: apothecaries' weight, alligation, aliquot parts, annual interest, cube root, cases in percentage, compound and complex fractions of more than two digits, compound proportion, dram, foreign money, folding paper, the long method of greatest common divisor, longitude and time, least common multiple, metric system, progression, quarter in avoirdupois table, reduction of more than two steps, troy weight, true discount, unreal fractions.

Wilson[36] in 1922 recommended the following eliminations "on the basis of the business requirements of the large majority of the community": long method of greatest common divisor; most of least common multiple; long, confusing problems in common fractions; long method of division of fractions (always invert and multiply); complex and compound fractions; apothecaries' weight; the furlong in long measure; troy weight, the root in square measure; the dram and quarter in avoirdupois weight; the surveyors' table; the table of folding paper; tables of foreign money, all reduction of more than two steps; most of longitude and time; cases in percentage (make one case by using X and the equation); true discount; most of compound and annual interest; partial payments, except the simplest; profit and loss as a separate topic; partnership; cube root; the metric system.

In the *Third Yearbook of the Department of Superintendence, N.E.A.,*[37] the following list for complete elimination is given: compound numbers, addition, subtraction, multiplication, division; greatest common divisor and least common multiple beyond the power of inspection; long confusing problems in common fractions; complex and compound fractions; reduction of denominate numbers; cases two and three in percentage; annual interest; compound interest, except savings; partial payments; true discount; proportion; ratio beyond the ability of fractions to satisfy; partnership with time;

[35] See W. A. Jessup, *Fourteenth Yearbook* of this Society, Part I, 1915, Ch. VIII.

[36] Wilson, G. M., and the Connersville (Indiana) Teachers. *Connersville Course of Study in Mathematics for the Elementary Grades.* Warwick and York, 1922, pp. 13-14.

[37] *Third Yearbook, Department of Superintendence, N.E.A.,* p. 41.

longitude with time; exchange, domestic and foreign; apothecaries' weight; troy weight; table of folding paper, surveyor's table, table of foreign money; much of mensuration—trapezoid, trapezium, polygons, frustrums, spheres; cube root; the metric system.

It is certain that these recommendations have done much to take from arithmetic useless and non-functional material. Distinction should be made, however, between the acquisition of detailed skill involved in the study of a topic and the understanding of the social meaning of a particular practice. It is unnecessary for a pupil to perform tedious examples in compound interest, yet he should know the meaning of the method and its significance in banking, and know that banks do their computing by means of tables previously prepared. A like analysis should be made by curriculum-makers of other topics suggested for elimination to determine whether the topic itself is wholly undesirable or whether there should be a different objective in its teaching than was formerly obtained. A number of the topics recommended in the *Third Yearbook* just cited for complete elimination should be subjected to this kind of scrutiny.

3. Future Eliminations

Besides a more careful consideration of the topics already recommended for elimination to determine what phases of the topic should be left out and what phases retained, much needs to be done in connection with eliminating bizarre problems which are manufactured for drill. Practically all arithmetic books still contain such problems. Many of them represent situations which rarely or never occur in life and many are unsuited to the pupil at his particular stage of development. It is quite probable also that there is much overteaching in certain of the easy combinations and much underteaching in others. Improvement here depends on a more accurate knowledge of relative difficulty and a more careful construction of textbooks. Much also remains to be done in individualizing arithmetic so that those who accomplish rapidly pass on to more advanced work or use their time for other activities.

4. Criteria for Elimination

The following criteria for determining eliminations are suggested:

1. Is the material of practical use in life?
2. Does the material coincide with the practices of the business world?

3. Can the material be understood by children and used, therefore, for a broadening of experience?
4. Are there certain phases of the topic which should be taught for informational value as opposed to skill value?
5. Does the topic contribute in a real way to the development of general quantitative concepts?
6. Is the material desirable as a foundation for later mathematical work?
7. Will the topic be of interest to superior pupils even if of little or no value to others?
8. Does the daily work in arithmetic contribute its full share of situations calculated to build habits of self-reliance and independence in the pupil?

5. Addition of Material

a. General Mathematics in the Upper Grades.—While elimination of useless and obsolete content has been operating to cut down the number of topics and the amount of material, there have also been certain additions. Perhaps the most outstanding single tendency has been the effort to break down the water-tight compartments between arithmetic, algebra, and geometry so as to include certain concepts from algebra and geometry in the arithmetic work of the upper grades.

The formation of junior high schools, with the consequent necessity of reorganizing the curriculum for these grades, has undoubtedly led to new ideas about 'general mathematics.' Consequently, many courses of study prescribe certain elementary algebraic concepts and a considerable amount of intuitive geometry in Grades VII and VIII. It should be pointed out, however, that this tendency is by no means universal and that many textbooks have made little attempt really to integrate the course for these grades. Arithmetic is often made the center of emphasis and a little algebra and geometry inserted into it. It is doubtless true that inadequate preparation of teachers has prevented any rapid realization of an integrated course in general mathematics for the upper grades.

b. Additions of Problems of Social Significance.—There have been certain other additions to arithmetic which are evident in several grades. Nearly all of these can be said to aim to broaden the subject so that the pupil will be able to apply number concepts to his social world. The treatment of banking, insurance, and other such topics attempts to picture by illustration and by problem just how business affairs are conducted and to make more evident the social significance

of these matters. Statistics and graphs are used to give the pupil skill in a kind of representation of facts which is becoming more common. Problems are drawn from all of the other fields of the pupil's curriculum.

These tendencies lead to the query as to how far arithmetic should go in the treatment of material which is obviously the application of number concepts and techniques to other fields of study. It is possible, for instance, to think of arithmetic as confined to the acquisition of mathematical concepts and of skills in the manipulation of number. The applications might then be handled by teachers of science, manual training, social studies, and the rest, as means for the proper understanding of the field under study. Such an organization would involve a very careful construction of courses of study so that overlapping and repetition would be avoided. It would also require teachers trained to see the mathematical concepts involved in other fields.

A large proportion of the work of the upper grades consists of the kind of application just mentioned. Yet the arithmetic teacher is usually unable to present the material so that it has real social meaning. Insurance, for example, is too often taught for the purpose of developing skill to compute premiums and to do a few other things which the pupil never performs later in life. The real problems of insurance, which involve the value of insurance, the kind which persons of varying situations ought to take, the returns on life insurance as compared with other investments, etc., are left untouched. Stocks and bonds, as now taught, do not begin to consider the real social problems involved. Just how much can be done with material which has no vital meaning for the pupil until several years later is in itself a serious question.

It is conceivable that the curriculum could be reorganized so that each teacher would deal with the mathematics needed in the understanding of a given field of study. It is doubtful if this will be done soon because teachers are not prepared for this kind of treatment and the task of curriculum construction would be too exacting.

Meanwhile, certain textbooks have attempted to include a considerable amount of explanation of the meaning and significance of the topic being considered. This sometimes runs into the use of several pages of explanatory matter. A certain textbook for the eighth grade copyrighted in 1928 devotes four and a half pages to the explanation of fire insurance, whereas a widely used book copyrighted in 1911

devoted about one quarter of a page to this matter. Another book published in 1927 gives seven pages to explanatory material about bonds, as compared with two and one-half pages in a book published in 1918. An examination of books used to-day shows a decided tendency to expand the treatment of many similar topics. Makers of courses of study need to make a careful investigation of how much explanation of such social topics needs to be given to the teacher. If not in itself a complete manual, the course of study should certainly give reading references to the teacher, so that she will know where to find information. If arithmetic teachers are made responsible for the presentation of material which has social and scientific significance, then they must be prepared to do more than merely present the techniques for manipulating the figures involved. If they can not achieve this result, the pupil certainly will never get from them the real meanings which arithmetic can give him of the relation of processes and institutions to his own welfare.

IV. Changing Methods of Instruction

Suggestions as to methods of teaching have varied, like the subject matter, according to the approved aims of the school, the concept held as to the function of the school, and the like.

1. More Concrete Work, Less Abstract Drill: The Place of Object Teaching

Dominated largely in the early years by a logical ideal and a disciplinary aim, the method in arithmetic was one of memoriter mastery wherein the study was begun by declaiming in concert numbers from one to a hundred and singing the multiplication tables to some popular or patriotic tune. Based upon the doctrine of faculty psychology, the task was one of amassing "lumber yards and stone quarries of atomic facts," and the cultivation of the attention was the immediate goal sought. The emphasis accorded 'mental operations' and the belief that the memory reached maturity more quickly than the perceptive faculties was gradually reversed in favor of the doctrine "that in childhood the activities of perception are greater than other mental activities" and "that both single and related perceptions must be clear and distinct in order that the memory may do its proper work."[38] This change in point of view gave, between 1860 and 1875,

[38] Boston, Massachusetts—*School Document No. 17: Suggestions Accompanying the Course of Study for Grammar and Primary Schools.* 1878, pp. 17ff.

new impetus to (1) the introduction of concrete work into the schools of this country, (2) the work with 'Gifts' and object lessons, and (3) in some cases a new emphasis upon sense training. The following excerpts from courses of study suggest the influence of object teaching (reaching its height in 1880-1900) and its changing emphasis upon (1) sense training, (2) work with the concrete rather than the abstract —the topics of instruction being selected frequently because they had certain mathematical interest rather than because they met some actual need of the pupils in either their present or later life—and later (3) activities of the pupil's everyday environment based upon the theory of 'learning to do by doing.'

1850

Object Lessons: One of the best methods of introducing these developing exercises is to teach the properties of objects separately. By directing the attention of a child for a considerable time to a single property, a distinct and lasting impression is made. Thus, one or more lessons may be devoted to form, others to size, color, weight, motion, number, taste, sound, etc. . . . . The first exercise may be devoted to length, or extension in one direction. The teacher draws a fine straight line on the blackboard, and explains that it has length, but no breadth or thickness. Then he measures it, and gives various illustrations of length, as the length of the floor, the height of a man, the distance across a field, etc.[39]

1870

Object Lessons will be given him through life. This is the way Providence teaches. The Great Teacher, so gladly heard by the common people, taught in parables. He spoke to the eye and ear. The habit of heeding such teaching is the habit needed through life. Their eminent usefulness is well illustrated in the career of Franklin. It not only gives readiness and correctness of observation, but it rouses, directs, and disciplines the powers of inquiry and reason.[40]

1890

Kindergarten—The Gifts. *The Fourth,* a two-inch wooden cube, divided into eight brick-shaped blocks. Number. Twos and multiples of two in combination. Quick recognition of ones, twos, threes, fours, and fives in groups of objects. Carry number into everything, bead stringing, marching, making of chains, borders, etc.[41]

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[39] Madison, Wisconsin—*Annual Report of the State Superintendent of Public Instruction, 1854.* Pp. 29ff.

[40] State of Tennessee—*First Report of the Superintendent of Public Instruction of the State of Tennessee, October 7, 1869.* P. 49.

[41] Saginaw—*Saginaw East Side, Public Schools—Manual of Instruction, 1896-1897.* Pp. 13ff.

1910

So strong is the necessity for objects that if the teacher does not require their use, the child will count his fingers or make marks on the slate or blackboard, thus showing the natural demand of the mind for concrete things in the primary exercise with numbers.[42]

Upper grades.—Make the work real in teaching percentage, interest, proportion, discount, taxes, insurance stocks, mensuration. Let the children dramatize the work, keep store, buy and sell, act as insurance agents, brokers, bankers, commission merchants. Get copies of all forms of business paper, copy and use in actual transactions. Let them measure, estimate, approximate, and plan for problems by actual measurements.[43]

These quotations point to the influence of object teaching and sense training extending over a rather long period of years—a period which, in so far as object teaching is concerned, again evidences a dualism of point of view, with perhaps even greater difficulties in a consistent application of theory to practice.

2. Larger Use of the Pupil's Environment and Increased Provision for Self-Activity

With the swing of psychology away from the faculties toward the direction of reaction and response, it was recognized (1) that "the procedure of the teacher may call into play chiefly the memory of the pupil, thereby neglecting training in power to judge values, initiate, organize ideas, and appreciate ends," and (2) that "mistakes made . . . and discovered by the pupils themselves have far more educative value than perfect results obtained by blind following of a dictation."[44]

Resultant upon this recognition one finds object teaching being interpreted so as to provide for a larger use of the pupil's environment and to give an opportunity for him to initiate activities, and to discover number facts for himself, to learn by doing things within his range of interests. The few suggestions from the courses of study selected to illustrate this trend also force one again to recognize the early introduction of the new point of view, sometimes misinterpreted, just as the preceding quotations have suggested the many artificial situations set up in an endeavor to harmonize the new point of view in practice.

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[42] Philadelphia, Pennsylvania—*The Course of Study in Arithmetic for the Public Schools of Philadelphia.* 1910, p. 4.

[43] Michigan—*District Schools of Michigan, Tenth Edition, 1912.* Pp. 142ff.

[44] East St. Louis—*Detailed Course of Study for Grade Classes in Arithmetic.* 1914, p. 16.

1850

Let it be understood by all, that the great secret of successful teaching is to awaken in the minds of the young a love of knowledge and of truth, to show them the advantage of mental activity, and to plant deep the conviction that all true progress must be the result of their own well-directed exertions through the master habit of close attention.[45]

1870

But many schools are beginning to remedy the defect by introducing into the course of study even for the lower grades, the natural sciences. Laws of health are being expounded, science of government and moral philosophy are crowding out less practical and no better disciplinary studies. Let Colorado join in the reform, by demanding of her teachers a higher grade of scholarship. We have little use for teachers who are never more than one lesson in advance of the pupils; and who, if some bright inquisitive pupil seeks information relative to the philosophy of fire extinguishers, or of the manufacture of lucifer matches, or of the bursting of water pitchers on frosty nights, or of the rebounding of a marble from the stone pavement, or why it is injurious to health to lace tightly any portion of the body, or to wear metallic bracelets and necklaces in a freezing atmosphere, is compelled, through ignorance to reply: "It will do you more good to find that out for yourself" or "I'll tell you some other time."[46]

1890

The ingenious teacher finds many ways of having pupils learn by doing.[47]

1905

Concrete problems should be drawn from the field of the child's interests and experiences.[48]

1920

It is relatively easy to give real meaning to number work if real situations, which arise in every classroom, are properly used.[49]

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[45] Springfield, Massachusetts—*Rules of the School Committee and Regulations of the Public Schools of the Town of Springfield, 1846-1847.* P. 9.

[46] Colorado—*Second Biennial Report of the Superintendent of Public Instruction of the Territory of Colorado, for the Two Years Ending September 30, 1873.* Pp. 10ff.

[47] Missouri—*Course of Study for the District Schools of Missouri.* Published by Missouri School Journal, November, 1891, p. 33.

[48] Chicago—*Chicago Public Schools, Course of Study for the Elementary Schools, Adopted August 31, 1904.* P. 11.

[49] Indiana—*Manual with Courses of Study for the Elementary Schools of Indiana.* 1919, p. 37.

1928

Integers[50]

Specific Objectives	*Suggested Activities*
1. Ability to sense meanings of numbers	Connective number with concrete, objective experiences through play, with materials, etc.
2. Ability to count by ones, up to 50 a. Rationally	Counting Members of family Toys—dolls, balls, marbles Bounces of ball Children in each row Objects in picture Pictures in room Children who bank Flowers in vase New kittens New children in the neighborhood Children in Sunday School class Pennies for show Candles for birthday cake Days of the week Books needed in each row Children present or absent Stripes in flag Scores in games Plates needed for party Characters in story
b. By rote	Counting by rimes Ten Little Indians One, Two, Buckle My Shoe One, two, three, four, five, I caught a hare alive, etc. Playing games involving serial counting Counting for sheer joy in the activity

3. Correlation: Relating the Several Fields of Instruction

The gradual movement away from the theory of mental discipline to child activity based upon a consciously recognized purpose made necessary the bringing of the materials of instruction together in a

[50] Kansas City, Missouri—*Course of Study in Arithmetic—Grades 1-6.* 1928, p. 52.

natural order. Correlative with this change is the recognition of the relationship between arithmetic and other fields of instruction. In reading the following quotations one is again forced to recognize the dualism of conflicting points of view resulting in the development of forced and absurd relationships, the use of devices, motivation, adult-judged and adult-imposed interests, and the like. The statements suggest the gradual growth and extension of the movement from the *drawing* of objects to be taught to such other related fields as science, spelling, and the like.

1850

Spelling . . . . thirty words selected from the terms and definitions used in arithmetic.[51]

1860

The letters used for numbers to be taught as they occur in the captions of the reading lessons.[52]

1900

Much of the work in arithmetic in our schools must necessarily be in the nature of drill work, and too much stress cannot be laid upon it. Rapidity and accuracy in the four fundamental processes can only be attained by teaching these processes, and not attempting to interweave them with some other branch of study and thereby losing the force that can only be obtained by the concentration of the mind on the one thing to be accomplished. The attempt to make every lesson in arithmetic a language lesson, a drawing lesson, an object lesson, or a lesson in nature study, or vice versa, will detract from the lesson and confuse the child. The one thing needed in arithmetical processes is practice. A child can only learn to add, subtract, multiply, and divide, by adding, subtracting, multiplying, and dividing; and no way is known by which greater mental activity can be acquired than by drill in the above-named processes.[53]

1915

VI. Mathematical problems which occur in other school work: (a) Manual training, (b) Cooking, (c) Sewing, (d) School gardening, (e) Geography.[54]

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[51] Chicago, Illinois—*Seventh Annual Report of the Board of Education, for the Year Ending February, 1861.* Superintendent's Report, p. 66.

[52] Boston, Massachusetts—*Annual Report of the School Committee of Boston, 1857.* P. 288.

[53] Louisville, Kentucky—*Course of study.* 1899, p. 7.

[54] New Hampshire State—*Program of Studies for the Elementary Schools of New Hampshire.* 1916, p. 74.

1920

There will be days, therefore, in upper-grade arithmetic on which business institutions will be studied without reference to the use of numbers or number processes, just as the institutions of government, or commerce, or transportation are treated in geography. Such work gives the child a better understanding of the situation in which he is to do his figuring. It is important in business life to know both what to do and how to do it. . . . . Explain stock quotations as given in daily papers. Distinguish between reliable and non-reliable bonds. Correlate with Civics.[55]

Still more recently, and especially since 1920, there have appeared larger units of instruction and 'wholehearted purposeful activities' in many of the better courses of study. In many cases conscious effort has been exerted to introduce activities which will eventuate in number facts and arithmetical processes. This is illustrated by such projects as the cafeteria, making a house, and the grocery store. In still other instances activities have been encouraged where subject-matter lines seem to have been completely forgotten in the original planning, the activity being carried on for its own sake. In these cases, however, arithmetical content has crept in either through the conscious planning of the teacher or, as some contend, through the natural turn of events. The following extracts from recent courses of study are illustrative of these two types of work:

1928

I. A CAFETERIA (Third Grade)[56]
  A. Reading of numbers
    1. Price of each kind of food
    2. Units based upon cafeteria activity
    3. Dollars and cents (Use of ¢ and decimal point)
  B. Writing of numbers
    Writing menu and price for each article
    Writing cents in estimating amount of lunch purchased

4¢	and	$.04
5¢	and	.05
3¢	and	.03
3¢	and	.03
15¢	and	$.15

[55] San Francisco, California—*Course of Study in Arithmetic for the Day Elementary Schools—City and County of San Francisco.* 1918.

[56] Kansas City, Missouri—*Course of Study in Arithmetic for Grades I to VI.* 1928, pp. 158f.

C. Vocabulary
Add
Subtract, in making change
Remainder, or difference
Multiply
Multiplication
Divide
Purchase

D. Combinations
All addition combinations may be used through changing prices of foods and through use of prices in reading units based upon the cafeteria activity.
Multiplication combinations may be used; as buying 3 sandwiches at 5¢ each.
Subtraction combinations may be used when estimating change.

E. Fundamental processes
Addition
Column addition when figuring amount of sales; as
Mary's lunch was 18¢
Jack's lunch was 15¢
Jane's lunch was 12¢
Tom's lunch was 21¢
How much did these children spend in all?
Subtraction
Making change
A box of cookies was bought from a salesman for $2.25. He was given a $5.00 bill. How much change should be received?
Estimating change when purchasing lunch
Lunch costs 14¢. Cashier received cash, 25¢.
Cashier says, '14, 15, 20, 25,' or '14, 15, 25.'
Multiplication (Finding total)
Five children bought lunches at 18¢ each.
How much did they spend in all?
Division (Short division)
Jack has 20¢ to spend for ice cream for part of his lunch for a week. How many times can he buy ice cream if it costs 5¢ for each serving?

F. Money experiences
Child handles toy money, buys lunch, and receives change.
Cashier estimates amount of lunch and makes change.

G. Measurements
Measuring by inches when making tags and paper trays.
Measuring by feet when making counter.

1928

A GROCERY STORE (Second Grade)[57]

Initial Indication and Interest.—After the class had made a cafeteria the question was raised by the teacher as to where the food in the cafeteria came from. There followed a discussion of dairies, bakeries, markets, ice plants and grocery stores. The possibility of building one of these was discussed. Since they were all familiar with a grocery store it was decided that the class should build one. The covered crates which had formed the walls of the cafeteria were used for the counter. The paper was cut from the openings of other crates and these were piled on each other to make the shelves.

Development.—Posts were put at either end of the store. A cross piece was put at the top of these and on this was put the name. Letters were cut for the same, pasted on brown cardboard and tacked to the board. . . . The oil cloth which they used when working with clay was spread over the counter. Empty boxes, cans, and bags were brought from home. Small pieces of wood were covered with soap wrappers and candy wrappers. Sand was weighed and put in sugar sacks. Small rocks were put in salted peanut bags. Scales were brought from the stock room and put on the counter. A telephone was made. A telephone book was made, also. . . . Paper money was made by the children. The children played in the store every day. An awning was made for the store. . . . A chair was made for the store keeper to sit in when not busy. A show case was made, also.

Materials Used

Tools—hammers, saws, rulers, needles, scissors, crayons, brushes.

Materials—(a) brought by the children—boards, boxes, thread, nails, soap wrappers, candy wrappers, empty boxes, empty cans, empty bags, broomstick (for part of telephone), wire (for telephone). (b) furnished by the school—colored paper, drawing paper, clay, paint (water colors), paste. (c) bought with money the children made at a candy sale—paint, 10¢; brush for shellac, 10¢; cloth for awning, 30¢; total, 50¢. (d) found on or near the school grounds—clay, vines for baskets, small rocks, sand. (e) left from last unit of work—boxes (covered with brown paper), nails, tacks, paint, brushes, shellac.

Outcomes in Mathematics

Use of foot and yard measure:

1. In measuring for the awning.
2. In placing the letters for the name of the store on cardboard. These were pasted 1½ inches from the bottom of the paper.

Use of the scales and familiarity with oz. and lb.

[57] Raleigh, North Carolina—*Teaching in Grades Two and Three.* 1928, pp. 69f. Outcomes in other fields of subject matter are eliminated in this excerpt.

Writing and reading numbers up to 1000, through making and using the phone book.

Use of 5¢, 10¢, 25¢, through making and playing with toy money.

Making change.

Writing and solving simple problems.

4. Special Methods of Instruction: Spiral, Grube, Topical, Problem

Contingent upon the movement away from formal discipline and sense training toward life applications of arithmetic, the special methods or types in instructional procedure have varied from the highly organized logical plan to the flexible unit of work based upon the needs, interests, and abilities of a particular pupil group. The long established method of teaching the four fundamentals in their regular order was replaced by the 'spiral method,' which, in general, consisted of dividing the numbers used in teaching addition, subtraction, multiplication, and division into classes, or so-called circles, and teaching each circle in relation to the four processes before passing on to the next higher circle. With increased emphasis upon sense training the method developed by Grube went beyond the principle of dividing into classes, or circles. Within the limits of the small numbers he took up each of them, commencing with 1, teaching the child all there was to know about it before passing to another number. Treating, for instance, the number 2, he had the children perform all the operations that are possible within the limits of this number, no matter whether in the usual classification they were called addition, subtraction, multiplication, or division. The whole circle of operations up to 2 was exhausted before the child progressed to a consideration of the number 3, which was to be treated in the same way. So Grube takes up one number after the other, and compares it with the preceding ones, in all imaginable ways, in regard to addition, subtraction, multiplication, and division. This comparing, or 'measuring,' takes place always in connection with external, visible objects, so that the pupil can see the objects, the numbers of which he is to compare with each other.

With the changing point of view giving increased attention to the child's reactions and interests, the "monumental errors" of such methods as those of Grube and Speer were recognized as "attempting to present primary numbers according to the 'logic' of the adult mind."[58] In the place of these methods there were developed the

[58] Augusta, Maine—*Course of Study in Mathematics for the Elementary School.* 1909, p. 13.

'topical' and the 'topical-spiral' plans. As the name suggests, the 'topical method' was one "in which a definite topic is assigned as a year's work" rather than subject-matter topics taught recurrently from grade to grade as in the spiral plan. The method was not applied in practice to any appreciable extent on the primary level; only one course of study of those examined through the 1920 period was organized according to this plan. The early years of the century beginning about 1905 more commonly used the 'compromise topical-spiral method.' The period since 1915-1920 placed new emphasis upon the child's part in the program of instruction, with the introduction, in a new way, of the 'project method' or the 'project-problem method' and its emphasis upon an activity program. Under the name of this method we find both the spiral and topical organizations of subject matter as well as a consideration of the interests of children which makes for an organization distinct within itself.

The Richmond, Virginia, *Arithmetic Course of Study* is the only one of the 119 courses prior to 1921 examined which was specifically organized in terms of suggested pupil activities. Illustrative of the general tendency to apply this method to the primary grades first, this course, while including 'type projects' in the upper grades, is organized by months with reference to specific subject-matter topics.

Each of the several major methods has had some merit. Thus, we see the spiral organization considered today in its bearing upon proper learning sequence; we see the topical method applied in the use by some schools of such class activities as the school store, post office, bank, and the like, carried on as central organizing activities for a year's work; and we see the project or problem method interpreted to include type studies, teacher- and pupil-suggested activities and problems, a series of units of work or one or two larger units for the year's program.

5. The Place of Drill: Games and Devices

In the field of a tool subject any discussion of method of instruction would be incomplete without a brief mention of the nature and place of drill. The major trends seem to have been fourfold: (1) increased emphasis upon the child's ability to think as well as his ability to memorize, (2) more concrete work and less abstract drill, (3) the motivation of even abstract drill through games and other devices, and (4) less dependence upon mechanical drill and more upon arith-

metic as a tool to be used in meeting a situation. The following excerpts from the courses of study suggest both the change in emphasis upon drill and the nature of the drill, from the early stress upon 'mental operations' and drill for drill's sake as a basis of disciplining the mind to emphasis upon the development of necessary pupil interest by means of games and devices, and later to the pupils' recognition of the need for drill through the carrying forward of selected activities. No attempt has been made to edit the English of these excerpts.

1850

Mental arithmetic is studied thoroughly, till the pupils can give the results of ordinary practical questions, without the use of slate or pencil. While studied for the purpose of cultivating the habit of rapid mental operations, it will also help to strengthen the reasoning powers and give "skill in the application of principles" in all the practical departments and business of life.[59]

1900

Mental operations are conducted for three distinct purposes: First, as an introduction to the processes in written arithmetic, and second, as an exercise in mental discipline. Pupils gain self-reliance and power by solving hard problems—independent of pencil and paper; third, as a test of the child's ability to reason. Therefore, the mental exercise is one of the most important exercises in mathematical teaching in the elementary schools.

Mental problems should be dictated by the teacher, as a rule but once, in order to cultivate attention, the question stated before the name of the pupil is given and the problem should be reproduced by the pupil.[60]

1920

Our principal business as teachers then is to get going in our children, as effectively as we can, such whole-hearted purposeful acts as will require the child to use actually the habits to be formed, the knowledge to be acquired, or the idea to be built up. In true "life situations" the desired subject matter (habit or information or idea) is generally present as a means to some end, as adding is necessary to playing beanbags. The child may be brought, however, to purpose directly the acquisition of some habit or the increase of some skill. In such cases we have certainly the best, possibly the only good kind of drill. Our main school reliance should probably be on the life-situation, purposeful act, where the children—under the

[59] New Haven, Connecticut—*Report of the Superintendent of Common Schools to the General Assembly, May Session, 1856.* P. 68.

[60] New Haven, Connecticut—*New Haven Public Schools—Course of Study in Arithmetic.* 1898, p. 4.

teacher—themselves purpose, plan, execute, and judge. But we may supplement this when needed by the "purposeful drill." [61]

6. Types of Arithmetic Problems

The changing nature of the drill materials is further seen by a study of illustrative problems taken from examination questions and discussions of problem materials in the course context.

1850

In 144 miles, 1 furlong, 8 yards, 1 foot, how many feet? [62]

Divide ⅔ of ¾ of 2/7 of ⅞ and explain each step of the process. What is the greatest common divisor of 125350 and 365? [63]

1860

Reduce ¾ of $\frac{7\,4/11}{3\,2/3} + 3/11$ to its simplest form. [64]

Sample problems:

3. Multiply 890870 by 900800.

6. Required the excess of nine hundred and twelve thousand and ten above 50082.

3. Reduce 2 rods, 35 poles, to the decimal of an acre. [65]

1880

1. Divide 144 A. 3 R. 18 rds. 3 yds. 1 ft. 36 in. by 11 and multiply the quotient by 5½.

5. If 7¾ cwt. be carried 36 miles for $9.50, how far might it be carried, at the same rate, for $320?

6. Divide the square root of 18740241 by the cube root of 111284641. [66]

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[61] Richmond, Virginia—*Richmond Public Schools—Arithmetic Course of Study.* 1921, p. 3.

[62] Chicago, Illinois—*Fourth Annual Report of the Superintendent of Public Schools of the City of Chicago.* 1858, p. 29.

[63] Chicago, Illinois—*Third Annual Report of the Superintendent of Public Schools of the City of Chicago.* 1856, p. 18.

[64] Cambridge, Massachusetts—*Report of the School Committee of the City of Cambridge for the Municipal Year Ending January, 1859.* P. 27.

[65] Cincinnati, Ohio— *Common Schools of Cincinnati—Thirty-Second Annual Report for the School Year Ending June 30, 1861, and Various Supplementary Documents, Exhibiting the Condition of the Schools.* 1861, pp. 88f.

[66] Louisiana—*Biennial Report of the State Superintendent of Public Education to the General Assembly of the State of Louisiana.* 1882, pp. 37f.

1900

Numerous practical problems must be drawn from the sources of permanent interest to the child from the social occupations. These must be of such nature as to impart interest to the otherwise dry facts of number.

It is not expected that teachers will prepare all the problems used in their work. Every teacher should have several primary arithmetics at her desk; but if problems are not suited to the needs of her class, she should thoughtfully adapt them to her purposes. She should, moreover, see to it that many of her problems grow out of the experiences of her pupils. If the pupils themselves can be led to give intelligent assistance in this work, so much the better.[67]

1910

The problems should also be as practical in their nature as possible. They should represent the operations of real life, and not some abstract or fancied view of what these operations might be. In denominate numbers the measures should be applied to the actual uses of the store, the shop, the market, the household, etc. The problems in percentage should not merely illustrate the theoretical principle, but represent the actual business of the store or office.[68]

1915

Subtraction:[69]

$$\begin{array}{r} 476892365 \\ \underline{298789253} \end{array}$$

Applied Problems:[70]

1. Industrial arts: book covers, window boxes, cold frames, seats for school yard, etc.

2. Geography: (a) clothing, (b) house furnishings, (c) factory problems, (d) farm problems.

1920

a. Problems based on factory activities (if factory has been visited).

b. Problems based on community interests.

c. Problems based on family purchases, at the stores.

d. Problems based on other school work, such as drawing, geography, manual training, etc.

e. Problems based on childhood activities that may not be directly connected with the school, as gardening, selling papers, pop corn, etc. Also, problems based on their play.

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[67] Jackson, Michigan—*Public Schools, Jackson, Michigan, Course of Study.* 1901, pp. 63f.

[68] Philadelphia, Pennsylvania—*The Course of Study in Arithmetic for the Public Schools of Philadelphia.* 1910, p. 44.

[69] New Hampshire—*Program of Studies for the Elementary Schools of New Hampshire,* 1915, p. 69.

[70] Atlanta, Georgia—*An Outline of the Course of Study for the Elementary Schools, 1916-1917.* P. 131.

f. Problems on measurement.

Type problems for Fourth Grade may be presented by the pupils themselves, suggested by their own activities, such as products of their gardens and orchards, gathering, marketing, etc., their work as paper boys, errand boys, nurse girls, playing store, keeping fruit and candy stands, keeping accounts of their school supplies, the school enrollment and many other sources which will be noted and supplied by the alert teacher.[71]

1927

Our class banked $9.71 last week. The week before we banked $9.46. How much did we bank in the two weeks? Which process should be used?—Addition, Subtraction, Multiplication, Division. Which of the following is the most reasonable answer?—25¢; $9.00; $19.17; $25.00.[72]

You know how much the material for your dress cost and how many yards mother bought. What can you find out?[73]

The preceding problems not only suggest changes in drill materials, but serve also as concrete expressions of a changing content and method in the direction of the needs, interests, and abilities of the pupil group essential to growth in handling number situations as they are met in daily experiences.

7. Use of the Textbook as a Phase of Method

Throughout the development of the course of study in arithmetic the textbook has played a prominent part. With the change in method has come a very definite change in its use from the formal work of the early days which demanded that absolute mastery of certain knowledges and skills, mostly computational, be secured through a rigid and formal process of drill and memorization. Under such a method the slavish following of a textbook was an essential; any divergence from it on the part of the teacher subjected her to severe criticism.

When among the earlier courses an effort was made to use several textbooks rather than one, the attempt led to much discussion and complaint relative to the dangers and difficulties resulting from a lack of uniformity in the textbooks used. The use of the text as a reference only, and later the use of a multiplicity of texts, is a relatively recent tendency coming with the change in method from the old mastery idea

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[71] Akron, Ohio—*Course of Study in Arithmetic—Elementary Grades.* 1921, p. 30.

[72] Toledo, Ohio—*Supplementary Problems in Arithmetic, Grade Four,* p. 33.

[73] *Ibid.*, p. 88.

to that now suggested of presenting materials in situations familiar to the child with stress on knowledges and appreciations rather than on computational skill only.

This changing conception of the place of the textbook in the teaching of arithmetic is shown by the following quotations:

1850

Textbooks: Another serious obstacle to the greater efficiency of our common schools, is the greater diversity of textbooks in use. By the reports made to this office it appears that the books which are most used in the different towns comprise a list of fifteen different spelling books, eighteen readers, ten geographies, *fifteen* arithmetics, and twenty grammars; and it is believed that if all the school books in use were known, they would comprise a list nearly as extensive as that reported in Connecticut a few years since, which, in the five studies named, included the works of one hundred and ninety-one different authors. The same diversity of textbooks extends to . . . Algebra, Philosophy, Chemistry, etc.[74]

Our schools suffer less from *want* of books than from a *multiplicity*—a multiplicity not only from various editions of the same author, so altered that scholars cannot use them with any sort of concert in their recitations, but also a multiplicity of works by various authors, upon the same subject and branches.[75]

1870

Uniformity of Textbooks—The variety of textbooks on the same subject, which I find in the very many schools, is a great hindrance to classification and progress. Each new teacher is allowed to introduce his favorite authors. New scholars from other districts, towns, or States, retain their old books. In one school I found seven classes in Geography, where with uniformity of books, they would be reduced to three, to the great improvement of the school. In another school, with nineteen scholars, there were six classes in spelling, seven in reading, five in Arithmetic and seven in Geography. The result is 'confusion confounded,' the embarrassment of the teacher, and discouragement of the pupil. Instead of system, classification, and thoroughness, the teacher's time is wasted on a medley of textbooks. When no two have the same book, the recitations of each individual by himself must be hurried and superficial.[76]

In teaching arithmetic to the several classes, every teacher shall be at liberty to employ such books as he shall deem useful, for the purpose of affording illustration and examples; but such books shall not be used to

[74] Madison, Wisconsin—*Annual Report of the State Superintendent of Public Instruction.* 1854, p. 26.

[75] Connecticut State—*Report of the Superintendent of Common Schools to the General Assembly, May Session, 1856.* P. 54.

[76] Quoted in *Tennessee State: First Report of the Superintendent of Public Instruction.* 1869, p. 100.

the exclusion or neglect of the prescribed textbooks, nor shall the pupils be required to furnish themselves with any book but the textbook.[77]

1915

In the earliest grades the teacher will find it necessary to conduct the work without reliance upon any particular textbook, but in the upper grades a series of arithmetics should be used. The course cannot, however, be conducted in the best manner without supplementing extensively any one book which may be adopted. The school board will find it desirable to supply teachers with several sets of arithmetics, at least with a number of additional single copies for supplementary material. The teacher will need to base much of her instruction on the experiences of her pupils in their home life, their play activities and their other school work.[78]

1924

Use of Textbooks: Textbooks are not to be used by pupils below the third year. From the third year, books are to be used simply as a tool just as the dictionary would be used. If the approach to every new topic is to be motivated material, then the textbook should be used *only for drill work,* after the new topic has been taught from situations with which the children are familiar, illustrated by objects when possible.[79]

1928

The Meaning of Percentage:[80]

References:

Gugle, Book I	pp. 77-78
Schorling and Clark, Book VII	pp. 24-25
Anderson, Book II	p. 212
Hart, Book I	p. 48

The First Principle of Percentage:

References:

Schorling and Clark, Book VII	pp. 22-27
Stone, Book I, Graphing—Chap. II	pp. 218-221
Hart, Book I	pp. 49-51
Anderson, Book II	pp. 213-216
Breslich, Book I	pp. 215-217

[77] Boston, Massachusetts, 1866, quoted in *District of Columbia: Department of Education (U. S.) Special Report of the Commissioner of Education on the Condition and Improvement of Public Schools in the District of Columbia submitted to the Senate June, 1868, and to the House with Additions, June, 1870.* P. 475.

[78] New Hampshire State—*Program of Studies for the Elementary Schools of New Hampshire.* Third Edition, 1916, p. 53.

[79] Montana State—*State Course of Study for Montana City Elementary Schools.* 1924, p. 226.

[80] Houston, Texas—*Course of Study in Arithmetic, Grades Six and Seven, 1928-29.* P. 54.

8. Providing for Individual Differences

The changes which have been noted in the method of instruction of necessity raised the question of ways and means of providing for the individual pupil. The first provisions were administrative only and dealt with the problems of size of classes, promotion, and a graded classification of pupils. The following statements suggest the emphasis placed upon managerial aspects of the problem, in keeping with the then accepted goal of mental discipline, especially the cultivation of the attention.

1850

The great want of our school, at the present time, we conceive to be, a better classification. . . . By a better classification, we mean a greater uniformity of age and attainment in each school. The great diversity in these respects, in our Grammar schools now, makes it necessary to divide a school into a large number of classes. One school for instance, has nine classes in the single department of Arithmetic, five in Reading, etc., making in all twenty-one classes. It will be seen, at once, that the teacher can give but little time to each class; can do but little if any more than hear them recite from the textbook; that he cannot stop to ask them questions of his own.[81]

1860

This is a school in which everything is taught by one teacher, from the A B C scholar up to those advanced in geography, grammar, and arithmetic. Frequently in six hours a teacher will have to hear thirty recitations, saying nothing of interruptions from recess, asking of questions, and rendering personal assistance in mathematics and other studies. To go through with all his labors requires the force of a steam engine. He has no time for explanation and illustration—to cross-question his pupils, and draw out and develop their own powers of thinking—to galvanize their latent energies into life. For the most part it is an effort to see what the child has learned from the textbook. It is for the pupil a swallowing without digesting. Such is the character of most of our common district schools.[82]

1870

Grading of the Schools: In 1866-7, the schools were graded. In the grammar schools there are eight grades, the first being to the lowest. In the High School, there are three classes called respectively, Junior, Middle, and Senior. The time devoted to the studies of each grade and class is

[81] Dorchester, Massachusetts—*Annual Report of the School Committee of Dorchester, March, 1874.* Pp. 5f.

[82] Kansas State—*The Third Annual Report of the Superintendent of Public Instruction of the State of Kansas, 1863-1864.* Pp. 28f.

generally one year, so that a pupil that enters the first grade, at the age of six years and passes all the examinations for promotion to higher grades, will graduate at the High School at the age of 17 years. When practicable, the pupils in each grade are divided into three classes, those of nearly equal scholarship being put in the same class. Sometimes the pupils in the first, or highest class, pass over two grades in one year. An effort is always made to have the grading so close that the progress of pupils of good capacity is not retarded by those having dull and sluggish intellects.[83]

1910

The course of study is divided in two ways: (1) into six sections; (2) into four sections; each section covering a year's work. Pupils taking the course in six years are classified in six grades, called the fourth, fifth, sixth, seventh, eighth, and ninth grades. Those taking it in four years are classified in four grades, called grades A, B, C, and D. When pupils are promoted to the grammar schools, they begin the first year's work together. After two or three months they are separated into two divisions.

One division advances more rapidly than the other, and during the year completes one-fourth of the whole course of study. The other division completes one-sixth of the course.

During the second year the pupils in Grade B are in the same room with the sixth grade. At the beginning of the year they are five months (one-half of the school year) behind those in the sixth grade. After two or three months grade B is able to recite with the sixth grade, and at the end of the year both divisions have completed one-half of the course of study —the one in two years and the other in three years. The plan for the last half of the course is the same as for the first half, the grades being known as the seventh, eighth, and ninth in the one case, and as C and D in the other, etc.[84]

1915

Group Work in Number:

It is not to be expected that all of the children in a class of forty or forty-five will grasp each new number idea presented to them with the same degree of readiness, or that the same method of approach can be used with all alike. For this reason, group work is necessary.

The groups for number work will change frequently as the teacher sees the need for additional drill for some of her pupils on the work of the day or the week before. Desk work will be provided for those who can work independently and more time and attention thus provided for those who need special help. In the case of the presentation of new work, how-

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[83] Nashville, Tennessee—Quoted on page CLX of the Appendix of Tennessee State—*First Report of the Superintendent of Public Instruction of the State of Tennessee, October 7, 1869.*

[84] Cambridge, Massachusetts—*Course of Study for the Cambridge Schools, September, 1904.* P. 25.

ever, the groups should be thrown together and the work presented to the class as whole rather than to groups.[85]

As the last quotation suggests, size and organization of classes is but one phase of the problem, and educational leaders have not been unmindful of the need for differentiation in method and content of instruction. Both the development of point of view and theory and its more tardy application in practice have been gradual, moving from (1) the recommendation that "all in the same class attend to precisely the same branches of study"[86] to (2) a recognition of difference in "character, ability, development and attainment,"[87] providing for varying rates of progress over the same instructional material, to (3) a but little recognized principle that the "ability and needs of the pupil must determine the matter and method of all teaching,"[88] providing for differentiated content of instruction as well as rates of progress. The quotations which follow suggest the changing point of view away from teacher-recognition of differences for purposes of management to teacher-study of pupil needs, to a coöperative consideration of pupil needs by the teacher and pupil.

1870

Again and again would I impress upon others my conviction that whatever of instruction we impart to a pupil, we should be guided only by the consideration of the pupil's age, health, and capacity. We can make very few too enlightened for their future stations.[89]

1890

The work of each semester is merely indicated, leaving teachers free to choose their own methods. Uniformity of results is desired only so far as is consistent with individual interests. Irregular or indolent pupils should not be permitted to retard the progress of the class, and quick faithful pupils, who outstrip the grade, should receive individual promotion.

The teacher should not lose sight of the individual in the care of the class, but should consider the peculiar traits of character and mental

[85] Atlanta, Georgia—*Atlanta Public Schools, Outlines for the Primary Department, Board of Education, 1916-1917*. P. 106.

[86] Connecticut State—*Report of the Superintendent of Common Schools to the General Assembly, May Session, 1856*. P. 25.

[87] Los Angeles, California—*Courses of Study for the Public Schools of the City of Los Angeles for the Year of 1906-7*. P. 7.

[88] East St. Louis, Illinois—*Detailed Course of Study for Grade Classes in Arithmetic*. 1914, p. 8.

[89] Colorado—*Second Biennial Report of the Superintendent of Public Instruction of the Territory of Colorado, for the Two Years Ending September 30, 1873*. P. 10.

capacity of each. Mental training should be the aim rather than the mere acquisition of knowledge.[90]

1905

No two classes are exactly alike. Each should be handled according to its needs. Some need the drill more at one point; others, at another. Select your problems and train your class according to their evident needs, and as soon as one difficulty is mastered pass on to the next.[91]

1915

The ability and needs of the pupil must determine the matter and method of all teaching.[92]

1924

Individual and Group Instruction: In a group of pupils in any semester's work there is a wide range of ability. The teacher determines this range by diagnostic tests. The remedial work which follows must be largely individual or in homogeneous groups. Thorndike, "Exercises in Arithmetic," the "Courtis" and the "Studebaker" Practice Exercises, are very helpful in the fundamental operations. In no case should the teacher expect all to do exactly the same amount and quality of work. Each pupil should be kept busy to his maximum capacity. Some pupils should solve but few problems, these being selected from the easy ones, while others might profitably solve the most difficult ones and propose new ones. The textbooks usually give in any one lesson made up of problems a variation in difficulty.[93]

1926

General Suggestions:[94]

Plan for individual differences:

a. Give more difficult questions under the same topic to the more advanced pupils.
b. Let advanced pupils help to direct and check the work of the backward ones.
c. Develop initiative and leadership in the less aggressive of the advanced pupils by appointing them as leaders.
d. Give maximum and minimum assignments.

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[90] Minneapolis, Minnesota—*Thirteenth Annual Report of the Board of Education of the City of Minneapolis for the Fiscal and School Year Ending June 30, 1890.* P. 161.

[91] Dayton, Ohio—*Dayton Public Schools Course of Study: Syllabus and General Directions for the Guidance of Teachers, Adopted by the Board of Education August 25, 1904.* P. 183.

[92] East St. Louis, Illinois—*Detailed Course of Study for Grade Classes in Arithmetic.* 1914, p. 8.

[93] Indiana State—*Manual with Course of Study in Arithmetic for the Elementary Schools of Indiana.* 1924, p. 12.

[94] Baltimore County, Maryland—*Course of Study in Arithmetic, Grades One to Eight.* 1926, p. 272.

1927[95]

Required Course 'C'	Supplementary B	Supplementary A
D. Multiplying		
1. All multiplication facts of previous grades . . . Practice for a speed rate of 100 in 5 minutes....	100 in 4 minutes	100 in 3 minutes
Two-digit number by two-digit number—11 in 5 minutes	11 in 4 minutes	11 in 3 minutes
Three-digit number by two-digit number—6 in 5 minutes	6 in 4 minutes	6 in 3 minutes

1927-1928[96]

X Group	Y Group	Z Group
Reading and writing Arabic numbers to 999,999	Reading and writing Arabic numbers to 99,999	Reading and writing Arabic numbers to 9,999

1928

II. Adjustment for Fundamental Processes[97]

A. Subject Matter

It is impossible for the borderline type ever to handle the same amount of subject matter that the normal child handles. Hence it behooves the teacher to select for this child the subject matter within his capacity that is of highest social value and interest to him. A comparative study of courses for normal and subnormal children determined the following suggestive outline for borderline children. In this outline that subject matter which is necessary to the independent social life of all individuals makes up the common core.

For borderline children the common core is vitally necessary, as it is the foundation of all quantitative thinking. It comprises the fundamental processes in integers, fractions, and decimals, and their applications to life activities. The regular course of study should be referred to for further guidance in the teaching of each process.

The following experimental outline suggests by topic—

The common core, which all children should acquire if possible.

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[95] Chicago, Illinois—*Tentative Course of Study in Arithmetic, Grades One to Six.* 1927, p. 21 of Sixth-Grade Course.

[96] Cleveland, Ohio—*Tentative Course of Study in Arithmetic, Kindergarten and Grades One to Four, 1927-1928.* P. 66.

[97] Denver, Colorado—*Differentiation of Study in Arithmetic for the Borderline Child, Grades One to Six, Inclusive.* 1928, p. 5.

The limits of difficulty, within which most borderline children should remain.

Most frequently used type, which should receive most emphasis in drill and problem solving.

The growing recognition of differences among pupils in needs, interests, and abilities found expression in (1) graded textbook material, (2) a new and increased use of inventory tests, diagnostic tests, and remedial materials, and (3) more suggestions in the course of study as to ways and means of providing for these differences.

Based upon analyses including a study of (1) intelligence, (2) achievement in specific fields, (3) social and economic background, (4) community surveys and analyses, and (5) teachers' judgments, courses of study to-day suggest varied means of providing for differentiated content and methods of instruction within the suggested administrative organizations of homogeneous grouping, special classes for the superior and underprivileged, and individual instruction.

The following provisions for differentiation within a homogeneous or non-homogeneous group are among those mentioned in courses of study in arithmetic:

1. Varying rates of progress—adjustment on a time basis
2. Differentiation through special courses
3. Use of minimal, average, and maximal assignments
4. Varying degrees of efficiency required for the same assignment
5. Use of large units of work or activities in which each pupil carries out that part of the activity suited to his interests and abilities (coöperative division of labor)

The field is one which is primarily conditioned by the findings of psychology and experimental investigation and one which is at present replete with many unanswered questions, including such problems as those of specificity of learning, extensive *vs.* intensive material for the superior child, and the like.

V. Topics Which Should Be More Adequately Treated in Future Courses of Study

Preceding sections in this chapter have described the changes in aims, time allotments, content, and methods of instruction which have been witnessed in the field of arithmetic during the past few decades. Despite the recent activity in scientific study and in constructing

courses of study, it is not an exaggeration to say that much more remains to be done than has yet been accomplished. Any student of this field could select a large number of items which need more adequate treatment in future courses of study than they now receive. The limits of space permit the treatment here of only three such topics; namely, (1) Content and Methods for the Primary Grades, (2) Grade Outcomes, Methods, and Content for Different Levels of Ability, and (3) Learning Steps in Various Processes.

These three topics are, in the opinion of the writers, topics concerning which teachers need and desire much more aid than is now given in the average course of study.

1. Content and Methods for the Primary Grades

a. Lack of Agreement in Current Practice.—Tables V and VI summarize certain details regarding the provisions for instruction in primary number in six cities. These tables are intended to illustrate in concrete form the wide divergences which now characterize the amount of time allotted to arithmetic in Grades I and II, the quantity of assistance furnished the teacher of primary number through courses of study, and the specific items of arithmetic knowledge and skill which it is proposed to teach children in these grades. Thus, in three of the cities (Table V) instruction in arithmetic is purely incidental during the first year; children are merely taught various types of counting in connection with the other activities in this grade. In Baltimore and Philadelphia, on the other hand, teachers are expected to develop, besides skill in counting, ability to deal with a considerable number of the addition and subtraction combinations. In the latter city also, ten of the addition and ten of the subtraction combinations are taught as early as the first half of the first year.

Table VI shows that the variations in current practice exhibited in the case of Grade I are equally prevalent in the case of Grade II. Thus, Richmond continues incidental teaching throughout the second grade and makes no attempt to develop ability to deal with any of the number combinations. At the other extreme are Baltimore and Philadelphia, the former city devoting 30 minutes per day to arithmetic, and both cities expecting children at the end of the second year to know not only the 100 addition and the 100 subtraction combinations, but a large share of the multiplication and division combinations as well. Another fact to be noted in Table VI is that Denver and Dallas give

TABLE V.—ARITHMETIC INSTRUCTION PROVIDED IN GRADE I AS INDICATED IN SIX RECENT COURSES OF STUDY

City and Year of Course of Study	Time Allotment in Minutes per Week	Pages of Treatment in Course of Study	Specific Items of Knowledge and Skill Developed							
			Reading Numbers up to	Writing Numbers up to	Counting by				Number Combinations	
					1 to	2 to	5 to	10 to	Addition	Subtraction
Richmond, Virginia, 1926...	None: incidental	3	25	25	100	20	25	100	0	0
Berkeley, California, 1923....	None: incidental	5+*	132	132	132	20	100	50	0	0
Baltimore, Maryland, 1926 ..	No statement...	29	99	99	100	0	100	100	54	30
Denver, Colorado, 1926......	None: incidental	7*	100†	100	100	?	0	100	0	0
Philadelphia, Pa., 1925......	No statement...	26	100	100	100	0	100	100	36†	36†
Dallas, Texas, 1924...	No statement...	2+	200	200	100	0	0	100	0	0

*Preceded by a number of pages of general principles, typical procedures, and the like.

†Of this number of combinations, 10 are taught as such in the first term or semester of the first grade.

TABLE VI.—ARITHMETIC INSTRUCTION IN GRADE II AS INDICATED IN SIX RECENT COURSES OF STUDY

City and Year of Course of Study	Time Allotment in Minutes per Week	Pages of Treatment in Course of Study	Specific Items of Knowledge and Skill Developed							
			Addition		Subtraction		Multiplication		Division	
			Combinations	Examples†	Combinations	Examples*	Combinations	Examples*	Combinations	Examples*
Richmond, Virginia, 1926....	No statement...	1†	0	0	0	0	0	0	0	0
Berkeley, California, 1923 ...	60	3†	60	Yes	60	0	0	0	0	0
Baltimore, Maryland, 1926 ..	150	46	100	Yes	100	Yes	40	Yes	40	Yes
Denver, Colorado, 1926......	No statement...	37†	100	Yes	100	Yes	0	0	0	0
Philadelphia, Pa., 1925.....	No statement...	34	100	Yes	100	Yes	68	Yes	44	Yes
Dallas, Texas, 1925.........	No statement...	34	100	Yes	100	Yes	0	0	0	0

*Uses of abstract numbers beyond the simple combinations.

†Preceded by a number of pages of general principles, typical procedures, and the like.

to their teachers, who are directed to teach only the 200 combinations in addition and subtraction, about the same amount of explanatory material, as measured very crudely by pages in the course of study, as does Philadelphia, where the teachers train the children in the use of 112 more combinations in multiplication and division.

One may next turn to a comparative study of arithmetic texts. In the first place, it is to be noted that very few serious efforts have been made to prepare texts specifically for number work in Grades I and II. Furthermore, these texts are not widely enough used to secure anything like uniformity in offerings or methods. Still again, only two of these texts are planned to articulate well with the texts which are intended for use in the grades following. In the second place, an examination of texts designed for Grade III reveals, as in the case of the courses of study, lack of agreement relative to items to be taught, the time for teaching them, and the instructional devices and methods to be employed. Thus, one text, after devoting seven pages to counting and concrete number, provides for the teaching of all of the 100 addition combinations in the next 11 pages, the method of teaching consisting almost entirely in drill on the abstract formulas. Another text develops each of the addition combinations, except the zero combinations, through concrete numbers and intersperses instruction on a large part of the subtraction combinations and a smaller part of the multiplication combinations before (at the end of about 55 pages) the last of the 100 addition combinations has been presented.

A country-wide inventory of the items of arithmetic ability possessed by children at the beginning of the third school year would undoubtedly yield data that would be amazing in variability. All degrees of ability would almost certainly be uncovered, from that represented by counting to that represented by considerable skill in performing complicated examples in column addition and simpler examples in multiplication and division. Anything else than this extreme variability would scarcely be possible. Because of the absence of expert guidance through helpful courses of study and because of the limited number of commonly used texts especially prepared for the first two grades, teachers too frequently are left to accomplish what they will and can, and by any method that presents itself.

The explanation for the state of affairs described in the foregoing paragraphs lies fundamentally of course in ignorance concerning children's knowledge and uses of number on entering school, their capacity

for profiting from instruction in number, the manner in which they learn arithmetic, the most effective and economical methods for teaching number facts and processes, and so on. Searching investigations of these problems have been slow to appear, and in the meantime it has been necessary to adopt some policy of instruction. It was inevitable that marked differences should develop.

b. Significance of Current Disagreements for the Course of Study.—The present lack of texts in primary number and of exact knowledge regarding the most vexing problems of instruction by no means should be interpreted as an excuse for slighting the course of study in arithmetic for Grades I and II. On the contrary, these lacks make it even more imperative that teachers be provided with careful statements relative to the particular abilities which they are to develop and relative to the materials and methods which they are to employ in developing them. Until the time when rigorously controlled experimentation shall have produced a sound basis for teaching it is probably true that almost any plan of instruction which represents a systematic approach is better than no plan at all. This statement would seem to be fully applicable to those schools which prefer to reduce to a minimum the arithmetic facts which are taught in Grades I and II and to make such instruction purely of the incidental sort. Even when instruction is incidental in the sense that it is combined with instruction in other types of subject matter, it should be systematic in the sense that teachers should know what they are to attempt to do and why they are to attempt to do it. Otherwise little else than confusion and bungling can be anticipated.

c. General Suggestions for the Course of Study in Primary Number.—The paragraphs which follow are not written for the purpose of stating exactly what is to be taught and how it is to be taught. Limitations of space would prevent the listing of the most desirable specific objectives and the describing of the best instructional procedures even were these known. Until they are known it would appear to be preferable to outline briefly certain considerations which may well be borne in mind in framing a course of study. The reader who is in need of concrete recommendations will do well to examine some of the better courses of study now available in primary number.

The first suggestion offered relating to arithmetic in the primary grades is that *the course of study should be based on a psychological analysis of the learning situation.* At the same time that due recog-

nition is given to the philosophical and sociological principles involved, emphasis may well be laid on the notion that the course of study can be most helpful only when it is based upon a psychological analysis of the situation which arises when children begin the systematic study of arithmetic in school.

Any analysis of the learning situation which is made from the point of view of the adult is certain to minimize greatly the difficulties which face young beginners. For example, there is for the adult little mystery and little cause for hesitancy in supplying the answer 15 for $8 + 7$. Long continued practice has given to the fact so much significance and has produced such facility in its use that the answer is given with accuracy and confidence and without delay or the necessity of thinking out the solution. There is a simplicity, an inevitableness, and an obviousness about it that quite unfits the adult for any true appreciation of the beginner's confusion. All the struggles which he himself underwent in learning the fact have long since been forgotten; all the temporary aids which he relied upon to assist him in securing the desired answer have been superseded by mature methods; and the adult is apt to believe that the fact, so uncomplicated in terms of usage, must be equally uncomplicated in terms of learning.

In the case of more overt forms of behavior, such as walking, talking, and the like, the situation seems at first glance to be quite different. The adult, or mature performer, is able to observe directly the processes employed by the beginner; furthermore, he is able to compare his own expert adjustments with those of the beginner, to analyze the source of the beginner's difficulty, and to recommend improved procedures which are capable of evaluation and redirection. In arithmetic the adult does not have the advantage of direct observation; moreover, his own processes resist any form of analysis which enables him to sympathize with the confusion of the beginner. The blundering efforts of the child to learn, recognized as excusable or even desirable in the case of walking and overt behavior in general, are apt, in the case of the arithmetic facts, to be regarded as unnecessary, if not wasteful and harmful, and the child may be urged to master the facts by a process of associating them directly and of repeating them until they can be produced almost automatically on demand.

And yet fundamentally there is no reason to believe that the process of learning in the case of arithmetic differs in any important

essential from the process of learning in the case of the more overt forms of behavior. Whatever apparent differences there may be found can be accounted for in terms of the nature of the responses involved.

A second suggestion is that *the course of study can safely assume that number experience is not as foreign to first-grade children as it has frequently been thought to be.* All instruction in number is sometimes postponed to the second or even to the third grade on the assumption that children when they enter school are incapable of learning arithmetic. The contention is that children up to the time they first come to school have had no experience with number, that they do not need it in Grades I and II, and that to teach them anything about number is to violate their very nature and to force upon them something for which they are not ready.

Admittedly the evidence on this point has up to the present been meager and unconvincing. Buckingham and MacLatchey, however, in the extensive study which is reported in Chapter IV of Part II, demonstrate beyond reasonable doubt not only that children on entering school do know something about number but that they know far more than has been suspected. At the risk of repeating these facts, it may be said here that careful systematic individual testing has shown that more than 90 percent of children on entering school (there were 1290 children in this study) can count to 10, and 60 percent can count to 20; that 50 percent can count by 10's to 40; that over 50 percent can count the number of twenty exposed subjects; that about 50 percent are able to supply any number of objects from 1 to 10 on demand; and that approximately 25 percent to 75 percent are able to combine in some way easy pairs of numbers like $3 + 5$ and $5 + 1$. When all these facts and others like them are assembled, it is clearly absurd to insist longer that children entering the first grade have had no number experiences, that they have no knowledge of number, and that they cannot profit from instruction in number.

The results of the Buckingham and MacLatchey investigation are of great significance for the course of study in primary number. There is now evidence that it is possible for the course of study to err in the direction of requiring too little as well as in the direction of requiring too much. A plan of instruction, for example, which allots an hour or two a week throughout the first year to the task of teaching children to count to 100 is clearly wasting time and, more seriously,

is probably engendering in the children an attitude toward arithmetic that may have harmful and lasting consequences.

A third suggestion is that *the course of study must insure assistance to pupils in bridging the gap between concrete and abstract number.* If a group of nine concrete objects, arranged, for example, in the familiar domino pattern—one group of five and one group of four—is exposed to pupils in the primary grades, appropriate techniques of investigation disclose the fact that children apprehend the total number of objects by means of a variety of methods. The adult and the advanced pupil, it is true, require but a glance to discover that there are nine objects in the exposed pattern, and their processes approximate something like "two fives less one" or "five and four." Among the children, however, there are those who, even after some practice with the domino cards, count the dots or other objects one by one. There are others who grasp the first five objects as a unit of five, and count the other four, thus "five, six, seven, eight, nine." And there are others who apprehend the final four objects by breaking them up into two two's and adding them to the five, thus "five and two and two."

It is apparent that these various methods of apprehension differ materially in their degree of maturity and abstractness. In the case of counting, each object is grasped independently and is identified by a number name and thus marked, as it were, by a tag. The successive number names refer less to aggregates than to individual objects successively apprehended. The final number name, "nine," means that the last object has been told off and a limit has perforce been reached for counting. There is very little of the abstract about such a child's notion of nine. It may be said that he deals with concrete objects by means of a concrete method. In marked contrast to these immature procedures are those of the finished performer who apprehends the number picture by thinking $(2 \times 5) - 1$ or $5 + 4$. To him the individual items of the objective patterns have entirely lost their identity; for them he substitutes symbols, and apprehension consists in a process of manipulating artificial products of his own invention. His procedures are essentially abstract.

Such investigations make it clear that children's number ideas require time for development. There is no magic by which the child who has just learned to count objects one by one suddenly displays proficiency in dealing with groups of objects and with abstract num-

bers. Progress is seen to be, not a matter of jumps, but a matter of orderly advance through a series of successively more complicated processes. This principle, recognized in some measure in the case of advanced types of arithmetic ability, has not yet been properly appreciated in the case of primary number. Too frequently instruction in counting is followed immediately by drill on the number combinations.

The course of study should point out to teachers the nature and the seriousness of childrens' difficulty in passing from concrete to abstract number and should warn teachers not to expect the transition to be made instantaneously. While providing training in counting because of the undoubtedly valuable outcomes of such instruction, the course of study must emphasize the fact that training in counting needs to be supplemented by other types of instruction before children are introduced to the abstract number combinations as such. Counting can give children little more than the positional value of numbers in the notation system; it can give them little notion of the content of the numbers, and it is the latter that makes the combinations meaningful and something other than empty verbal abstractions. Teachers are justified in looking to the course of study for suggestions regarding methods and materials to provide them a basis for the intermediate kinds of instruction which are here stated to be necessary.

A fourth suggestion is that *the course of study should set forth clearly both the uses and the abuses of drill as a method of instruction.* Under conditions of adequately graded steps of instruction and adequately devised teaching procedures children should hardly be aware of the time when they leave concrete numbers entirely behind and pass once for all into the field of abstract numbers. The transition should be so easy and gradual as not to be recognized at all by the children. As a matter of fact, they should have been dealing with the basic facts represented by the addition combinations for some time in the form of concrete numbers; and on this basis they should have developed such abstract processes of dealing with the concrete numbers that the withdrawal of the objective clue to their processes should occasion no difficulty.

'Formal drill,' defined as repetition, now becomes an appropriate method of instruction. Introduced too soon as the chief method of instruction, drill is at the best futile, and at the worst positively harmful. Premature drill is futile because it cannot develop meanings.

A child can no more learn the meaning of '$7 + 2 = 9$' by merely repeating '$7 + 2 = 9$' than he can learn the meaning of 'horse' by repeating 'horse.' And premature drill is liable to be harmful for the reason that it may block progress toward more effective, more mature methods of dealing with numbers. Classroom drill in arithmetic merely constitutes an opportunity for the child who counts to learn to count more rapidly and more accurately; it cannot provide for him better procedures. Furthermore, in case his processes lead him to faulty answers, drill comes to be drill on errors. The function of drill is best conceived as that of fixing something which has previously been comprehended; it cannot take the place of instruction which is responsible for the comprehension.

2. Methods and Content for Different Levels of Ability

a. The Modified Course for Pupils of Superior Ability.—Courses of study in arithmetic for pupils of superior ability present questions still to be answered. Modification of instruction to fit the abilities of pupils below the average in ability has received much emphasis. The public demand that all children go through the schools at the same rate has made special help and differentiation for slow children a necessity. The bright pupils have so far received scant attention. They are able to do the work of each grade without great effort. Their talents may remain little noticed; in fact, they often develop habits of carelessness and make records much lower than their abilities warrant. In recent years an attempt has been made to make some provision for their needs and capacities, but little actual progress has been made. The social pressure for differentiation for bright pupils is not great. The psychology of superior children is not well understood. Little research is available regarding the work and capabilities of superior children.

In a general way the learning characteristics of superior children may be thought of as the converse of those of subnormal children.[98] The bright child will deal with the abstract more quickly than the normal. He requires less drill for fixing habits. The bright child will hurdle many of the steps in learning difficulty that obstruct the progress of the normal child.

Modification of the course of study for superior children should be both quantitative and qualitative. The bright child can cover more

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[98] See *Learning Characteristics of Borderline Children.*

subject matter than the normal. He can also be given more difficult examples and problems to solve.

The type of enrichment for the bright child in a given mechanical process should consist of a great variety of types of examples. He can easily learn to add numbers not placed vertically one over the other, subtract numbers when the subtrahend is not placed under the minuend, and the like. He can give attention to the more involved types of examples in the various processes, such as circulating decimals in the quotient in long division, adding lists of uneven numbers, and dividing when the dividend, divisor, and quotient are not in their usual positions in the example.

The course for the bright child should also be enriched in the application of processes in actual situations. Problem-solving as generally taught is very difficult for the normal pupil but fairly easy for the superior. When a new process is taught, its use may be illustrated at once in problems of two steps or more. Superior children are able to see quantitative relationships in the study of elementary science and social science. They may be placed on their own initiative much more than normals. General and comprehensive questions are meaningful to them. They have the ability to analyze their own work. If they are to be kept to their highest capabilties, the teacher must use a critical attitude in dealing with them and their work. They need to be challenged as regards their work and methods. Enrichment for the bright child means quite a different thing from enrichment for the slow and dull child. Both require enrichment, but of a different character. Teaching material is enriched for the superior by the use of more involved situations and by offering a variety of form and difficulty for practice. For the dull, material is enriched by simplifying processes, by making use of all steps in learning, by a variety of appeal and attack, by furnishing much practice on each process, and by making concrete applications.

Some recent courses of study have presented material and methods intended for normal pupils that have proved to be much more suitable for superior children. The degrees of difficulty set as standards of attainment are too great for normal children. The processes presented and explained in detail are involved. The hypothesis that normal pupils should be able to use a process in problem-solving as soon as learned, especially when presented in two- or three-step problems, has not proved practicable. Too great a variety of applications to adult

situations bewilders the normal pupil. Such courses seem to be well fitted to superior pupils.

A commendable practice in drill material for the superior as well as the normal child is well exemplified in a recent work-book series in arithmetic. This work book presents material for practice under two sections. The material in Section I is intended for normal and less than normal pupils. The examples are simple, varied in form and difficulty and numerous enough to afford sufficient drill. When verbal problems are used to apply a given process, a simple one-step form involving child experiences is generally used. Section II is intended to furnish drill for superior pupils. Examples are longer and more involved. The process being studied is used in verbal problems. Much use is made of problems without numbers where thinking and generalization are necessary for reaching the correct solution. Many applications involving adult uses of processes are given, such as the reading of frequency tables, study of sectioning land and number of acres in each fractional part of a section, estimating volumes, making one's own problems, and the like.

b. The Modified Course for the Lower Levels of Ability.—The need for modification of courses for slow and backward children is widely recognized. In actual practice teachers are making some modification for slow pupils. In several cities courses of study have been worked out for pupils in the so-called 'opportunity classes,' and in a few cases textbooks have been written and placed in the hands of the children in these classes. These courses of study and textbooks are generally based upon the experience and judgment of teachers and supervisors in this field, infrequently on the results of research. It is not uncommon to find a teacher with children of the borderline or feeble-minded type drilling a class on long examples in fractions or involved exercises in long division.

This section gives a brief description of a modified course of study in the Denver Public Schools intended for a certain type of subnormal children. The course was prepared by the Committee on Special Classes in the Denver Elementary Schools, Mrs. Gladys May Maclin, Chairman, but was intended for use in regular classes where children of the borderline type are found.

A tabulation of standards of attainment was first made in nine courses of study for subnormal children. The tabulation showed that these courses of study expect dull and borderline children to meet as

high standards of attainment as are set up for regular classes in our most modern textbooks and courses of study. It was therefore decided not to use the standards of attainment found in these courses, but rather to make studies of what subnormal children in Denver were really able to do and so discover the levels at which the standards should be fixed.

In order to be able to think clearly on the question of modifying the course of study in arithmetic for subnormal children it was considered advisable to limit such modification to a given type of subnormals. The borderline type with intelligence quotients from 70 to 80 on the Stanford Revision of the Binet Test was chosen for this purpose. An assumption was made, which was later confirmed in part by statistical evidence, that pupils of the same mental *age*, whether they were of the normal, dull, or borderline type, would be able to do approximately the same grade of work. The selection of a narrow range in intelligence quotients made it possible to compare pupils of the borderline type of a given chronological age with normal pupils of the same chronological age. Since the difference in ability between the borderline pupil and the moron, or definitely feeble-minded, is as great approximately as the difference between the borderline and the normal type, it seemed unwise to attempt to make one course of study for all slow and backward children. The modification of the original course of study for children of the borderline type was intended to suggest further modification downward for feeble-minded children and upward for dull and backward pupils with intelligence quotients from 80 to 90.

The first evidence collected on the question as to whether borderline children of a given mental age can do the same grade of work in arithmetic as normal children of the same mental age indicated that borderline children do as well as normal children of the same mental ages in the simple mechanical phases of arithmetic, but are more handicapped than the normal children in the complex phases of problem-solving.

Table VII gives a short summary of this study. The median scores of normal pupils in the table are the results of city-wide curriculum tests in arithmetic given to all children in the various grades.

The median mental age of the normal pupils in each grade was then computed. Borderline children (I.Q.'s 70 to 80) in each grade with the mental age for the grade were then selected for comparison.

Thus in Grade 3B the median mental age of normal children (median I.Q. 100) is 8 years and 11 months. Borderline children (I.Q.'s 70 to 80) much older chronologically were selected for study. A comparison was made of children of the same mental ages, although they differed in chronological ages and intelligence quotients. The same tests were given to borderline children of the same mental ages as the normal pupils. The table indicates that the assumption is true that children of the same mental age in the normal and borderline groups do practically the same grade of work until the mental age of eleven is reached. Beyond this mental age there were very few test scores of borderline children available for comparison.

TABLE VII.—MEDIAN SCORES IN FUNDAMENTALS AND INTEGERS AND PROBLEM SOLVING FOR NORMAL AND BORDERLINE CHILDREN OF THE SAME MENTAL AGES

Grade of Test	Mental Age* of Normal and Borderline Children at Time of Test	Number of Normal Pupils	Number of Borderline Pupils	Median Scores in Fundamentals		Median Scores for Problem-Solving	
				Normal 100 I Q	Borderline 70-80 I.Q.	Normal	Borderline
Low 3...	8–11	1935	23	30.39	30.00	6.92	4.00
High 3..	9– 5	1330	24	40.64	29.00	9.62	5.00
Low 4...	9–11	2080	21	24.29†	24.00†	5.35†	5.00†
High 4...	10– 5	1352	23	30.43	30.00	6.57	3.00
Low 5...	10–11	2149	10	31.94	31.50	8.76	3.50

*These mental ages are translated from the actual chronological ages of Denver school children, hence are more accurate for Denver standards of attainment than other age-grade tables would be.

†Different tests were used for the first two and the last three half grades.

Such a table has value, because it indicates that in the mechanics of arithmetic the mental age is a useful index of the ability to do school work whether the intelligence quotients of the pupils concerned are normal or below normal, but it does not indicate what the limits of difficulty are for pupils of various mental ages in the various operations and processes of arithmetic.

In order to secure some evidence on the latter question, a study was made of city-wide test results in arithmetic to discover at what mental ages normal pupils reached 90 percent efficiency on the various types of examples in the tests given in January, 1928. Table VIII should indicate to the teacher the limits of difficulty to be set for the standards of attainment by pupils whose abilities correspond to the mental ages given.

TABLE VIII.—MENTAL AGES OF PUPILS IN THE DENVER PUBLIC SCHOOLS WHO REACHED 90 PERCENT EFFICIENCY IN CERTAIN TYPES OF EXAMPLES

Process: Addition	Grade Reaching 90 Percent Efficiency (or less where indicated)	Corresponding Mental Age at End of Semester*
4 0 6 7 9 3	Low IV	9 years—11 months
6958 6045	Low V	10 years—11 months
65849 70378	Low V	10 years—11 months
82 40 65 74 37	Low V	10 years—11 months
597 853 932 368	High VI (88% Efficiency)	12 years— 6 months
742 968 583 403 339 427 920 897	High VI (75% Efficiency)	12 years— 6 months
23 146 7 8793 18 786 893 4269	High VI (72% Efficiency)	12 years— 6 months

*This is a mental age based on the assumption that the median intelligence quotient of all Denver pupils is 100; in other words, these are actually chronological ages, translated into mental ages, on the basis of the assumption.

TABLE VIII—(Continued)

Process: Subtraction	Grade Reaching 90 Percent Efficiency (or less where indicated)	Corresponding Mental Age at End of Semester*
967 636	Low IV	9 years—11 months
742 417	High IV	10 years— 5 months
734 498	High IV	10 years— 5 months
4735 2817	Low V	10 years—11 months
7424 6539	Low VI	12 years— 0 months
704 496	High VI (86% Efficiency)	12 years— 6 months
786 98	High VI	12 years— 6 months
80008 3799	High VI (86% Efficiency)	12 years— 6 months

Process: Multiplication	Grade Reaching 90 Percent Efficiency (or less where indicated)	Corresponding Mental Age at End of Semester*
822 3	High IV	10 years— 5 months
803 5	Low V	10 years—11 months
34 32	Low V	10 years—11 months
638 6	High V	11 years— 5 months
312 403	High VI (86% Efficiency)	12 years— 6 months

TABLE VIII—(Continued)

Process: Multiplication	Grade Reaching 90 Percent Efficiency (or less where indicated)	Corresponding Mental Age at End of Semester*
7894 × 8	High VI (86% Efficiency)	12 years— 6 months
89 × 76	High VI (78% Efficiency)	12 years— 6 months
493 × 487	High VI (64% Efficiency)	12 years— 6 months

Process: Short Division	Grade Reaching 90 Percent Efficiency (or less where indicated)	Corresponding Mental Age at End of Semester*
6)312	Low VI	12 years— 0 months
6)3168	High VI	12 years— 6 months
9)4775	High VI (82% Efficiency)	12 years— 6 months
8)2432	High VI (88% Efficiency)	12 years— 6 months
30)600	High VI	12 years— 6 months

Process: Long Division	Grade Reaching 90 Percent Efficiency (or less where indicated)	Corresponding Mental Age at End of Semester*
51)306	Low VI	12 years— 0 months
42)882	Low VI	12 years— 0 months
72)1872	High VI (87% Efficiency)	12 years— 6 months
47)4653	High VI (88% Efficiency)	12 years— 6 months
79)3558	High VI (69% Efficiency)	12 years— 6 months
37)21830	High VI (57% Efficiency)	12 years— 6 months
46)52276	High VI (75% Efficiency)	12 years— 6 months
36)15485	High VI (50% Efficiency)	12 years— 6 months

It should be kept in mind in studying Table VIII that pupils of borderline intelligence in the elementary schools rarely reach a higher mental-age level than eleven years. This would indicate that in the Denver Public Schools it is probably uneconomical to spend much time in teaching to borderline children addition in integers more involved than two columns of five numbers each, subtraction in integers more involved than four places and without zero difficulties, multiplication in integers more than three places in the multiplicand and one place in the multiplier. This table would also seem to indicate that it is very difficult to secure high efficiency in any type of division in integers. The study was not continued in fractions and decimals, but such a study would probably indicate that very little may be expected of pupils of the borderline type in anything but the very simplest examples in these processes.

The learning characteristics of borderline children are based on the psychology of subnormal children. The following list has been made up after a study of such psychology. An attempt was made to state these characteristics positively rather than by comparing borderline with normal children. The latter method of statement is negative in form and gives some teachers the impression that there is little that can be done for borderline children. The borderline child is not likely to succeed unless these characteristics are fully appreciated by his teachers.

1. The borderline child's ability to deal with arithmetic will in general approximate that of a normal child of the same mental age.
   (a) In fundamentals, where simple mental functions are involved, the borderline child responds much as the normal child of the same mental age.[99]
   (b) But in problems or in processes where complex functions such as reasoning, judgment, and generalization are involved, he responds more slowly than the normal child of the same mental age.
2. The borderline child understands and learns general processes through situations in which specific habits and automatic responses are formed.

[99] This arises from the fact that borderline children are older than normal children of the same mental ages. The borderline children have, therefore, had much practice beyond the amount given normal children.

3. He generalizes and applies processes only to problems well within his training and experience.
4. Complex functions, such as reasoning and judgment, must be developed step by step through a carefully graded series.
5. Interest and concentration can be secured through using only subject matter well within the child's experience and ability. Problems and processes must have significant value for him in reaching an immediate end.
6. Initiative and resourcefulness in the solution of problems can be fostered through success in solving many simple problems in his daily activities.
7. The borderline child must acquire through direct teaching much arithmetical knowledge that the normal child acquires incidentally. For instance, number concepts depending upon the power of association between abstract symbols and concrete objects must be developed through often repeated experiences with many familiar objects.
8. The borderline child thinks in terms of small numbers. His quantitative point of view can be broadened by arousing his interest in fields where larger amounts are involved.
9. He gains in ability to detect his own errors and correct his own mistakes when he has visible evidence of the benefits of doing so.
10. The borderline child is very susceptible to suggestion. He improves most through constructive criticism offered in a friendly encouraging spirit. He learns best when the application of principles of instruction is in harmony with his nature.

To insure the proper kind and amount of subject matter for borderline children, both qualitative and quantitative differentiation was used. This resulted in the subject matter of the modified course being organized around three main divisions under each operation and process: (1) the common core which all children should acquire if possible, (2) the limits of difficulty within which most borderline children should be taught, and (3) the most frequently used type, which should receive the most emphasis in drill and problem solving. The studies reported in Tables VII and VIII served to influence greatly the limits of difficulty and the most frequently used type.

Since it is characteristic of the borderline child that he learns best by doing, that he learns only subject matter that comes well within

his training and experience, that he understands processes by observing and using them in concrete situations (not by listening to abstract explanations), it naturally follows that an 'activities' type of procedure is the most effective method of making arithmetical concepts and processes function in his actual needs. The following are suggested activities for borderline children in connection with problem-solving and other applications of arithmetic. While such activities are useful also for normal children, they constitute the chief type of approach for borderline children for insuring the transfer of arithmetic skills in actual situations. Many other equally good or better activities might be listed. These are included because they indicate the types of activities that should be used in teaching borderline children.

1. Some Classroom Activities Involving Arithmetic
   Recording and comparing ages and birth dates of pupils
   Recording and comparing size of families
   Learning addresses and telephone numbers
   Learning system of house numbering
   Figuring distance of each child from home to school
   Comparing distances from home to school
   Computing carfare
   Computing cost of school lunches
   Keeping attendance records
   Making and using calendar
   Observing and recording number of clear and cloudy days in a week, a month
   Recording temperature, indoors and out
   Telling time of intermissions, classes
   Computing length of school periods
   Planning weekly order of milk, crackers
   Computing cost of milk, crackers
   Budgeting and arranging supplies
   Passing books, papers
   Buying personal school supplies from store
   Planning window or outdoor gardens
   Ordering seeds or bulbs from catalogues
   Computing postage for letters and packages
   Buying food for school pets
   Making maps of schoolroom, suggesting rearrangements
   Making maps of school grounds
   Making maps of school neighborhood
   Planning size, number, and arrangement of material for bulletin boards
   Measuring materials used in practical arts
   Computing cost of materials used

Using measures in recipes
Scoring arithmetic and spelling papers
Comparing individual and class scores
Making graphs to show progress
Finding pupil and class averages
Selling tickets for school entertainments
Keeping scores for games and contests
School banking

2. Some Home Activities Involving Arithmetic
   (a) Activities arising from provision of food
      Buying and selling of groceries, milk, meat, fruit, and vegetables
      Computing daily, weekly, and monthly grocery bills
      Comparing cost of groceries at different stores
      Planning wholesome, inexpensive meals
      Planning picnic lunches; refreshments for parties
      Comparing cost of meals in cafeteria, restaurant, and at lunch counter
      Substituting inexpensive dishes for more expensive ones
      Computing wages of clerks, cashiers, and deliverymen in grocery stores, and meat markets
      Computing wages of cooks, waiters, waitresses, bus-boys in restaurants
      Computing saving from growing one's own vegetables; from keeping poultry
      Using dry and liquid measures
      Comparing the cost of baking one's own cakes and pies with cost of buying them
   (b) Activities Arising from Provision of Clothing
      Buying and selling of clothing, shoes
      Computing cost of wardrobes for various members of family
      Comparing cost of wardrobes of different seasons
      Buying from advertised sales in newspapers
      Ordering from catalogues
      Writing money order; paying postage
      Comparing cost of making clothing at home with cost of buying them ready-made
      Measuring amount of material needed for various garments
      Selecting least expensive and most serviceable materials
      Making family budget for clothing
      Computing cost of cleaning, pressing, and laundry
      Computing wages of dressmaker; of milliner; of shoemaker; of employees in dry goods store
   (c) Activities Arising from Provision of Shelter
      Computing cost of renting a house
      Computing upkeep of property; such as, taxes, fire insurance, repairs
      Buying furniture for various rooms

Comparing prices at sales
Estimating cost of fuel, gas, light, water
Computing cost of lawn, garden, flowers
Computing wages of carpenters, bricklayers, and other building artisans
Computing cost of apartments of rooms
Computing wages of janitors, gardeners, chambermaids, laundresses, domestic labor

(d) Activities Arising from Care of Health
Keeping height and weight records
Recording weekly or monthly gain or loss in weight
Budgeting time for sleeping, playing, working
Computing cost of illness; doctor, nurse, hospital, loss of time

(e) Activities Arising from Planning Recreation
Computing cost of movies, parties, picnics, concerts
Computing cost of playground equipment
Computing budget for magazines, books, newspapers
Computing cost of music or radio in the home
Buying musical instruments
Computing expense of musical education
Computing cost of celebrating various holidays
Planning Christmas presents within a certain sum
Computing cost of automobile and upkeep
Planning auto trips and sharing of expenses
Computing cost of trips by train
Planning vacations for certain sums
Computing wages of men on train crew
Figuring game scores, baseball percents

(f) Activities Arising from Saving Money
Planning a visit to a bank
Opening a savings account
Buying savings stamps
Using a check book
Budgeting children's allowance so saving is possible

3. Learning Steps in Various Processes in Arithmetic

The course of study in arithmetic should furnish the information needed to enable the teacher to gear together all teaching materials into the plan of instruction laid out by the course. This would include work in textbooks, commercialized drill materials, and tests. It is not always easy to do this. There may be, in both texts and drill books, plenty of certain types of materials but a lack of other important types of materials.

The teacher should not be expected to be able to decide on the adequacy of the textbook and drill materials furnished. The basis for

judging such adequacy should be furnished by the course of study. The course of study should exhibit the types of examples in the various processes that require emphasis because of their significance in the sequence of learning and because of their difficulty. The course of study should set forth the sequence of steps in learning in each process so that the teacher may be able to base instruction in a new process or a new phase of a process upon what has already been learned. A list of the steps is valuable for other reasons. If the teacher is to develop ability to diagnose difficulties by observation, he should be furnished with a book of reference containing the steps in learning. If the teacher is to carry on preventive work, he needs to know the steps in learning arithmetic. Probably one of the most important aids required by the teacher for dealing with pupils of low ability is this familiarity with the steps in learning. For many normal and superior pupils the observation of all steps in learning is not necessary, but for those who have unusual difficulty in arithmetic the need for the teacher to know the steps in learning is manifest if the teacher is to aid the pupil to progress from the known to the related unknown.

The steps in learning have been presented in various ways. The following are some of them:

a. Analysis of Unit Skills.—The analysis of unit skills and abilities is discussed and illustrated in Chapter IV. Such analysis attempts to present all possible arrangements of numbers that present differences in difficulty in given processes. The analysis begins with the simplest forms in the processes and proceeds step by step to the structurally more difficult.

A brief illustration from the "Analysis of Skills" in Part I of Research Monograph Number Two, Arithmetic—Elementary School, Denver Public Schools, follows:[100]

		Skill No.
(*B*) *Adding with Carrying*[101]		
(1) Two two-figure numbers................	48 27	(30)
(2) Two numbers containing three or more figures		
(a) Carrying in units' column...........	329 146	(31)

[100] *Research Monograph Number 11 (2) Arithmetic—Elementary School.* Denver Public Schools, Part I, p. 13.

[101] This is only a portion of the complete analysis of carrying.

(b) Carrying in other than units' column	1923 4716	(32)
(c) Carrying in two consecutive columns	2615 6187	(33)
(d) Carrying in two columns not consecutive	1917 2639	(34)
(e) Carrying in more than two columns..	1647 2598	(35)

It should be noted that the added steps in difficulty are structural in nature; that is, they are due to the form of the examples. The number facts involved are not concerned, except as they are the cause for carrying or not. It may be seen at a glance that this type of analysis has value. It shows definitely the possibilities of difficulties due to the structure of examples. It has at least three limitations:

(1) Structural differences do not always mean new and separate difficulties. When they do, such difficulties may be so slight as to be negligible.

(2) There are difficulties involved in the processes analyzed that are not indicated by the structural forms of examples. For example, the actual difficulties of carrying in addition are not presented, such as: (a) knowledge that the right hand figure of the sum is to be written in the answer as a part of the sum, (b) knowledge that numbers carried may be "1" or more than "1," (c) ability to hold in mind the number to be carried, (d) ability to add it to the top or bottom figure in the next column.[102]

(3) The analysis of skills presents more separate types of examples than it is possible to emphasize by drill. The analysis in addition, subtraction, multiplication, and division in integers, fractions, and decimals in the Denver monograph here quoted included over nine hundred separate skills.

The analysis of skills is extremely useful to the teacher for the preparation of drill material to fill gaps not provided for by printed material. Most school systems have not yet furnished adequate printed drills for the classroom. Until that is done, teachers will have to prepare some material themselves.

The analysis of skills is valuable for analyzing examples in the various processes. There is an extremely large number of possible

[102] Denver Public Schools. *Op. cit.*, Part II, p. 70.

varieties in the structure of examples in a given process, so that in the full sense they cannot be classified by types. Drill material can best be analyzed by checking the examples involved with skill analyses. By such checking teachers become aware of the difficulties that children have and are better able to discover by inspection the hard spots in drill materials.

b. Analysis of Examples in Tests.—An attempt to list examples in the order of learning difficulty has been made by Kallom, Knight, Ruch, Brueckner, Hillegas, investigators in the Denver Public Schools, and others.

The analyses use various types of examples in certain processes as a basis of organization. They begin with the simplest type of example in a process and proceed by short steps to more involved types. Opposite each type is a list of the new steps involved in learning it. When tests containing these types have been given and the percentage of success or error for each type has been tabulated, a useful set of data is available to suggest where drill and emphasis are needed. If a large percentage of error is found, a more detailed analysis of difficulties, like that found in Table X, may be of value.

The following examples in addition for Grade 2A in the Denver Public Schools will illustrate.

TABLE IX.—ANALYSES OF EXAMPLES USED IN TESTS, DENVER PUBLIC SCHOOLS

Type of Example	Percent of Error Grade 2A, Jan., 1929	Knowledge, Abilities, and Mental Processes Demanded
Add: 6 2 —	5	Interpretation of word "add" Knowledge of symbols Ability to write digits Knowledge of first group of addition facts Proper placement of answer
3 + 4	8	Ability to recognize addition process in equation forms
6 * — 7 | 4 + ? = 4	21 34	Ability to supply the missing number when one addend and the sum are given
13 6 —	31	Ability to add by endings up to 100 within the given decade
27 6 —	33	Ability to add by endings up to 100, including the bridging of the tens

TABLE IX—(Continued)

Type of Example	Percent of Error Grade 2A, Jan., 1929	Knowledge, Abilities, and Mental Processes Demanded
3 2 5	8	Ability to add a number to a remembered number through one or more additions with a sum of less than ten
4 9 2	13	Ability to keep place in the column Ability to keep in mind each addition until the next number is added
4 2 0 4 7	27	Ability to neglect zero in the column
53 65	20	Knowledge that in adding two-figure numbers the sums of the respective columns constitute the total sum including cases where the sum of the left hand column is more than 10
70 20 80	27	Knowledge that the sum of any number of zeros is zero.
746 143	28	Knowledge that in adding more than two-figure numbers the sums of the respective columns constitute the total sum
12 3 63	18	Ability to neglect empty spaces in a column
16 78	77*	Ability to carry which includes: (a) Knowledge that the right-hand figure of the sum is to be written as a part of the sum (b) Knowledge that the number carried may be "1" (c) Ability to hold in mind the number to be carried (d) Ability to add it to the top or bottom figure in the next column
4847 2093	86*	Ability to increase attention span to include carrying in two consecutive columns
92 42 86 57 68	92*	Increase in attention span due to longer columns as well as carrying

*Example not taught in Grade 2A.

c. Steps in a New Process.—A third method of listing the steps in learning arithmetic is similar to the analyses of examples. It is much more in detail, however, and each succeeding example differs from the last only in one particular. Thus pupils may learn a new process one step at a time. The examples in Table X are intended to prepare for carrying in addition in integers and illustrate the progressive difficulties involved. An attempt has been made to simplify the learning by using the same or similar number facts.

d. Error Studies.—Another method of arranging the steps in learning in the order of difficulty is by giving examples in a test and noting the percentage of error made for each of them. This method differs from those previously mentioned in that the order of difficulty is based on pupils' actual performance in a test instead of on inspection. It sometimes happens that simpler examples from the standpoint of logical arrangement are actually more difficult to the pupils than others that are logically more involved; for example, long examples in one-column addition may be more difficult than short ones in two-column addition.

Examples of determining steps in learning difficulty by errors made are the studies on "Errors in Percentage" and on "Errors in Fractions" reported in Part II of this *Yearbook.*

Table IX exhibits the percentage of error made by some three thousand Denver children in each example. An error study is most useful when it is accompanied by suggestions of the difficulties causing the errors.

e. Difficulties of the Combinations.—A fifth method of setting forth the steps used in arithmetical learning is to utilize the comparative difficulty of the various combinations. The application of this method may be seen characteristically in the rating of textbooks or drill exercises in order to determine whether the combinations have been presented to the pupil so that opportunity for drill has been made for all of them and opportunity for drill on the more difficult ones has been made more frequently than for drill on the easier ones.

Table XI, derived from Lutes and Samuelson,[103] for instance, shows the number of times that the digits 1, 2, 3, etc., were to be added to 1, 2, 3, etc., in a given exercise in a certain book. Thus, 5 was to be added to 2 seven times, 8 to 2 twice, and so on.

[103] Lutes and Samuelson—*A Method for Rating the Drill Provisions in Arithmetic Textbooks.* State University of Iowa, March, 1926.

TABLE X

I. EXAMPLES CHILDREN HAVE HAD WHICH PREPARE FOR CARRYING IN ADDITION

(1) Number Facts 9 7 —— 16	(2) Adding by Endings 29 7 —— 36
(3) Number Facts for (1) Broken Down (9 into 3 and 6) 3 6 7 —— 16	(4) Two-Place Adding Without Carrying 43 16 —— 69
(5) Three-Place Adding Without Carrying 142 457 —— 599	(6) Use of a Partial Sum over 10 (14) 83 63 —— 146

II. EXAMPLES ILLUSTRATING PROGRESSIVE DIFFICULTIES IN ADDITION

(1) Simplest Form of Carrying (one step harder than I, 2) 29 17 —— 46	(2) Second Step in Carrying 39 27 —— 66
(3) Zero in Units' Column 45 35 —— 80	(4) Zero in Tens' Column 65 48 —— 113
(5) Zero in Answer Caused by Carrying 65 38 —— 103	(6) Zero in Units' Place in Carrying 47 76 —— 123
(7) Adding Three Numbers 26 32 26 —— 84	(8) Zero in Answer with Three Addends 43 26 21 —— 90

TABLE X—(Continued)

(9) Adding by Endings 53 28 64 —— 145	(10) Zero in Answer, Three Numbers 35 34 36 —— 105
(11) Adding Three-Place Numbers Carrying in Units' Place 168 427 —— 595	(12) Carrying in Tens' Place 654 295 —— 949
(13) Carrying in Units' and Tens' Place 392 239 —— 631	(14) Carrying in Harder Addition 683 588 —— 1271

ZERO DIFFICULTIES					EMPTY SPACES	
(15)	(16)	(17)	(18)	(19)	(20)	(21)
238 342 —— 580	488 317 —— 805	565 497 —— 1062	427 804 —— 1231	208 670 382 —— 1260	44 28 7 —— 79	46 38 4 148 539 —— 775

The foregoing methods of listing steps in learning indicate that lifficulties may be of various kinds: (1) The structure of the example nay indicate a type of difficulty. (2) A given example or type of example may contain many difficulties. In order that the teacher may understand them, they should all be listed. (3) When one actually begins with the related known to aid in introducing a related unknown, there are many steps of difficulty involved. (4) Teachers often underrate the difficulty of various types of examples until actual results on tests with these types show the amount of error made. (5) The combinations used in computing examples in the various processes are important causes of difficulty. Well-organized drill material will contain all number facts and give more practice upon the more difficult ones.

Table XI.—Frequency of Use of Various Number Combinations in Addition Practice Unit I, Series A, Book One, Grades III and IV

Numbers Added	Numbers Added to								
	1	2	3	4	5	6	7	8	9
1	1	1	2	3	2	1	1	1	1
2	3	6	5	3	3	3	2	2	
3	4	5	4	5	4	2	1		
4	6	9	5	6	3	1			
5	5	7	8	5	2				
6	4	5	9	3					
7	3	7	3						
8	2	2							
9	1								

The structural analysis can be quickly used with a minimum of computation. It is easy to apply in the analysis of textbooks and drill materials.

The analysis of examples has practical value for the teacher. It is not a lengthy task and calls to the mind of the teacher the most significant difficulties of each type of example in a process, excepting that of combinations. The more detailed analyses of examples for the use of teachers in presenting a new process or phase of a process are very useful for giving teachers a more complete list of steps in learning.

The error studies are enlightening because they give objective evidence of the actual difficulties of types of examples in a process.

The analyses of combination difficulties have value because they make possible drill material scientifically constructed so far as combinations are concerned.

To understand fully the difficulties in arithmetic computation it is necessary for the teacher to be familiar with the research that has been carried on to date. Courses of study should recognize and make reference to the various methods of listing learning difficulties. Such printed material as is available in this field should be placed in the hands of curriculum committees and teachers.

VI. Problems in the Construction of the Course of Study

1. Revision and Construction of Courses Still Go On

There has been marked interest in course-of-study construction and revision in American schools in the past six or eight years. This interest seemed to reach its height some four or five years ago, at which time practically every school system in the United States that purported to be 'progressive' was making some provision for the revision of courses of study. Many of those systems in which committees were appointed to revise courses have to-day changed their plans and are no longer attempting to work out their own courses. They have discovered that teacher committees, working after school hours without the aid of subject-matter specialists, are generally unable to prepare good courses of study. Most school administrators now realize that curriculum revision is a very complicated and involved problem requiring expert service, familiarity with the contributions of sociology and educational philosophy, with the application of scientific method to education, with studies of methods of procedure, and numerous other activities in the fields concerned. School administrators have discovered that where scientific studies have been carried on or scientifically prepared drills are available, it is often more economical to use them than to attempt to prepare something probably inferior in the same field.

The size of the community or administrative school unit does not necessarily determine whether good or poor results will be secured in course-of-study construction. Wherever students of education and of curricular and teaching problems are at work, improved courses of study are likely to be found, whether in a city of one million population or in a county organization of rural schools.

Many superintendents, supervisors, principals, and teachers have developed into experts because of the curriculum movement. This may happen in any school system, but the larger school unit, with a director of curriculum, with a time budget allowing teachers to write courses of study, and with money for bringing in outside help is more likely to function in producing good courses of study.

2. Curriculum Construction Does Not Happen by Magic

The principles governing productive activity in this field are not different from those operating throughout the entire school organization. It is unlikely that much good work will be done unless a

special organization has been evolved for the work, unless superintendents and supervisors are trained and willing to give their time to the work, and unless boards of education are ready and willing to appropriate money for teachers and principals to work full time on the job and to import experts for local conferences.

Every type of public-school activity has its administrative aspect, and curriculum construction is no exception to the rule. It is possible to carry it on with a poorly organized group of workers and with inadequate financial support, but it can best be carried on when an adequate organization has been set up and when sufficient financial aid has been provided.

3. Some Characteristics of the Course of Study in Arithmetic

1. The course of study in arithmetic should lay out the general plan for instruction, including objectives, learning units, grade-placements, correlations and integration with other subjects.

A school system should have plans and policies, not only in administration and extracurricular activities, but also for the classroom instruction. Instruction cannot succeed without purpose and plans any more than any other type of activity. The purposes, aims, and provisions for meeting them should be embodied in the course of study.

2. Materials to be placed in the hands of the teacher and not those of the pupil should be in the course of study.

The old idea that teaching is a giving out process from teacher to pupil is passing. In highly socialized class periods the teacher more and more occupies the background. But this is no reason why the teacher should not still have preparation for the subject superior to the pupils. The teacher should be familiar with the general instructional plan in arithmetic. He should be aware of some of the researches in the field. He should read something of the controversies about arithmetic. He should not cease to study learning and teaching problems in the subject until his information is greater in every way than that needed by his pupils. Only by such superior knowledge, purpose, and ability can he act as a real leader of his group. The course of study in arithmetic is intended to furnish him this superior background of knowledge, purpose, and ability.

3. Materials that require scientific preparation, like drill materials, should not be placed in the course of study.[104]

[104] See the discussion of drill in Chapter IV.

The preparation of drill materials must be done scientifically if done properly. It requires much time and familiarity with certain technical fields. It is wasteful to ask teachers who have neither the time nor knowledge necessary for doing it properly to do this. Only those who have had special training for such work should attempt it. Much drill material has been prepared and is now available; it is better to purchase it. The chief reason why drill materials are prepared locally is that when drill books are purchased they do not precisely fit the local course of study. However, more drill material is available now than formerly, and there is a tendency for the makers of it to agree upon what constitutes good material and methods in the teaching of arithmetic.

4. The course of study should assume the chief responsibility for making local application of the skills acquired.

The problem of applying arithmetical knowledge and skill to school and community life has probably not yet been solved adequately except in a few cases. Local industry, social life, and interests should be taken into consideration in a systematic attempt to make such applications. The course of study is the instrument that is best fitted to assume the responsibility for this. In the actual operation of a program of application of arithmetic abilities, the teacher must play the chief part. Only the teacher is in a position to capitalize events and interests with his class and to develop them for arithmetic applications. But the course of study should treat largely of this problem and point out local possibilities.

5. The course of study should make adjustments in accordance with the emphasis placed on arithmetic and define its position in the school program with respect to other subjects.

Every school system has its individual set-up. This may be due partly to local attitudes toward public education and partly to the personnel of school officials and teachers. These set-ups often affect the emphasis placed on given subjects. The course of study should be flexible enough to provide for these things.

In some states subjects are required by law to be taught which in other states are not taught at all. Local attitudes may affect the number of subjects taught, the emphasis placed on certain subjects, and time-allotments for all subjects. The local course of study should be made with regard to these things.

6. The course of study in arithmetic should provide materials and suggestions for differentiation according to pupil abilities.

The need for differentiating instruction according to pupil abilities is almost universally recognized, and various plans are being tried. Whatever provision is to be made for a school system should be described in the course of study. Differentiation in arithmetic implies knowledge of the steps in learning the subject, additional drill, more provision for integration for slow pupils, and more involved and more advanced material for the superior, as has been suggested earlier in this chapter. The course of study should accept the responsibility for these things.

7. Materials that are to be left to the initiative of teachers, principals, and supervisors should be clearly indicated in the course of study.

Much material has been eliminated from arithmetic teaching in the past ten years. Some believe that there is still more to be eliminated. Whether certain processes or phases of processes are taught is frequently left for local decision. Additional material may be added by local initiative. Such provisions for local initiative should be suggested in the course of study. Teachers, principals, and supervisors should be made aware of the possibilities for elimination and additions, the conditions under which these should be made, and the reasons for such modifications.

4. Teacher Participation in Course-of-Study Construction

The teacher has the chief responsibility for carrying on public education. He works directly with the pupils. Administrative and supervisory provisions are intended for the aid of the teacher. The course of study serves the same purpose. Except for the improvement of instruction, these aids are useless. The best way to improve the teaching is to improve the teacher, and the teacher in turn can best be helped by aiding him to help himself. The course of study is for the use of the teacher. Its statement of aims and objectives, its outlines of materials to be taught, its suggestions of methods, all give the teacher something with which to work. But when provision is made for the teacher to prepare the course of study, a much higher type of teacher improvement is offered. The teacher must then use his own initiative; he must take responsibility for organizing materials for instruction. For this purpose he has not only the regular supervisory

force to aid him, but he should be able to call on assistance from outside the school system as well.

The reasons for teacher participation in course of study construction are so cogent, so striking, that they may seem to settle the problem of curriculum construction. Yet the facts would probably show that the vast majority of teachers who have worked on courses of study produced little or nothing, and only surrounded themselves with a mass of written material and outlines that were entirely impractical and not understandable to themselves.

Teacher participation has achieved highly satisfactory results, but only under favorable conditions. When those conditions are wanting, it is only the teacher who has already become an expert in his field that is likely to succeed in course-of-study building.

To build a course of study requires specialized abilities. An expert is required to build, or to direct the building of, a course. Whether this expert service comes from a college professor, a highly successful teacher, a supervisor, or a group of teachers who include a critically-minded teacher, a good compiler and organizer, and some sensible teachers of arithmetic—the principle is the same; expert service is necessary to build a course in arithmetic. It is possible to take strong, energetic teachers and by expert direction to train them to become good builders of courses of study. The question to be answered is: Is such training worth the effort? Probably the well-organized school systems that have constructed their courses in this way and have made full use of them have proved that it is worth while.

The contributions of research and of the specialist to curriculum-making are as important as those of the teacher. The most successful courses, whether constructed locally by teachers, principals, and supervisors or by specialists from departments of education, are those courses that have incorporated the best contributions in the field to date. Such contributions are made chiefly by research in arithmetic and by specialists who have given much time and effort to the development of the subject. The question as to whether classroom teachers, principals, supervisors, or specialists shall build the course is a question of procedure in course-of-study construction and not to any great extent a question as to what type of contributions to arithmetic shall be incorporated in the course. All good and progressive courses of study must incorporate the best and latest developments in arithmetic, whatever their source.

GUIDE TO CHAPTER IV

CHAPTER IV

SOME CONSIDERATIONS OF METHOD

F. B. Knight

Introduction

In a social institution as complex and intricate as modern education it is at times useful to recognize several divisions of the total enterprise. Thus, one sphere of endeavor in education may be thought of as dealing with aims, goals, and purposes. Another area of the total field may be the province of method, of techniques, of ways of prosecuting aims and attaining goals. The distinction between goals and methods need not be pushed too far, and it is obvious that the same worker may concern himself almost simultaneously with both. The distinction is useful here, however, because this chapter deals predominately with certain aspects of the problems of method. It assumes, for the moment, that one knows what he wants to do in arithmetic, or rather what he wants children to be doing, thinking, and feeling while under his direction. Considerations which have as their center of interest *means to be used* in contrast to *aims to be sought* are methodological in character. This chapter deals with some considerations of the methodology of arithmetic. No work in the methodology of arithmetic could safely be undertaken without continual and assiduous attention to the aims, goals, and purposes involved. Consequently, certain points of view relative to curriculum determination may well serve as an introduction to a consideration of aspects of method as such.

I. Principles of Curriculum Construction

1. A Useful Content

In arithmetic the specifics of the curriculum are selected less on philosophical speculation and the sheer inertia of tradition and more on the principle of utility. A first principle of curriculum determination in arithmetic is: *Teach what is useful to average citizens in the everyday affairs of life.* It must not be thought that this principle is

the sole dominating factor. Tradition and speculation still place many a specific in almost every published curriculum and certainly in every textbook. Neither should it be judged that we are ready in any scientific and dependable way to actualize fully in curriculum construction the principle of social utility. There is considerable uncertainty as to what is useful and what lacks the virtue of the practical. The arithmetic curriculum which children are actually studying is mixed and confused. Justifications for its content range all the way from fine hopes and stout beliefs concerning the cultural improvement of society to particular exercises calculated to enable one to check his grocery bills and to protect young wage earners from losing their jobs because of inability to add and subtract with economical accuracy and promptness. It seems evident that social utility as an idea is gaining strength and may in time dominate the actualities as well as the theories.

The principle of social utility is rendered useful by an impartial and penetrating examination of the arithmetical needs of competent living. Commendable attempts to discover the 'real' uses of arithmetic are already in the literature of education and others are being conducted. A typical example of the better sort of investigation already made is Wilson's *What Arithmetic Shall We Teach?*[1] It is easy to underestimate the difficulty of any final research of this type and to overestimate the value of any particular research.

2. Present Research Tentative

The apparent finality of present research of this type is in possible danger of attracting undue and naïve faith. The arithmetical needs of common life can certainly be hinted at by discovering to what uses adults put arithmetic. It is to be remembered, however, that we will not find in use to-day skills and abilities which adults do not possess, be they ever so useful. One is forced to get along with what abilities he has. One must work within strict limits of his own skills and powers. What uses of arithmetic would to-day be common practice had the adults of to-day been taught arithmetic twice as well as they were taught is an important unknown in the problem of what to teach for preparation of adult living to-morrow. If the curriculum is cast in terms of what adults can do when trained in less effective schools

[1] Wilson, G. M. *What Arithmetic Shall We Teach?* Houghton, Mifflin Company, 1926.

than those of to-day, it is obvious that past failures may impede progress. The probability that life practices to-day are not to be taken as sole guides for curriculum determination should be pointed out. At any time, needs discovered from an inspection of what adults do are a function of what adults can do as well as what it would be useful for them to do if they could. Progress is made in many ways, one of which is to break through present capabilities (practices) to future possibilities. When all adults were reading by kerosene lamps, no investigation of life's activities would have led us to include a knowledge of safe electric wiring in a preparation for competent domestic management. But the first users of electricity broke through a curriculum determined on adult practice at the time and went beyond a repertoire of abilities which represented common practice at the time. It is possible that the curriculum in arithmetic, while much influenced by life's needs as discovered from adult practices, should also be affected by prudent discovery of other uses (needs) not now present in adult life. The absence of such possible uses is presumed to be caused by the incapability of adults due to faulty education. Curricula may well be cast in the main in terms of what adults do, but they should be well seasoned by what it would be useful for adults to do. To-morrow must not be entirely determined by the practices of yesterday and to-day. Even yesterday's child had no need for an ability to *speak* numbers clearly over the telephone or read them accurately from a telephone directory.

A further caution relative to the influence of adult practice upon the curriculum may be suggested. The idea of discovering adult practice is fundamental, but its actual realization is quite another thing. While excellent first studies are available and should influence the curriculum, no one has yet more than scratched the surface of the problem. Studies in the social, economic, and private uses of arithmetic far more penetrating than those yet available must be made before this attack upon curriculum determination has exhausted its gifts to us. And such studies must go far beyond sheer computation to include competent quantitative thinking. What the average citizen *should* know, in at least a semi-scientific way, about the arithmetic of saving, budgeting, investment, insurance, and governmental arithmetic might not be ascertained by averaging what 10,000 average citizens do know about such topics.

3. Willingness to Change

A second principle underlying the construction of curricula pertains to the behavior of those responsible for them. It may be stated: *In addition to intellectual assent, a willingness to practice the new curricula must be widespread.* In other words, a successful waging of the age-old war between status-lovers and improvement-seekers must be vigorously prosecuted. How much schools are held back through the dead weight of custom, tradition, and habit is anyone's guess. But obviously the pressure of less-than-the-best simply because it is also customary is very great. For example, it is doubtful whether there could be found twenty-five readers of this discussion who would not agree that the abandonment of our curious and clumsy system of denominate numbers and the substitution of a universal metric system would be a clear gain. But that such abandonment will be exceedingly slow is almost certain because our present adult generation has gone to the trouble of learning our present system of denominate numbers. It hesitates to suffer the great temporary inconvenience of learning a new system in order to rid future generations of the permanent inconvenience of a less-than-the-best system of denomination. It is quite possible that a complete abdication of common fractions and the use of decimals only would be a real gain. But adults hold the whip hand and, having gone to the trouble of mastering common fractions, would rather have the rising generations learn them also than give up strong but less-than-the-best habits themselves. However, improvement in curricula by the simplification of the demands of adult life is highly desirable, and whatever can be done to push the slow wheels of useful change is a worthy endeavor. It is not becoming humility in our profession, however, to be overcritical of the perpetuation of clumsy ways on the part of society in general or to be overloud in cries for lightening the load of teaching by requiring instruction in as simple and efficient number systems as possible. Until we teachers rid the schools of clumsy and inferior ways of learning what we teach, we should be tolerant of others no more handicapped by tradition than we are. We can ill afford to criticize the indefensible demands of the adult world when we tolerate equally awkward and futile methods of learning for no other reason than habit and ignorance.

4. A Critical Attitude

A third principle of curriculum construction and practice deals with the evaluating of proposed modifications. *With the astounding*

growth in the expression of thought and study in written and oral form, we must have improved methods of distinguishing between the fundamental and the trivial, between the true and that which is merely plausible, and between thorough and superficial research. Educational practitioners must not be too suggestible. Unless there is at least some critical ability on the part of those who translate theory into classroom activity, we may suffer by mistaking enthusiasm for truth and mere garrulousness for words of wisdom. Statistically speaking, there is danger of being partially hypnotized by central tendencies with either unknown or oversized probable errors, and by zero-order correlations which await but a partial or multiple correlation to be reduced to nothingness.

Many school systems have been quick to abandon indefensible units of content and to concentrate on better work on fewer and more important units. However, there may be danger of responding too readily to the slogan 'minimal essentials only.' A better curriculum cannot be built by a process of subtraction only. The determination of valuable content to be added is of comparable importance to the elimination of indefensible material.

The fact that school practice is becoming more and more sensitive to educational investigation and research is a clear, but unstable, asset. This increasing plasticity of educational practice creates a situation demanding wise heads. If practice is following theory with increasing eagerness and decreasing resistance, what are the consequences of vicious and inadequate theory? In the old days it did not much matter what the theorists said because no one paid much attention to them. To-day it does matter because more and more the practitioner is believing what the theorists tell him. It is perhaps needless to point out that with the growth in influence of research and theory two governors are needed. First, simple prudence requires that professional guidance be only of the highest quality. Investigations (now that they really matter) must be exhaustive, impartial, scrupulously honest, and unquestionably competent. Secondly, the practitioner must add to his abilities an increased power of discrimination. When one forms the habit of taking advice, he must in sheer self-protection also exert the power of discrimination between good advice and poor advice. There must be an increased power of critical discrimination on the part of teachers, supervisors, and superintendents to match their own growing inclination to accept advice and

the growing amount of advice available. It is, of course, obvious that the customary is not to be presumed superior to the suggested simply because it is customary. The practitioner should be as critical of his present curriculum and present methods as he is of new methods and other contents suggested to him. A clear recognition of the need of discrimination, and a technique for use in the process of inclusion and exclusion of content and method may be considered a basic principle of curriculum construction.

There is a need for improved criteria for judging the pros and cons of many crucial controversies. Not the least of these is the nature of the curriculum. And the controversy extends to possibilities of informational arithmetic as well as to restrictions of computational arithmetic.

5. The Influence of Method

A fourth principle of curriculum construction deals with one effect which method exerts on the curriculum. In treatises on arithmetic an itemization of what is to be learned may be kept quite separate from descriptions of how children are expected to learn. But no such separation is possible in the classroom. At least the fundamental principles which underlie the method of learning greatly affect the curriculum itself. Shall arithmetic be taught as it would be in a teacher-driven school, or in a child-centered school, or in a school in which many virtues of discipline are mixed with the humanizing, liberalizing, and enlightening influences of dynamic psychology? Shall curricula be constructed in sympathy with the enervating effects of lockstep formalism, or shall they assume the newer methodology based on a psychology which uses all the important aspects of human learning rather than a few? The interdependence of curricula and methods of teaching can be neglected only by the thoughtless.

Perchance the arithmetic curriculum must do more than teach the youth to 'figure' and to solve correctly the little stories which are customarily called verbal problems. It must contribute to self-realization, to the habit of quantitative thinking, to the custom of thinking of one's own performance in the light of objective scores. It must do its share in partnership with other subjects to lead children to powers of self-reliance as well as to effective systems of attacking problems. A curriculum of the slave school may in a sense contain the same topics as does one for use in schools of activity, of expression, and of creative

endeavor. But curricula for these schools of contrasting ideals will be written in quite different ways. A curriculum based on the theory of learning with problem situations as the basic principle of organization will differ in form from one built on the theory of an orderly development of mathematical principles and processes as the main foundation, with problem situations used as enrichments and as applications.

Thus a principle of curriculum construction which will be acceptable to improvement-seekers, if not status-lovers, will be: *Build the curriculum so that it will not only make possible, but will actually facilitate, initiative and self-direction by the pupil.*

6. Respect for Possibilities

Fifth, to build skills, to gain judgments, and to acquire attitudes takes time. Teachers are prone to forget that learning is not instantaneous. It is easy to plan for pupils more than they can accomplish well. If it requires a given number of hours for a nervous system to change itself in certain ways called for in a curriculum, then no more should be called for than can be accomplished. For example, if it takes twelve hours of properly distributed practice for an average child of the proper age to master acceptably the one hundred addition combinations by methods now used, it is folly to construct a curriculum which provides but five hours for the accomplishment of the task. If it can be demonstrated that to handle with insight and a tolerable absence of strain certain types of problems a mental maturity of thirteen years is required, then such problems should not be placed in a curriculum for ten-year-old children. It is evident to even semi-careful thought that a genuinely scientific curriculum simply cannot be written at the present time because the facts of human learning needed for its construction are not now available. Yet within our present knowledge we can accept the principle: *Construct the specifics of a curriculum in strict accord with the known facts of learning and in their absence examine with care the assumptions relative to learning upon which every bit of the curriculum is based.* That this principle of curriculum construction is grossly and flagrantly neglected with great frequency at present is sun-clear to those acquainted with the facts. The sociology of curriculum construction is dependent upon the psychology of curriculum construction.

It will be evident to the reader that the five principles noted in the foregoing pages are but a few of those basic to sound curriculum-building. They serve the purpose here of making quite clear that no method in arithmetic can be worked out except in close relation to the ends in view, and that these ends deal both with subject matter and with the type of educational activity in which the subject matter is to be presented.

7. The Sources of Method

Method in arithmetic is the application of material drawn from those sciences which bear particularly upon the problems involved in its teaching. Arithmetic itself is a body of knowledge within the science of mathematics. From the standpoint of method, materials from the science of psychology dominate the enlightened teaching of most other school subjects as well as of arithmetic. Other sciences also contribute. For example, physiology contributes useful information about the hygiene of the eyes, about fatigue in general, and other factors which affect the bodily condition of pupils and, hence, their learning.

No method can be any better than its own scientific and theoretical foundations. In fact, the actual practice may not be equal to its own theoretic base. As a science grows and matures, methods based upon it are or may be improved. While practice often lags somewhat behind knowledge, there ought to be, in the teaching profession, sufficient energy and courage to permit practice to keep at least within gunshot of expanding scientific knowledge.

It is possible for one to practice—to apply—without having a very clear notion of just what scientific principles he is applying. It is possible to muddle through by using, even unwittingly, enough correct principles. Even when practice is reduced in effectiveness by failing to use all the important principles, or by using some false principles, or by using correct principles incompetently, it is still possible to produce a fair product. That is, it is not a question of using an absolutely correct method or an absolutely futile one. In real life methods of varying worth are used, methods based with varying amount of security upon the scientific considerations underlying them. A clear gain would be made in the methodology of arithmetic by an abandonment of the notion that the truly practical may be in opposition to sound theory. The most practical is always the application of adequate theory.

II. Absence of Scientific Information an Embarrassment

The teaching of arithmetic is handicapped at the present time by ignorance. There are many problems which will not be solved until much more knowledge is available. For example, until recently little attention has been paid to discovering the knowledge children possess on entering school. Consequently, many children are taught what they already know. A terse report of items of information basic to method in arithmetic which ought to be known, but which simply are not known, would require a volume. Examples of unfortunate ignorance will occur to any serious student of the subject. Perhaps the five that follow are fair illustrations.

1. Interference Factors

Very little is known about the rôle of interference factors in learning. In fact, it is not certain that all the most serious interference factors have yet been reported. Suppose a child during informal work in fractions in the lower grades learns that the way to get a fractional part of anything is to divide by the denominator of the fraction. Thus to get $\frac{1}{2}$ of \$20, divide by the whole number 2; to get $\frac{1}{4}$ of 16 sticks of candy, divide by the whole number 4, etc. How much interference is set up by inadvertently trying 'of' to the process of dividing by whole numbers? Later on the child must learn that 'of' means multiplying, as $\frac{3}{4} \times 16$, or $\frac{5}{6}$ of 30, etc. It is true that even in multiplication the division by the denominator is still used. But an interference factor of a serious nature is developed in the case of the young child who learns at one time to solve $\frac{1}{2}$ of 16 by neglecting the fraction idea and translating to the older experiences of dividing by the whole number 2, and who then meets $\frac{3}{4}$ of 16 which becomes $\frac{3}{4} \times 16$.

How much interference is there between the essential idea of subtraction (in contrast to the essential idea of addition) and solving a subtraction problem by the additive method? Even a cursory examination of errors during learning will satisfy one that immature minds do not find subtraction as an idea and subtraction as a novel type of addition entirely compatible. The adult distinguishes between the meanings of the word *subtraction.* He knows that subtraction as an idea means to take away, or deduct from, but that subtraction as a process is used to take away, to find the difference, or to discover how

much more is needed. That such variations of meaning are possessed by the word *subtraction* is not quite clear to many children.

What types of interference factors are involved in the basic ideas of percentage and in their inherent language requirements? For example, consider the likeness and unlikeness of: $\frac{1}{3}$ of N, .33$\frac{1}{3}$ of N, and 33$\frac{1}{3}$% of N.

Transfers of wrong procedures may be considered as interference factors. To illustrate, many children will perform correctly, *before* they have studied multiplication, the example $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$. After they have studied multiplication they will often, if not cautioned, find the answer $\frac{3}{10}$. Here they remember to add, but they now add *both* the numerators and the denominators. A possible explanation is that having in recent work in multiplication treated both numerator and doniminator alike, as it were, they do similarly in subsequent addition. In general, it may be said that what a child does at any time is a result of all of his past experience. Some of his past experience may lead him into error, and many children possess quite limited amounts of ability to analyze from their past experience only the correct and useful things to use in any given situation.

A further example of possible interference factors may be found in the case of denominate numbers. Here the relations between lower and higher orders of denominations are so contradictory, or at least different, that habits learned in one system of numbers may interfere when using another system.

Samples of such possible interference are:

1. (a)	$\begin{array}{r}223\\6\overline{)1338}\end{array}$	$13 \div 6 = 2$ and 1 to carry, and carry it as a *1*.
(b)	$6\overline{)13 \text{ ft. } 2 \text{ in.}}$	$13 \div 6 = 2$ and 1 to carry but carry it as a *12*
2. (a)	$\begin{array}{r}432\\-217\\\hline 215\end{array}$	In changing the form but not value of 432, we borrow one ten and use it as ten ones, or 422. Or, if the Austrian Method is used, we add ten ones to 432 and one ten to 217.
(b)	$\begin{array}{r}4 \text{ gal. } 1 \text{ qt.}\\-2 \text{ gal. } 2 \text{ qt.}\\\hline\end{array}$	Here we borrow *1 gal.* but in the form of *4 qt.*, or we add *4 qt.* to 4 gal. 1 qt. and *1 gal.* to 2 gal. 2 qt.
3. (a)	$\begin{array}{r}21\\34\\26\\\hline\end{array}$	Here the 11 ones are separated into 1 ten and 1 unit, and we write the 1 unit and carry the 1 ten.

(b) 4 bu. 1 pk. Here the 11 pk. are separated into 2 bu. 3 pk. We write the 3 pk. and carry the 2 bu.

3 bu. 4 pk.

2 bu. 6 pk.

How severe the interferences of the decimal system are in whole numbers (where 10 of one unit always makes 1 of the next higher unit) with the varieties of analogous relations in denominate numbers, is unknown. But that they would be a fruitful source of investigation is highly probable.

A final example of a possible interference factor is instructive. To many children the idea of multiplication contains the notion of getting larger—in a wholesale fashion. Thus $48 + 9$ gets larger, but after all 57 is not so very much more than 48. But 48×9 results in 432, and this is getting larger in a wholesale style. Every child has many experiences in multiplication of whole numbers, all of which contribute to a notion that multiplication means getting larger. The following multiplications, however, behave in curious ways. Forty-eight times 1 results in forty-eight—even less than the effect of addition. Forty-eight times 0 results in 0—to many children a perpetually curious affair. Forty-eight times one-fourth results in twelve—a great reduction in magnitude. And after perhaps enough experience with the effect of multiplying by a proper fraction to become accustomed to its contradiction of previous experience, multiplication by mixed numbers in which a fraction is still a potent element is introduced. Here, as in the case of $48 \times 2\frac{1}{4}$, the result begins to get quite a bit larger again in spite of the fact that multiplication by fractions only a few days ago lead to diminishing products. That $\times 9$, $\times .9$, and $\times .09$ lead to progressively smaller products even while the visual picture of these multipliers is getting progressively larger, presents difficulties to the child which are not relieved by a steadfast refusal on the part of the teacher or text to mention them. Interference factors in arithmetic are not the only problems of the growing child which are wrongly handled by an anxious silence.

Reversal of behavior in division seems more vexatious. Any child who has waded through long division will have been impressed by the thought that division means getting smaller. With this mental set he goes on to learn that $9 \div \frac{1}{2}$ results in 18, or that $12 \div \frac{1}{4}$ results in 48. It would be a safe wager that the assertion "$50 \div \frac{1}{5}$ is 250" seems sheer hocus pocus to the average fifth-grade child.

A majority of pupils are at present struggling with many gross and subtle, definite and hazy interference factors, with practically no guidance based on adequate scientific information concerning them.

2. Data on Difficulty

Perhaps of greater damage than our relative ignorance of interference factors is our relative ignorance of the learning difficulty of many of the important items to be learned. Amounts of practice needed to learn most items are yet to be determined or even approximated. Under describable conditions of learning how many practices should be given in translating certain fractions to decimal form, or certain percents to decimal form?

To change 4% to .04 is very much easier for typical pupils than to change 4¼% to .0425. Is this difficulty an inherent one or is it largely due to inequalities of practice provided? Consider the following illustration of the whole problem of relative difficulty of items and distribution of practice on the same items.

In the total computation of percentage, a changing of decimals to percents is not of equal difficulty. A defensible classification of levels of difficulty is given in Table I. It is quite probable, to be sure, that certain items in one class of difficulty are really harder than certain others in another class of greater average difficulty.

TABLE I.—TYPES OF DIFFICULTY OF DECIMALS TO BE CHANGED TO PERCENTS

Type A:	Decimals less than .01, as .005 of N = .5%, or ½% of N.
Type B:	Decimals, even hundredths, as .07 of N = 7% of N. (Includes from .01 to .09)
Type C:	Decimals, with fractions, as .075 of N = 7½% of N. (Includes from .01 to .09)
Type D:	Decimals, even hundredths of two digits, as .24 of N = 24% of N. (Includes from .10 to .99)
Type E:	Decimals, of two digits, with fractions, as .195 of N = 19½% of N. (Includes from .10 to .99)
Type F:	Decimals, of unity or over unity, as 1.25 of N = 125% of N.
Type G:	Decimals, written as even tenths, as .8 of N = 80% of N.

Here is an analysis of types of decimals unsupported by adequate experimentation, but nevertheless quite useful in studying the relation between practice provided on different types of decimals and pupils' performance on them.

In a study of difficulty in percentage by Edwards and Knight, reported in Part II of this Yearbook, the following data on levels of difficulty in changing decimals to percents appear:

Type B, Type D, and Type F decimals are most easily changed to equivalent percents. Less than twenty percent of the pupils used in the study failed to change the decimals .55, .06, .01, 1.24, 1.13, and 1.63 to equivalent percent form correctly.

Types C and E decimals present stubborn difficulty. From 40% to 59% of the pupils failed to change such decimals as .0325, .0475, and .065 to correct percent forms. Of the 200 responses on each of these decimals studied, 38 reported .0325 as 325%, while 46 changed it to 0325%; the decimal .065 was changed to 65% by 52 pupils and to 065% by 41 pupils; the decimal .0475 was changed to 0475% by 41 pupils and to 475% by 43 pupils. The frequency of wrong types of responses in the three decimals used should be noted.

Type G decimals proved the most difficult of all. Of the 200 pupils, 111 changed .4 to 4% and 103 pupils changed .7 to 7%. That definite wrong suggestion is at work here seems clear.

With a clear contrast in mind between the difficulty of changing such decimals as .55, .06, .01, and .24 to percents, and the difficulty of changing such decimals as .0325, .0475, .065, .4, and .7 to percents, consider the amounts of practice on these two contrasting sets of decimals in textbooks.

From Mrs. Rice's study on drill in percentage, as reported elsewhere, the data for Table II are obtained. This table reports

TABLE II.—PRACTICE ON CHANGING DECIMALS TO PERCENTS

School Grade	Types	Average of Eight Texts*
VI	B, D, and F (Easy)	38
	C, E, and G (Hard)	27
VII	B, D, and F (Easy)	44
	C, E, and G (Hard)	25
VIII	B, D, and F (Easy)	11
	C, E, and G (Hard)	13
Total	B, D, and F (Easy)	93
	C, E, and G (Hard)	65

*The texts represented here are all in current use. Their average is held to be a fair expression of the present status of the topic under consideration.

the actual practice a pupil will get in changing decimals of the types noted in Table I to percents, if he does all work in drill and problem work on percentage once, as such, in the texts studied.

From the Edwards-Knight study it can be seen that different types of decimal-to-percent changes vary in difficulty. From the Rice study it is evident that texts vary in amount of practice given to these types. By combining the facts of both of these studies there appears at least some of the explanation of the causes of difficulty. In studying Table II, the reader may well ask himself if relative difficulty and relative amounts of practice are not related. If he suspects so, he can also question the soundness of the method underlying this section of current teaching of percentage.

A consideration of the problem of instruction (or lack of it) and difficulty merits a moment's time. Instruction (or its lack) and difficulty are obviously coördinate with practice (or its lack) and difficulty. From Miss Mahoney's study of instruction in percentage (Section IX of this chapter, especially Table XVII) it is clear that relative emphasis in instruction on various types of decimal-percent changes is not in proportion to the difficulty of these types.

Relative to a very narrow aspect of the general ability, percentage, the following statement is supported by verifiable data (the data for difficulty of types are from the Edwards-Knight study, those for amounts of practices on various types from the Rice study, and those for instruction on the various types from the Mahoney study): *Both instruction and practice on various types of decimal-percent relationships are, on the whole, decidedly out of proportion to the difficulty of these types.*

This statement is not offered as sufficient proof that the present teaching of percentage is lacking in respect paid to the varying difficulty of the items which go to make up that total ability. It is suggestive, however, of the possibility that much of pupil difficulty may be traceable to careless and uninformed construction of drill and instructional material. Space forbids the inclusion of tables showing the difficulty in other sections of percentage and the relative paucity of drill and instruction provided in typical texts. Examination of all the available data supports the point at issue, namely, the present lack of correspondence between difficulty and emphasis in instruction and drill material.

In general it is not known, except in a few instances, and there imperfectly, what amounts of practice are needed for the learning of the specifics of the curriculum. That the difficulty of an item and the amount of practice it should be allotted are related is perhaps axiomatic. But how amounts of practice required vary with other conditions of learning is unknown. And to the real embarrassment due to ignorance of how much practice should be provided, the fact must be added that in many instances practically nothing is known about how much practice actually *is* provided, be that amount right or wrong—a situation not peculiar to arithmetic.[2]

3. Fitting Methods to Abilities

Present ignorance of how best to vary methods in accordance with the ability of learners is also impressive. There is rather good evidence that a slow child needs more practice than a bright child, but what else he may need is a matter of unsupported judgment. Let us say that explanation X is perfectly constructed for a child of learning ability represented by 100. How should this explanation be changed to be a perfect one for a child with a learning ability represented by 75? Again, how should it be changed for a learning ability represented by 130?

4. Control of Motivation

Present knowledge relative to techniques of motivation and effects of varying amounts of interest in learning on the part of the learner leaves much to be desired. Some scientific knowledge bearing on this problem is, however, available. The research reported in Part II, Chapter XI, deals with one important technique of motivation.

5. Precise Histories of Learning

While there is an astounding quantity of data on pupil and class scores on tests, there is still a lack of sufficiently precise information as to the *causes* of such scores. A search of the literature will reveal a surprising lack of detailed studies showing an analysis of the learning

[2] Digests of actual drill provisions presented on pages 239-262 clearly show degrees of disagreement in amount and distribution of practice in representative texts and obvious inadvertencies in many which amply reinforce the contention that precise control in practice in arithmetic is greatly handicapped by sheer ignorance of both how much practice ought to be provided and how much actually is supplied.

which leads up to the making of the scores on tests. Only in rather superficial ways is it determined why children made errors or did work correctly in terms of the learning experiences which preceded their performance on tests. Control over learning and the merit of methodology will be increased, we are confident, when there is available sound interpretation of exact case histories of pupils and classes. With such exact information of what has happened *during* learning at hand, causes of weakness and strength in final performance may be better understood and classroom practice in its various aspects better effected.[3]

Even such excellent diagnostic studies as those of Buswell,[4] yielding as they do more insight than do scores on tests, although they tell what abilities children have or have not, hardly give information about the causes of the arithmetical status so well described—and the causes must be known to control the changes in the status.

To summarize, it has been suggested that much more scientific information is needed. Many fundamental problems of method stand in pressing need of investigation. In this connection a word of caution may be in place. An overestimation of the contributions of sporadic research should be avoided. At present the field of arithmetic is being very actively worked. So many things are being 'discovered' so fast that a sharp distinction between fundamental research and clever experiments which are suggestive but not conclusive should be insisted upon. It is useful to record definitely the belief that method must be based on scientific investigation. It is equally useful to caution the practitioner that pseudo-science may be as misleading as personal opinion. Progress (in contrast to mere change) will be made more surely if high ideals of scientific research and patient practice of scientific method permeate investigations in arithmetic. Anything less will promise more than it can give.

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[3] The literature on this point is not entirely barren, however. Such studies as that of Mr. O. O. Cook, who organized in readable fashion practically everything that five fifth-grade children did in arithmetic during the year, and the study of Miss Rose Carr, who wrote a similar careful history for a fifth-grade class, provide detailed histories of learning of real value. Unfortunately, the mass of material in such studies forbids their publication at present, and summaries of them would be of questionable use. They may be obtained from the Library of the College of Education, The State University of Iowa, Iowa City, Ia.

[4] Buswell, G. T., and John, Lenore, *Diagnostic Studies in Arithmetic*. Chicago: Department of Education, University of Chicago, 1926.

III. Mathematical Knowledge versus Pedagogical Knowledge

Anyone at all acquainted with the personnel of workers in the field knows the distrust that many mathematicians, as mathematicians, feel concerning pedagogues, as pedagogues. It has been asserted in many quarters that only trained mathematicians should be responsible for the view of arithmetic given teachers in their professional training. What can a pedagogue know about arithmetic?

It is just possible that the amount of pure mathematics needed for competent work in arithmetic is far less than that needed in the teaching of advanced algebra, solid geometry, and college mathematics. A somewhat assiduous search of the literature fails to discover just how a mastery of advanced mathematics functions in the teaching of arithmetic. On the other hand, it is interesting to inquire into the possibility of several types of knowledge of arithmetic which may be useful. Let us assume for the moment that there is a useful distinction between a knowledge of arithmetic from the standpoint of mathematics as mathematics and knowledge of arithmetic from the standpoint of its teaching. This latter type of knowledge implies that when arithmetic is studied from the standpoint of children learning it, a somewhat different set of criteria might be used in judging one's mastery of it.

An example of teaching knowledge vs. mathematical knowledge of arithmetic may be found in the case of long division. From the mathematical point of view, the accompanying examples are the same. They are all long division examples, and that is all there is to be said.

(a)	$\begin{array}{r}213\\42\overline{)8946}\end{array}$	(b)	$\begin{array}{r}23\\26\overline{)598}\end{array}$	(c)	$\begin{array}{r}23\\53\overline{)1219}\end{array}$	(d)	$\begin{array}{r}57\\93\overline{)5301}\end{array}$
(e)	$\begin{array}{r}57\tfrac{2}{93}\\93\overline{)5303}\end{array}$	(f)	$\begin{array}{r}57\tfrac{1}{3}\\93\overline{)5332}\end{array}$	(g)	$\begin{array}{r}593\\46\overline{)27738}\end{array}$	(h)	$\begin{array}{r}370\tfrac{5}{74}\\74\overline{)27385}\end{array}$
(i)	$\begin{array}{r}76\\306\overline{)23256}\end{array}$	(j)	$\begin{array}{r}58\tfrac{27}{78}\\78\overline{)4551}\end{array}$	(k)	$\begin{array}{r}615\\29\overline{)17835}\end{array}$	(l)	$\begin{array}{r}42\\17\overline{)714}\end{array}$

From the standpoint of teaching, however, there are important differences between them. Mathematical knowledge is not enough nor is it of the right kind to exhaust the information needed for a good basis for the teaching of this process. Thus in computation:

Ex. (a) requires no carrying in the multiplication and contains no borrowing or carrying difficulty in the subtraction.

Ex. (b) contains a carrying difficulty in multiplication but no difficulty in subtraction.

Ex. (c) contains no carrying difficulty in multiplication but a subtraction difficulty.

Ex. (d) contains both multiplication and subtraction difficulties.

Ex. (e) contains a remainder which, if written in fraction form, cannot be reduced.

Ex. (f) contains a remainder which, if written in fraction form, can be reduced.

Ex. (g) contains a 'middle' zero in the quotient.

Ex. (h) contains a final zero and a remainder in the quotient.

Ex. (i) contains a three-digit divisor.

Ex. (j) contains difficulty in finding the correct quotient figures which does not appear in examples before (j). In estimating the first quotient figure it seems that 6 would be the correct digit since 7 is contained in 45 six times, yet the correct quotient figure is 5.

Ex. (k) contains an aggravated form of quotient difficulty. By estimating, it would seem that 8 would be the correct first quotient figure, since 2 is contained in 17 eight times. However, the correct figure is 6, which is two digits away from the estimated 8.

Ex. (l) contains an even higher level of quotient difficulty. By estimating, the first quotient figure 7 would seem to be correct. The number 4, which is quite a distance away from 7, is the correct figure.

These examples of long-division difficulties are not complete, but sufficient to illustrate a very important point of view. *A mathematical description of an arithmetical process does not yield the kind of information about that process which is an essential basis for its instruction to children. Analysis in terms of learning difficulty is essential.* The literature of arithmetic is somewhat incomplete because in many processes such learning analyses are conspicuously absent. Too often vigorous sermons about things in general are given, when decisions about method need precise and quantitative data. A discussion of the uses of analyses in terms of learning difficulties will be postponed until representative analyses of various types can be presented.

1. An Analysis of Long Division

The following analysis of a section of long division is of no particular importance from the standpoint of mathematics but from the standpoint of teaching it possesses real merit. It shows levels of complexity and difficulty which have bearing upon problems of teaching.

Table III[5] accounts for all long-division examples with double-digit

[5] This table was first printed in the *Fourth Yearbook of the Department of Superintendence* and is the work of H. A. Jeep.

TABLE III.—CLASSIFICATION OF ALL POSSIBLE TWO-DIGIT DIVISOR, SINGLE-DIGIT QUOTIENT LONG-DIVISION EXAMPLES*

Examples Which Involve No Quotient Difficulties			Examples Which Involve Quotient Difficulties—Other Difficulties Neglected		
Number	Percent	Nature of Difficulties†	Number	Percent	Nature of Difficulties
171	0.4	ND-I-NC-NB-NR	2489	6.2	ND- I
4238	10.6	ND-I-NC-NB- R	14,202	35.4	D-NI
4681	11.7	ND-I-NC- B- R	1454	3.6	D- I
186	0.5	ND-I- C-NB-NR			
6391	15.9	ND-I- C-NB- R	18,145	44.6	
6283	15.7	ND-I- C- B- R			
			40,095—Grand Total		
21,950	55.4				

*Examples in which the units' digit of the divisor is zero are not included in this study. See Footnote 6.

†C means there is carrying in the multiplication.
NC means there is no carrying in the multiplication.
B means there is borrowing in the subtraction.
NB means there is no borrowing in the subtraction.
R means there is a remainder.
NR means there is no remainder.
ND means there is no quotient difficulty when the first digit of the divisor is used as the trial divisor.
D means there is a quotient difficulty when the first digit of the divisor is used as the trial divisor.
NI means there is no quotient difficulty when the first digit of the divisor increased by one is used as the trial divisor.
I means there is a quotient difficulty when the first digit of the divisor increased by one is used as the trial divisor.

divisors and single-digit quotients (other than zero) with the exception of those examples[6] with divisors in which the units' digit is zero. There are a total of 40,095 such division examples. It is not contended that this table accounts for all the skills which go to make up the total long-division ability. It does provide, however, for a sufficient number of the total skills to make possible a clear picture of the distribution and classification of many of the skills of long division. The following comments will make this more clear.

The classification in Table III is on the basis of carrying, borrowing,[7] remainder, and quotient skills. Examples in which there are

[6] Double-digit divisor examples in which the units' digit of the divisor is zero are omitted from Table III because of the similarity of such examples, speaking from the psychological point of view, to single-digit divisor examples. Zero as a quotient is omitted since its difficulty is not the type of quotient difficulty here considered.

[7] When the additive method of subtraction is used, this would refer to the bridging in such subtraction.

quotient difficulties are grouped according to the type of quotient difficulty, regardless of the carrying, borrowing, and remainder difficulties they may or may not involve. The following illustrations are given for the purpose of clearing up any questions as to the nature of this classification.

$$\begin{array}{r} 21 \\ 42\overline{)882} \\ \underline{84} \\ 42 \\ \underline{42} \end{array}$$

This example involves no carrying, no borrowing,[8] no remainder, and no quotient difficulty.

$$\begin{array}{r} 7 \\ 85\overline{)595} \\ \underline{595} \end{array}$$

This example involves *carrying,* but no borrowing, no remainder, and no quotient difficulty.

$$\begin{array}{r} 7 \\ 85\overline{)597} \\ \underline{595} \\ 2 \end{array}$$

This example involves *carrying,* a *remainder,* but no borrowing and no quotient difficulty.

$$\begin{array}{r} 7 \\ 85\overline{)603} \\ \underline{595} \\ 8 \end{array}$$

This example involves *carrying, borrowing,* a *remainder,* but no quotient difficulty.

$$\begin{array}{r} 2 \\ 39\overline{)106} \\ \underline{78} \\ 28 \end{array}$$

This example involves a *quotient difficulty.* The correct quotient is not obtained when the first digit of the divisor is used as the trial divisor.

3 is the trial divisor
$10 \div 3 = 3$
3 is not the correct quotient.

$$\begin{array}{r} 6 \\ 59\overline{)359} \\ \underline{354} \\ 5 \end{array}$$

Some authorities advise the use of the increase-by-one rule[9] for such examples; i.e., the using of the first digit of the divisor increased by one as the trial divisor. There is a *quotient difficulty* in this example even though the increase-by-one rule is used.

$5 + 1 = 6 =$ the trial divisor
$35 \div 6 = 5$
But 6, not 5 is the correct quotient.

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[8] See Footnote 7.

[9] There are two rules for estimating the quotient in long division. First, by using the first digit of the divisor as it stands as the trial divisor.

$63\overline{)294}$ $6 =$ trial divisor
$29 \div 6 = 4 =$ true quotient

Second, for those examples in which the second digit of the divisor is large it has

It is apparent that such examples as $24\overline{)48}$, $76\overline{)161}$, and $27\overline{)172}$ are the direct concern of Table III. Each of them is a double-digit divisor, single-digit quotient example, and therefore each is accounted for in Table III on the basis of the carrying, borrowing, remainder, and quotient skills practiced in it. For instance:

$24\overline{)48}$	Practices:	No carrying No borrowing No remainder No quotient difficulty
$76\overline{)161}$	Practices:	No carrying Borrowing Remainder No quotient difficulty
$27\overline{)172}$	Practices:	Carrying Borrowing Remainder A quotient difficulty

For purposes of this study a two or more digit quotient example is considered as a complex of two or more single-digit quotient examples. For instance, such an example as $23\overline{)483}$ is accounted for by the examples $23\overline{)48}$ and $23\overline{)23}$. Likewise, $41\overline{)861}$ is accounted for by the examples $41\overline{)86}$ and $41\overline{)41}$.[10]

It will be seen from a study of Table III that as far as finding quotients is concerned there are 21,950 which cause no especial difficulty where the apparent-quotient or the increase-by-one rule is used. There are 18,145 divisor-dividend combinations in which there is a conflict of interest between the apparent-quotient and the increase-by-one rule. Of these, 2489 cause no difficulty if the apparent-quotient

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been found that many errors in estimating quotients can be avoided by use of the increase-by-one rule. When this rule is used in estimating quotients the first digit of the divisor increased by one is used as the trial divisor.

$28\overline{)193}$ $\quad 2 + 1 = 3 =$ trial divisor
$\quad\quad\quad\quad 19 \div 3 = 6 =$ true quotient

In this study the increase-by-one rule is considered for only those examples in which the second digit of the divisor is 6 or larger.

[10] It is obvious that when a two-digit quotient example is broken down into two single-digit divisor examples, psychologically the sum of the two latter examples is not exactly equal to the one two-digit example. For instance, the situation $23\overline{)23}$ is not exactly the same situation as the second division in the example $23\overline{)483}$.

rule is used, but do cause difficulty if the increase-by-one rule is used. There are 14,202 divisor-dividend combinations which give difficulty with the apparent-quotient rule but not with the increase-by-one rule. But a considerable number of these 14,202 combinations have divisors with unit digits of 2, 3, 4, or 5, so that the increase-by-one rule would not ordinarily be used. There are 1454 divisor-dividend combinations with difficulties with either rule. There is disagreement on the economy of teaching the increase-by-one rule. There is by no means a clear case for it. For a further discussion of the pros and cons of the increase-by-one rule, see pages 41-48 of the *Second Yearbook of the National Council of Teachers of Mathematics.*

The question can justly be raised as to the possible use of an analysis of the divisor-dividend combinations as reported in Table III. An analysis seems warranted by such considerations as these:

It is always safer to know the facts than to be ignorant of them. A teacher or textbook writer who is not impressed with the vastness of the number of divisor-dividend combinations may very well underestimate the sheer extent of the field concerned. Such underestimation leads to compression and terseness in initial instruction. Furthermore, a clear picture of the possible combinations, ordered by presence or absence of varying combinations of computational difficulties, increases the chances of giving a better sampling of difficulties in drill work. It is easy to construct drill work in long division which consciously or unconsciously avoids certain combinations of computational difficulties common in long-division examples. If, however, examples are to be selected in accordance with the relative frequency of combinations of difficulties contained in the total field, analysis alone will give us the needed facts. Obviously, if division, rather than only the easy section of the total possibilities, is to be taught, a table such as Table III is almost necessary—first to show what relative emphasis should be given, and then to check drill work to ascertain whether it is provided. In addition, until some analysis lays out the total function in an orderly fashion, prudent sequence of instruction as well as wise sequence in drill is rendered haphazard and uncertain. That such an analysis is useful will be apparent to one who compares the drill and instructional material printed before this analysis was first published (or before similar unpublished analyses were made) with instructional and drill material appearing subsequent to 1924. It is, of course, obvious that a listing of the 40,095 combinations, indexed

for types of difficulties present, is the true source of material for drill material made to specifications. Table III is but a statistical summary of such a list. The important thing to hold in mind at the moment, however, is that such an analysis is not mathematical but pedagogical in nature. It is offered here to support the point of view that mathematics, as such, is not a sufficient basis for the teaching of arithmetic.

2. Characteristic Inadequacies Arising from Stressing the Mathematical versus the Pedagogical Point of View

Addition, subtraction, multiplication, and division which involve the use of whole numbers only are relatively simple affairs from the standpoint of mathematics. When their teaching is dominated by the mathematical point of view in contrast to the learning point of view, the instruction given them is built on a general assumption that the rapid presentation of the procedures as a whole, with a few cautions here and there, is sufficient. Such presentations are models of preciseness in definition and dignity in composition. Instruction so dominated seems to treat children as if they were miniature adults. Printed instructional material is presented in great compactness. It is highly economical of space, and an adult who is already familiar with the topic is prone to judge it to be economical of the pupil's time. An instructor dominated solely by the mathematical point of view often gives the impression of being more familiar with the subject than with the pupil. Unfairness to teaching procedures dominated by the mathematical point of view must be avoided. They have been somewhat influenced by notions that children were not instantaneous and perfect learners, yet the mathematical view has yet to deal seriously and persistently with learning difficulties. This failure is evidenced in several ways. As just mentioned, the criteria of good instructional material have been compactness, a dictionary type of terseness, and an ungrounded faith in the efficacy of rules and definitions. Such instructional material is very meaningful to the adult—who brings to its reading a mastery of the process. But what it looks like to the child, who with an immature mind is learning something he does not yet know, is quite another question.

Instruction dominated by the mathematical point of view almost always has assumed large amounts of transfer between processes and parts of them which are closely allied in the mind of the adult. Such instructional units have been summaries of procedures rather than

learning explanations, and they have given little attention to the exact nature of the difficulties which children actually do encounter in their learning.

Some examples of unfortunate assumptions relative to the learning difficulties inherent in the four processes using whole numbers are presented in the next few pages. These are probably the result of slighting the learning point of view in favor of the mathematical point of view.

a. Omissions of Fact.—Instruction and drill which are faulty because of omissions of facts useful in learning fairly permeate present teaching. These faults in method seem traceable to erroneous theory which assumes that the learning of certain facts will cause the mastery of others—extra booty, as it were—by way of transfer. This assumption presupposes amounts of inferential thinking which typical pupils may not possess. It is obvious that the products of inferential thinking should not be assumed until such thinking is known to exist.

(1) An example of Fact Omissions is the assumption that a child who knows that $4 + 6 = 10$ will thereby know that $6 + 4 = 10$. Specific instruction on $7 + 9$ is assumed to carry over to $9 + 7$. If we speak of $4 + 5$ as an addition fact and $5 + 4$ as its reverse, there are forty-five primary addition facts and forty-five reverses. Ten combinations have no reverses. If we neglect the idea of transfer, there are one hundred primary addition combinations.

Until quite recently, practically all work in beginning addition has been based on teaching the forty-five primary addition facts and rather trusting to luck or transfer for the mastery of their reverses. Now, logically and mathematically, a real identity may exist between $4 + 5$ and $5 + 4$, and similarly between the other forty-four facts and their reverses. However, that these facts and their reverses possess psychological and learning equivalence is a gratuitous assumption which is flatly contradicted by substantial evidence. To-day we teach the one hundred combinations. Satisfactory transfer from an addition fact to its reverse is not assumed. Of course, teachers for many years have taught all the one hundred combinations, but in doing so they exceeded the provisions in method books and in printed material.

(2) A careful examination of theoretical discussions of method will show that instruction in addition has frequently been based on the assumption that if a child knows that $4 + 5 = 9$, he will also know that $14 + 5 = 19$, $24 + 5 = 29$, $34 + 5 = 39$, and so on

(presumably through inferential thinking). That the possibilities of such inferential thinking on the part of children can be easily overestimated need not be argued.

(3) Zero in mathematics may not trouble the adult, but to the learner it is a veritable demon. Even to-day one is rather 'modern' if he teaches with care the combinations involving zero. An example of rather widespread neglect of zero in instruction is the case of a well-known text which tells the child, in its reference to a block of drill which deals with the primary subtraction facts, that the material in question contains all the subtraction facts that the learner needs to know. Still the following combinations do not appear in the material: 0-0, 9-0, 8-0, 7-0, 6-0, 5-0, 4-0, 3-0, 2-0, 1-0. Probably from the standpoint of mathematics, zero in subtraction can be omitted, but in second- and third-grade classrooms keen attention to such combinations is not omitted except by quite inexcusable oversight.

Assuming that reverses of combinations in addition, higher decade addition combinations, and combinations involving zero can be omitted as unnecessary elaboration of the obvious, are but chance illustrations of many instances of fact omissions.

b. Inadequacies of Procedure.—Thinking of procedures as ways of doing things in arithmetic, it is evident that much instruction does not deal with arithmetical combinations, but rather shows pupils how to use these facts. For example, carrying in addition uses the addition combinations, but is a procedure because it shows the child what to do with the results of certain combinations. Here the instructor often explains and illustrates procedure with a restricted set of data, and then expects the child to apply the procedure, unaffected by the data with which he learned it, to all proper situations. It is easy to overestimate the ability of children to indulge in such relatively high types of generalizations, especially when for lack of space or time the explanation of the procedure is terse. Assuming generalization from a too restricted instructional base is a harmful procedure inadequacy.

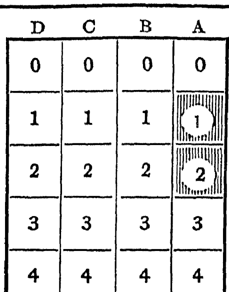
Illustrations of this type of inadequacy will be found in much of the basic instruction in carrying in addition. Altogether too often the explanation of carrying and the teaching illustrations of it will be all couched, or predominantly couched, in terms of carrying 1, as in the examples shown on the next page.

It is proper to introduce carrying with the case of carrying 1 because relatively small numbers will be involved, the computation will

127	33	623
312	50	47
424	2	405
115	48	212

be easy, and the child can put his whole mind on the new procedure. However, if the teaching of carrying uses only the carrying of 1 as illustrative material, and follows with practice which favors the carrying of 1, what is really being done is teaching how to carry *1*, rather than how to carry. Here essential generalization is left to the inferential powers of the pupil, handicapped by overemphasis on carrying 1. If the first teaching and practice on the procedure concentrate on 1, teachers must not be amazed at the stupidity of the children if they carry 1 when they ought to carry 2, 3, or nothing at all. If a child gets the right answer the first time he carries by carrying 1, he is hardly to be blamed if he builds up considerable confidence in carrying 1. Teaching a procedure in a very restricted setting, and then assuming that it will free itself from that setting and be of general use is an unfortunate inadequacy.

Figure I presents in schematic form the possibilities of carrying when the difficulty of the work is reasonable. The shaded portion shows where much of the instruction and first practice has been put. The unshaded areas are mathematically identical with the shaded, but psychologically are mastered either by specific instruction or by inferential thinking.



Carrying 1 from the units' column = A1; carrying 1 from the units' column and 2 from the tens' column = A1B2; or carrying zero from units' column, 1 from tens' column, and 3 from hundreds' column = A0B1C3, etc. We concentrate our teaching on A1 or A2—ruled area—and then expect a perfect transfer to such varieties of carrying as: A3, A0B1, A1B0C1, A0B2C0D3, etc.

FIG. I.—SCHEMATIC ILLUSTRATION OF REASONABLE CARRYING POSSIBILITIES*

* The reason for including zero in this drawing is that after the child has overlearned to carry 1 or 2, the carrying of zero, or no carrying, takes on real meaning and often becomes a real difficulty. Error studies show that children often carry 1 or 2 when there is nothing to carry. It is obvious, however, that the carrying of zero cannot become a difficulty until the child has learned to carry 1 or more.

Assuming, then, that it is reasonable to expect that a child who has received satisfactory instruction in carrying in addition should be able to carry from 0 to 5 in any one or all of the units, tens, hundreds, and thousands columns, it is evident that such a child must have mastered a total of 625 possible and reasonable carrying combinations.

It is not for a moment contended that all carrying types need be specifically taught or perhaps even practiced. Luckily the child can generalize, but to generalize from a concentrated experience on carrying 1 from one's column to the ten's column is to attempt to generalize from an inadequate base. Other samples of technical inadequacies of the procedure type can easily be found. While they are all too present in the curriculum of the third and fourth grades, they fairly infest the teaching of fractions and percentage.

The harmful practice of presenting instruction on too narrow a base of illustration is seemingly built on the assumption that the more compact an explanation is, the less time it will take a child to learn it, and further that explanations sound or look to the child first approaching a new topic as they sound or look to an adult long familiar with that topic. Terse explanations may save space or breath, but the saving is always at a heavy cost to the first learner in both time and understanding. He must learn somehow the many items not present in a compact explanation; he must generalize procedures by some hook or crook. An expansive treatment of a new topic, be it spoken by the teacher or written in a text, makes it possible to avoid many difficulties inherent in compressed explanations. The advantage of the so-called 'expansive treatment' is double in nature. It not only makes possible the identification of many inherent difficulties, but it also avoids the necessity of *unlearning* many wrong things which a child often infers from compressed explanations. Thus, a compressed treatment of carrying, stressing the carrying of 1, not only leaves the child to struggle with items of difficulty, but it likewise aids and abets an over-confidence in 1 as the digit to carry. This the child must unlearn. An advance in teaching is made when the ideal of explanation is not how short can the explanation be, but how helpful. Questions such as the following lead one away from procedure inadequacies which are at present quite unnecessary burdens upon arithmetic: What difficulties does the child meet when first learning a given topic? Which of these can be *safely* left to a child's

inferential powers? What types of illustration will help the child to possess genuine insight into the process and real power over it?

c. Sensori-Motor Inadequacies.—A third type of inadequacy in teaching may also come from basing arithmetic solely on mathematics instead of on mathematics and a psychology of learning applied to mathematics. This third type may be called sensori-motor inadequacies. Here the assumption is that if a child can do a given thing when arranged in one order, a change of order is inconsequential and need not disturb instruction. For example, various sequences of eye-movements need not be thought about if only the mathematics is the same.

A fruitful illustration of this type will be found in the case of multiplication used in division.

Before long division or even short division is taught, considerable skill in multiplication is first built. That this sequence is defensible seems apparent from the fact that while multiplication must be used in division, division is not needed in multiplicaton. Hence, before a child studies division he is able to perform such multiplications as the following:

Ex. A	Ex. B	Ex. C
46	46	46
8	2	9

The pupil is instructed in division by having identified the procedures and their proper sequences. Then he may very well practice upon examples like D.

Ex. D

$$\begin{array}{r}
829 \\
46\overline{)38134} \\
\underline{368} \\
133 \\
\underline{92} \\
414 \\
\underline{414}
\end{array}$$

Note that in this division example the pupil uses the same multiplications as appear in Examples A, B, and C above. Our teaching has assumed that his ability to multiply in Examples A, B, and C carries over completely to the uses of multiplication in Example D. Mathematically, the multiplications in Examples A, B, and C are equivalent to those in Example D. However, from the learning standpoint this

equivalence is not so evident. In Examples A, B, and C the multiplier is under the multiplicand in all instances. In division the multiplier is above the multiplicand (the divisor). The position of the products in multiplication is quite different from that of the partial products in division. In multiplication the drift of the numbers the child writes is to the left; in division it is to the right, as shown in Examples E and F.

Ex. E

$$\begin{array}{r} 829 \\ \underline{46} \\ 4974 \\ \underline{3316} \\ 38134 \end{array}$$

Ex. F

$$\begin{array}{r} 829 \\ 46\overline{)38134} \\ \underline{368} \\ 133 \\ \underline{92} \\ 414 \\ \underline{414} \end{array}$$

In multiplication, the multiplying is an uninterrupted process; in division, multiplication is sandwiched in between quotient estimation, subtraction, and bringing down of figures from the dividend. The multiplier in multiplication is present from the beginning; in division the pupil writes the digits of his multiplier as he goes along. The habits built up in multiplication, as multiplication, are enough to carry multiplication in division, only when considerable adaptability can be assumed in the learner. *Whether such adaptability can be assumed is a problem to which the mathematical point of view is not sensitive. Division as a process is mathematics. Learning division is a problem of educational method as well as a problem of mathematics.*

d. Other Inadequacies.—The careful reader will at once see that the three types of technical inadequacies underlying instruction which have just been cited are but representatives of a much longer list. In a perfect method such factors as the following deserve consideration: the reading difficulty of instructional material (spoken[11] and written); the use and abuse of illustrations in learning; the rôle of various motivation devices in the learning process (many of the present devices are obviously absurd); the questions of how much of a process to teach at any one time and how much of a process to teach at all;

[11] Very little is known about the comprehension difficulty of oral explanations. They doubtless vary in an extraordinary fashion. When investigations of the comprehension factors in oral material equal even present investigations in the field of comprehension difficulty of printed material, the professional training of teachers will perhaps include far more specific training in the technique of oral explanation than is now the case.

the problem of the effective order of presenting elements in a total process; which procedure to use when several are mathematically correct; the advantage of active attitudes of the learner in contrast to passive attitudes, and so on.

The purpose of the foregoing paragraphs is to point out the possible danger in the theory that he who knows a subject from the standpoint of subject matter is adequately prepared to teach that subject. It is almost certain that the interests of pupil learning require something more of the teacher than skill in the subject. Something in addition to mathematical insight of subject matter may prove useful.

IV. A Process Is an Organization

A broad view of arithmetic might distinguish two main divisions. In question form these could be: (1) What are children to learn? (2) In what ways are they to learn? The first is the problem of content; the second is the problem of method. These two problems can be divorced only in adult discussions of them. In actual learning they are thoroughly interwoven. Further, content as such cannot be studied from a single point of view. It is never enough to think of content as subject matter dealing with a variety of topics. Content must also be viewed from the standpoint of its psychological constitution. From the standpoint of learning, no skill is simply a collection of steps, no content only a list of subtopics. The next pages will be concerned with a discussion of the nature of arithmetical content from the standpoint of children's learning. And in this connection it will be helpful to describe a process to be taught, not as a single unitary affair, but as a complex or an organization of many units of skill all working together in proper order and in proper interrelationships. That is, a process such as addition or subtraction, when viewed as behavior of children actually at work, is the correct functioning in proper order of many units of skill dealing with or using a rather broad body of arithmetical facts. There are interrelationships between these units of skills. Some of them may be grouped together as variants or specifics of a given principle, procedure, or ability, others have relatively few mutual interrelations. The facts which a pupil manipulates as he uses skills of various types are not all of one kind and are not on a dead level of difficulty.[12]

[12] Consider in this connection the partial analysis of long division and the suggestion of levels in difficulty in decimal-percent relations presented earlier in this chapter.

Thus, a mastery of a process may be described as an organic whole composed of lesser abilities, and each lesser ability composed of units of skill. The material upon which a child operates his skills is the body of arithmetical fact, or those parts of it pertinent to the problem at hand. A child in mastering addition, for example, must learn many facts and must master many skills needed in the proper manipulation of these facts.

Many children can find the correct sums of the following addition examples:

Ex. A.	Ex. B	Ex. C.
324	670	921
57	325	6
308	47	473
531	852	980

In finding the sum for Example A, the child will use certain facts. Thus, if he adds up the columns, he will need to use these facts:

$1 + 8 = 9$	$2 + 3 = 5$	$1 + 5 = 6$
$9 + 7 = 16$	$5 + 5 = 10$	$6 + 3 = 9$
$16 + 4 = 20$	$10 + 2 = 12$	$9 + 3 = 12$

In actual work, many compressions of these facts would serve the worker satisfactorily.

But his successful finding of the sum is not the result of knowing in isolated bits the skills which compose the procedures, which are quite distinguishable from mere addition facts. Correct work involves the proper integration of the skills. Samples of such skills which he uses are: keeping unseen numbers in mind as he adds them to the next number seen in the column, and then promptly forgetting them; splitting a number such as 20 (the sum of the one's column), writing part of it in a given place and carrying the rest of it into the next column; not splitting the sum of the hundreds column (12), but writing all of it, and so on. Schematically a total ability to add or subtract or whatnot is like Figure II.

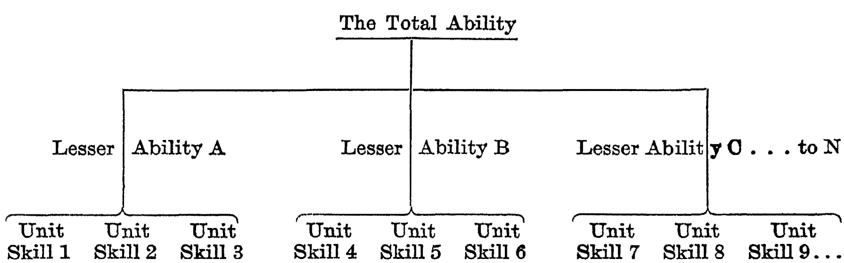


FIG. II.—A SCHEMA OF A TOTAL ABILITY

The point to be made is that *total skill demands the integration of facts and procedures*. A pupil never adds or subtracts in general, but always uses certain specific combinations at any given moment. A list of facts as well as an organization of skills or abilities must, then, be present in a satisfactory knowledge of any process. Let us consider for the moment any process as a hierarchy or organization of abilities. In any particular example some abilities may be used, but no others; a given unit of skill may precede or follow another, and certain facts but not others may be used. The abilities or units of skills or elements of the total solution may be analyzed and recorded in various ways, and of several such analyses one may be better than the others or all may be good enough. An example may contain some, but not all, of the total array of facts inherent in any total process.

The preceding paragraphs attempt to make clear an important view of content. In addition to its social and mathematical natures it possesses, for learning, a psychological nature. A process in arithmetic, such as addition of whole numbers, the subtraction of fractions, the multiplication of decimals, or the division of compound denominate numbers, is not a unitary affair on the one hand, or a simple collection of skills on the other. For purposes of controlling learning, it is not unitary since it must of necessity be learned little by little and a child may very well know part of the total ability but not all of it. It is not a simple collection because in actual work a child must constantly select some and reject other abilities at the moment. Addition is more than a collection of skills just as an engine in working order is more than just its parts. Addition in its constituent elements is so interrelated that the process works. A process is, then, a hierarchy or organization of abilities. Many of these abilities are in turn groupings or complexes of units of skill. When a child has so mastered the constituent elements of the total ability that they are well integrated, that some (but not others) may be properly selected for use at the moment, and when the operation of several skills proceeds in the correct order, it may be said that the child has learned the ability or process if at the same time, of course, he has satisfactory knowledge of the number facts to be used.

It is important to know what a process really is in terms of its demands upon learning, and such knowledge will be facilitated by attempts at analysis.

It may be of interest to consider next some sample analyses of the important processes—analyses which have as their aim knowledge of such a nature and so organized that it may form the basis for teaching and learning.

V. Addition of Whole Numbers

The following analysis is based on an investigation by Dr. F. L. Wells, Department of Education, Indiana State Teachers College, Terre Haute, Indiana. It views the general dexterity in addition as the result of the proper functioning of abilities (which are composed of units of skill in the manipulation of addition combinations). It is useful to base the methodology of instruction in addition upon some analysis like the one that follows. Improvement in this analysis is surely possible and highly desirable. The justification of any learning analysis is, of course, the desirability of identifying the learning units which compose a total process.

The analysis of addition of whole numbers displayed in Table IV conceives of the ability to add as a hierarchy of skills using the addition facts (following the general schema shown in Figure II). Addition of whole numbers is regarded as a total ability formed by a func-

Table IV.—Essential Abilities, Judgments, and Procedures Involved in Addition of Whole Numbers

Abilities	Units of Skill	Number
a. Ability to recognize the addition operation	Numbers expressed as in 1, 2, or 3	
	Pictures or objects	1
	Words	2
	Figures	3
	Form of statement as in 4, 5, or 6	
	With + or with + and = signs	4
	With words as 'add,' 'find the sum,' etc.	5
	In problem setting	6
b. Ability to write the addition operation in correct form	With + and = signs	7
	In column form	
	Addends in example	
	Two addends	8
	Three addends or more	9
	Addends in relation to each other	
	Addend with same number of digits as preceding addend	10
	Addend with number of digits greater than preceding addend	11
	Addend with number of digits less than preceding addend	12

TABLE IV.—Continued

Abilities	Units of Skill	Number
c. Ability to copy correctly numbers to be added	Units of Skill Numbers 7, 8, and 9	
d. Ability to write numbers in addition form with neatness sufficient for reading purposes	Units of Skill Numbers 8, 9, 10, 11, and 12	
e. Ability to identify numbers to be added as addends	Units of Skill Numbers 7 and 8 or 9	
f. Ability to form basic addition combinations	Basic addition facts without bridging, as 2 + 3	13
	Basic addition facts with bridging, as 7 + 6	14
	Basic zero facts, as 4 + 0	15
	Higher decade addition facts without bridging, as 24 + 1	16
	Higher decade addition facts with bridging, as 24 + 8	17
g. Ability to add single column examples (keeping one's place in the column)	Two digits	18
	Three or more digits	19
	Sums less than ten	20
	Sums ten or greater	21
h. Ability to add without carrying	In example	
	Two columns	22
	Three columns or more	23
	In column	
	Two digits	18
	Three or more digits	19
	Sums less than ten	20
	Sums ten or greater	21
i. Ability to add with carrying	In example	
	From ones' column only	24
	From columns other than ones'	25
	Double carrying	26
	More than two carryings in example	27
	In column	
	Carrying one	28
	Carrying two	29
	Carrying three or more	30
j. Ability to neglect zeros and empty spaces	Zero or zeros in column	31
	Column all zeros	32
	Empty space or spaces in column	33
	Empty spaces and zeros in column	34
k. Ability to identify columns	Single columns	35
	Units of Skill Numbers 22 and 23	

Table IV.—Continued

Abilities	Units of Skill	Number
l. Ability to write, in correct position, that part of sum which should be added	Position of sum	
	With + and = signs	36
	In column form	37
	Nature of sum	
	In example	
	Sum single column	38
	Sum two columns or more	39
	In column	
	Sum less than 10	40
	Sum greater than ten and written	41
	Sum zero	42
	Sum greater than ten, ones' digit only written	43
	Sum multiple of ten, zero digit only written	44
m. Ability to identify results of numbers added as sum	Units of Skill Numbers 36 and 37	
n. Ability to understand sum as answer or result sought	Units of Skill Numbers 4, 5, and 6	
o. Ability to add in reverse order for checking	Units of Skill Numbers 36, 38, and 39	
p. Ability to understand that answer is probably correct if answer and check agree	Units of Skill Numbers 36 and 37	
q. Ability to verify addition operation through counting	Units of Skill Numbers 4, 5, and 6	

tional organization of seventeen lesser abilities (Abilities *a* to *q*) and these lesser abilities in turn are composed of forty-four units of skill (Numbers 1 to 44).

In actual work some abilities will be used but not others; certain units of skill compose on occasion elements of several lesser abilities. Of course, an analysis committed to type becomes a stationary affair which actual adding never is. When a child is adding, his behavior contains movement, constant selection, and evidences of synthesizing various combinations of abilities and whatever units of skill composing these abilities that are useful at the moment. No analysis can catch the actual mental 'movement' of children as they add. The realities which are paramount in teaching the actual habits and knowledges of the children, and these exist in the neural and muscular systems of the child, not on a printed page. All that verbal analyses can do is to report in an artificial way a sort of shadow of the real thing. This

inability of analysis in arithmetic to do more than report a shadow of the reality is but an example of a common inability of all verbal reports to get to actual grips with the process described. A moving picture of a boxing match is much nearer to the real thing than a verbal description of the contest. Limited as analyses are in their ability to report the actualities, they still possess much use. At least they can guide us in creating situations calculated to facilitate attempts by children to build up in themselves the proper habits and knowledges. Without such analyses our teaching may be partial and fragmentary, leaving the child to learn as best he can by rough trial and error.

The analysis in Table IV does not report the learning elements in their order of difficulty nor in the order in which they are presented in learning situations. Considerable disagreement no doubt exists as to just what the best learning order is, and there is little scientific evidence on this aspect of the problem which goes beyond common sense and good judgment.

To make clear the application of the analyses shown in Table IV there follow two illustrations from simple drill examples.

Example A

$$2 + 4 = 6$$

Example A uses Skill 3 rather than Skill 1 or 2, since figures rather than words are used. Skill 4 is used, rather than 5 or 6, since the signs, not words, indicate the process and there is no problem setting. Unit Skill 13, and not Skills 14, 15, 16, or 17, is used since only a basic combination without bridging is involved.

Example B

Copy the following numbers in column form and find their sum: 214, 36, 400, 21, 119.

To work this example will require the proper integration of the following abilities and units of skill as presented in Table IV (if the columns are added upward):

Ability	Units of Skill
a	3, 5, but not 1, 2, 4, 6
b	9, 11, 12, but not 7, 8, 10
c	9 but not 7, 8
d	9, 11, 12, but not 8, 10
e	
f	14, 16, 17, but not 13, 15

g	19, 21, but not 18, 20
h	19, 21, but not 18, 20
i	26, 28, 29, but not 24, 25, 27, 30
j	31, 33, but not 32, 34
k	35, 23, but not 22
l	37, 39, 43, 44, but not 37, 40, 42

Abilities m to q are not called for in Example B.

The analyses may be further illustrated by reference to instructional material. For one familiar with this sort of material there is little difficulty in visualizing the meaning of the abilities and units of skill listed in Table IV or in finding in any text instructional material dealing with them. Others may find it profitable to scan the illustrations below which portray samples of instructional material dealing with the abilities and units of skill.

Illustration A.[13] Consider the instruction contained in the instructional unit labelled 'Illustration A.' It deals with the following items in Table IV: Ability a, Unit Skills 3 and 4; fragments of Ability f, Unit Skill 13. It identifies the answers to the combinations $2+2$ and $4+1$. It does not provide instruction on Ability f, Unit Skill 14, because the result of $7+3$ is not identified.

Illustration A

Whenever you see arithmetic examples like these you know that you must add.

$2+2=?$ $\quad 4+1=?$ $\quad 7+3=?$

The $2+2=?$ is read: two plus two equals how many? $+$ stands for plus and means add. The correct answer to $2+2=?$ is 4.

$4+1=?$ means four plus one equals how many? We must add. $4+1=5$.

What does $7+3=?$ ask us to do?

What is the correct answer to $7+3=?$

$+$ means plus and is the signal to add.

$=$ means equals.

Illustration B provides instruction on an important procedure and supplies identification (possibly not needed) on certain combinations. The abilities dealt with are ordinarily taught only after considerable instruction on more elementary aspects of addition has been given.

[13] The illustrations used here are original and are used as illustrations only.

Illustration B

Here is a new kind of example which all boys and girls are expected to know. Look at it carefully and read the Notes about it several times if you wish.

A

$$\begin{array}{r}52\\14\\\underline{27}\end{array}$$

B

$$\begin{array}{r}52\\14\\\underline{27}\\3\end{array}$$

C

$$\begin{array}{r}52\\14\\\underline{27}\\93\end{array}$$

Notes for You to Read

The first thing to do is to add the right-hand column. You have learned how to do this already. $7+4=11$. $11+2=13$. So 13 is the sum of the ones' column. But how are we to write the sum 13? 13 is really 1 ten and 3 ones. We write only a part of the sum 13. We write the 3 ones of 13 under the ones' column. The 1 ten of the 13 we will add to the tens' column. Picture B shows the 3 of the 13 written in the proper place.

Now we must add the tens' column. We have 1 ten carried from the 13. We think: carry 1, plus $2=3$, $3+1=4$, $4+5=9$. 9 is the sum of the tens' column. It is written correctly in Picture C.

Illustration B provides instruction on Ability i, Unit of Skill 24, and Ability l, Units of Skill 37, 39, 43. Fragments of Ability f, Units of Skill 13 and 16 are also provided. It is evident that Ability g, Units of Skill 19 and 21, Ability a, Unit of Skill 6, and fragments of Ability f, Units of Skill 14 (if adding upward) and 16 are assumed in this instructional unit.

The point at issue, of course, is not the quality of these Illustrations A and B, but rather the demonstration that definable aspects of the general ability to add can be and are identified.

A few comments on Table IV may be in order. It is granted that other and different analyses may be made of equal or superior merit; these comments apply to any analysis.

1. Analyses, being committed to a printed page and hence static in nature, cannot show the movement inherent in actual behavior.

2. Analyses do not show, with convenience, what parts of the total process are involved in the working of a given example—it is evident that any one example need not contain all the elements in a process of which it is representative. Thus an example or instructional material may deal with single carrying but not double, or may deal with

higher decade combinations with no bridging but not with those involving bridging.

3. Some combinations of elements in an analysis of a process are in any given example mutually exclusive; that is, if certain units of skills are called for in an example, certain other units of skill will not appear in that example. Thus, in Table IV Units of Skill 7 and 8 are ordinarily mutually exclusive. How much the learning of any one element facilitates the learning of any other represents an important problem of transfer upon which very few data are available. Probably such facilitations vary from nothing or even negative effects to practically perfect transfer.

4. Many would guess that considerable amounts of transfer are the general rule, and hence actual learning situations need not be as exhaustive as a complete analysis of the function. However, until the facts about the transfer are known, sins of too great elaboration in instruction are probably less damaging than those which lead to unduly meager instruction. It may be the part of prudence to underestimate rather than overestimate the influence of transfer.

5. Pending searching experimental studies in actual learning, we must rely a great deal upon insight and experience in determining analyses. All analyses must then be considered as tentative in nature.

The important consideration is that all attempts at instruction should be founded upon careful analysis of the content to be taught. A first question to be answered in all teaching is: What is the nature of the content or skill to be taught; or, what difficulties does thoughtful analysis show to be in need of explanation?

VI. Analyses of Actual Instructional Material

Space limitations in this section compel a choice among several possibilities:

First, other analyses of the addition of whole numbers might be presented and compared with that presented in Table IV. A penetrating defense for, or assault upon, any detailed analysis of as broad a function as addition of whole numbers or a genuinely painstaking comparison of several such analyses would, however, involve a lengthy and technical treatment and would draw upon criteria as bases for judging deficiencies which have but little except personal opinion to support them. It is admitted that the analysis in Table IV is imperfect, but at the same time it is held (with the exception of objec-

tions duly recorded) that such a table supplies a useful foundation upon which to base instruction in addition.

Second, similar analyses of other important sections of the skill or process side of the arithmetic curriculum might be presented. Analyses for certain important processes are available; some of them will be given farther on in this chapter.

However, it seems wise just here to follow out a third possibility; namely, that of making clear certain possible uses of the analysis already given in Table IV and of similar analyses in general.

1. The Uses of Analyses

There are several uses of learning analyses. Two are: first, they may become a schedule or a guide for the construction of teaching material, be that material presented in oral form by the teacher or in written form by a text; second, they may be used as a score card upon which to check some, though not all, of the important aspects of teaching material. In other words, satisfactory learning analyses may be used by one who creates instructional material or by one who wishes to determine which of several sets of available instructional materials he desires to use. That textbook material represents one of the major factors in actual classroom procedure is self-evident. It seems sensible to illustrate a use of learning analyses by checking through, with Table IV as a criterion, certain typical instructional material now available.

The point of view which is being presented is this: The learning aspects of any process should be set forth as competently and as clearly as possible. Instructional material should be constructed in the light of such analyses. In the practical situation many who are responsible for teaching do not write or prepare instructional material —they select one or two sets of instructional materials from a large list of possible treatments. This selection should be based upon accurate information relative to what any set of instructional materials teaches and what it does not teach. Satisfactory knowledge of the content of instructional material cannot be gained by casual inspection. Material must be carefully analyzed if estimates of it are to be trustworthy. It is held that basic elements of good method are: (1) at least semi-scientific knowledge of the psychological nature of the content taught, and (2) at least an approach to objective evidence relative to the nature of the instructional material used. Table IV

presents a learning analysis of addition of whole numbers. Materials like those presented in Tables V to XII supply objective and verifiable data on the nature of instructional material used in typical classrooms at the present time.

If Table IV (taken in connection with the specific number facts needed in work) be regarded as an acceptable basis for a teaching of addition, it will be profitable to study the instructional material supplied in typical current textbooks.

2. Limitations of Analyses of Text Material

It may occur to the reader that textbooks are not the proper sources to study in an attempt to discover what pupils are actually experiencing in moments of learning. It may be objected, for instance, that the teacher so supplements the text, as far as instruction is concerned, that a true picture of a process as learned cannot be gotten by reference to instructional material in texts. If this is the case, the prefaces of many texts which are statements of methodology underlying the material are sadly askew. They doubtless often are. However, that the average teacher goes much beyond the text in the right direction is questionable. Again, it may be objected that the actual teaching material which operates in the classroom is more accurately presented in printed courses of study than in the textbooks. This would be true only if printed courses of study were given preference over texts by teachers and pupils, and when the course of study worked out a treatment superior to the text. Further, it might be objected that it would be better to use as evidence of what is happening in the classroom the better known 'method' books which teachers study in their professional training. But here, again, we should have to be certain that the method books were really superior to the texts and that this superiority was fully recognized by the teacher.

In any event, textbooks have been chosen for analysis because they seem to be nearest the pupil and because they furnish objective data which pertain to learning. The pupil certainly looks at his text and its instructional material, whereas he almost certainly does not look at printed courses of study or method books. There is probably far less leakage between text material in the hands of the pupil and his learning reactions than there is between material in courses of study and method books and the actual responses of the children. Whether in the long run instruction will be significantly better than the instruc-

tion in the pupil's text itself, is unknown. The actual effect of an explanation of a process taken by the teacher from some source, translated into her own words, listened to, and reacted to by pupils, is a bit uncertain. How much of the printed instruction a child can get from his text is also unknown, and amounts of it competently reacted to by the pupil are doubtless a function of the form of presentation as well as of the actual content. Text material, however, is within the sight of the child and it is objective enough to make its study possible.

From the foregoing discussion it is clear that, given time and industry, it is quite possible to give rather reliable answers to the questions: Does a specified text teach or does it not teach any given ability? If it does something with an ability composed of several units of skill, does it identify certain units but not others? To take a concrete example, does it teach only the carrying of 1 from the ones' column as a unit of the ability to carry or does it identify more of the units of the ability to carry?

Accurate analysis for what texts teach and do not teach is possibly no more important than corresponding analysis of oral instruction presented by the teacher. How far it is profitable to push analysis may very well be a matter of disagreement. Accuracy of impression in judging texts probably varies directly with adequacy of analysis. It can be, and is in this report, quite impersonal and objective.

In the summaries of text analysis which follow nothing has been done with the problem of *quality* of instructional material. It is possible that two texts may teach the same units of skill but one may do a much better job of teaching than another. It is evident that a text which omits instruction on an ability assumes transfer or facilitation which may or may not exist. While omission of instruction on an element is not necessarily a weakness, it may be judged to be so unless other means for learning that element are present.

While, then, many limitations to text analysis may be admitted, the position that penetrating analysis, while it does not reveal the whole truth, does provide a much more trustworthy picture of the competence of instructional material than can be provided by more casual methods of appraisal, is a strong one. The limitations of analysis which appear to be the most serious are: (1) an analysis shows whether a given unit is taught or not. But in learning, it is not enough to teach in isolation; elements must be fused into patterns of skill of various types. The success of instructional material in helping this

dynamic fusion of elements into broad abilities is not shown any too well by present analysis. (2) Analysis does not show whether a given element ought to be taught or omitted. It simply shows whether a given element is taught or not. With present knowledge still fragmentary in the field of the specifics of learning, personal opinion must be used in interpreting the data yielded by analysis.

VII. An Analysis of Instructional Material in Addition

The following report on the teaching of addition of whole numbers is the work of Dr. F. L. Wells greatly condensed. The topic under discussion here is instructional material, rather than drill and problem material. It is important that this distinction be kept in mind.

1. General Methods Employed in the Analysis

Dr. Wells used the analysis of addition presented in our Table IV. The method of scoring and tabulating instructional material for the abilities which are reported in subsequent tables was based on careful study of material as suggested by Illustrations A and B previously presented.

The text materials used were analyzed by workers independently to make sure that all material that could in fairness be judged as instructional material was used. All instructional material from all texts used in the analysis to be reported was cut from the text and mounted on heavy paper. In this way all instruction on any process could be studied carefully. In analyzing such material, while due care was taken not to read into printed composition ideas which might be in the minds of the investigator but not on the page, omission of specific instruction was recorded only after it was shown clearly that it did not exist.

Our experience with this sort of detailed study of instructional material has shown clearly that even a careful reading of this material fails to give a correct impression of its specific content. Only by objective checking can the appraiser of instructional material get valid answers to crucial questions as to the worth of the material.

A next forward step in supervision of elementary education is the use of appraisal techniques by which the worth of materials may be more competently judged. It is the details of instructional material that function in actual learning situations, and instructional material tends not to be any better than is demanded. Demand for better

material will surely grow out of the more exact knowledge of material that will come from objective and thorough analysis of it.

The reader is reminded again that the following tables refer to instructional material and do not attack the problems of drill and problem material.

2. Data Presented Are Medians

All the data on instructional material in texts are given in terms of the median of the texts studied, unless otherwise noted. The texts used are those which are well known and presumably supply a large majority of instructional material in the classroom. The original data upon which the medians are based are deleted from this report in order to avoid the possibility of identifying any text and thus of suggesting undesirable comparisons. That such comparisons of texts can be made and that they are important to the supervisor and teacher as well as to pupils is obvious, but it is not the intent of the Yearbook's Committee to make them. In perusing the next pages, then, the reader should bear constantly in mind that while only medians are reported, variability amongst the texts is also an important matter. The verifiable data warrant the statement that the texts used differ almost beyond belief. The difference in adequacy of instruction, as here measured, between the median of the texts and the text giving the most or the least instruction is uniformly great.

3. The Identification of the Fundamental Addition Combinations

Any analysis of the addition of whole numbers would, it is assumed, contain references to the addition combinations themselves. The analysis in Table IV lists this body of essential information as Ability f, Units of Skill 13, 14, 15, 16, and 17.

The need of specific instruction on any or all of the addition combinations in any given grade would, of course, depend upon pupil mastery of preceding grades. But whatever the needs here may be, it seems desirable that the teacher using any instructional material should know what that material actually does by way of specifically identifying the combinations themselves. In this study, it is held that a number combination is taught (in contrast to drilled upon) if its sum is given or if a teaching situation is provided where it is

reasonable to assume that the child will find its sum from a study of the material in question.

Table V shows the basic addition facts taught in Grade III in eight current arithmetic texts. The addition facts reported here are facts taught as such; the data on the higher decade addition combinations appear in Table VII.

TABLE V.—BASIC ADDITION FACTS TAUGHT IN GRADE III ACCORDING TO THE MEDIAN OF EIGHT CURRENT ARITHMETIC TEXTS

Number Added To	Number Added 0	1	2	3	4	5	6	7	8	9
0										
1		×	×							
2		×	×		×	×	×		×	×
3			×	×	×	×		×	×	×
4		×	×	×	×	×	×	×	×	×
5			×	×	×	×	×	×	×	×
6			×	×		×	×	×	×	
7				×		×				
8									×	×
9			×							
Total	0	3	7	5	4	6	4	4	6	5

A check (×) means that the basic addition fact in question would appear in the instructional material in the third-grade textbook if that book represented the median of eight current textbooks. Thus, the median text gives the sum of 6 and 8 or provides a teaching situation distinctly facilitating the finding of 14 as the sum. Such is not the case for 7 + 8.

In examining Table V the reader must remember that it deals with instruction, not with drill. All texts supply a quantity of drill as a matter of course. He must also remember that the table deals with medians. Some comments on the variations among the texts may well be added here.

The data behind the medians show clearly that, as far as the specific identification of the one hundred addition combinations is concerned, the eight texts are constructed on widely diverse assumptions.

Some of them assume either that the child knows these facts before beginning third-grade work or should master them quite independently of his third-grade text; that is, the text provides drill but not the information needed to carry on the drill. Others of them agree that about half of the combinations should be identified, but do not agree on which half. The texts providing the most instruction use the theory that practically all of the combinations should be identified.

Lest the reader is shocked by the paucity of instruction reported and hence thinks Table V in error, he is assured that while all texts *drill* on all combinations, the median text *instructs* on them as shown in Table V. A conversation between a pupil and the median text might run:

> *Pupil:* What do you do about $9+9$?
> *Text:* I provide drill upon it.
> *Pupil:* But I must know its sum in order to practice upon it.
> *Text:* I do not tell you what the sum of $9+9$ is, nor do I provide a situation from the study of which you may reasonably find its sum for yourself. I assume you know that $9+9=18$, or that you will find this fact from some source other than your text.

No text seems to deal with the nineteen zero combinations in a convincing manner—a point that warrants some discussion here.

a. The Treatment of the Zero Combinations.—Two texts identify four, and one text, three, of the zero combinations.

Here seems to be a beautiful example of faith in inferential thinking which may not be quite justified by the facts. Let us give a moment's consideration to the amount of inferential thinking which texts (and other teaching instruments) consciously or unconsciously assume. The amounts may be indicated in four typical cases:

Case 1. If no inferential thinking is assumed, all nineteen zero combinations would be identified:

$0+0=0$,	$0+1=1$,	$0+2=2$,	$0+3=3$,	$0+4=4$,
$0+5=5$,	$0+6=6$,	$0+7=7$,	$0+8=8$,	$0+9=9$,
$1+0=1$,	$2+0=2$,	$3+0=3$,	$4+0=4$,	$5+0=5$,
$6+0=6$,	$7+0=7$,	$8+0=8$,	$9+0=9$.	

Case 2. Here some inferential thinking would be assumed. Perhaps these facts would be identified:

$0+0=0$,	$0+1=1$,	$0+3=3$,	$5+0=5$,	$0+7=7$,
$9+0=9$,	$2+0=2$,	$0+4=4$,	$6+0=6$,	$0+8=8$.

From these ten identifications the pupil would be expected to infer the correct answers to the following:

$1+0=\quad 3+0=\quad 0+5=\quad 7+0=\quad 0+9=$
$0+2=\quad 4+0=\quad 0+6=\quad 8+0=$

Case 3. Much inferential thinking is assumed. Teach these facts:

$0+0=0,\quad 0+2=2,\quad 4+0=4,\quad 7+0=7.$

The answers to these examples are gained by inference:

$1+0=\quad 2+0=\quad 0+3=\quad 5+0=\quad 6+0=$
$0+7=\quad 0+8=\quad 0+9=\quad 0+1=\quad 3+0=$
$0+4=\quad 0+5=\quad 0+6=\quad 8+0=\quad 9+0=$

Case 4. No zero combinations identified or only a very few. Here, if the text is properly required to provide instructional material, the theory is that the zero combinations require no instruction, presumably on account of their ease.

Most of the eight texts studied are built on a feeble use of Case 3 or frankly on Case 4 in their presentation of the zero combinations.

b. Which Combinations Should Be Identified?—Since the median text identifies some, but not all, of the combinations, it becomes important to know which combinations are identified and for which a text takes no responsibility as far as instruction is concerned. For present purposes the one hundred combinations may be divided into these three classes:

Class	*Description*	*Illustration*	*Number*
I.	Basic addition facts without bridging	$6+2=8$	36
II.	Basic addition facts with bridging	$6+5=11$	45
III.	Basic zero facts	$6+0=6$	19

TABLE VI.—CLASSIFICATION OF BASIC ADDITION FACTS TAUGHT IN GRADE III, FOUND BY AVERAGING EIGHT CURRENT ARITHMETIC TEXTS (BASED ON TABLE V)

Class	I	II	III	All
	Without Bridging (36 in all)	With Bridging (45 in all)	Zero Combinations (19 in all)	Total (100 in all)
Average of Eight Texts...	21	23	....	44

Table VI shows how instruction on combinations thus classified is provided in the 'average' textbook—meaning by 'average' a hypothetical text that did conform to the data obtained by averaging

eight current texts. It will be seen that the average text would identify 44 of the 100 combinations—well over 50 percent of the 36 combinations which do not involve bridging, about 50 percent of the combinations involving bridging (these are presumably harder but receive less instruction), and none of the zero combinations. One may well question whether the treatment of the 100 primary addition combinations in the average third-grade text has been the outcome of care and deliberation.

c. *The Addition Combinations above Grade III.*—Whether or not a sufficient number of pupils above Grade III need instruction on the one hundred addition combinations to warrant instruction upon them, figures not here exhibited make clear that the average text assumes no responsibility for such instruction above Grade III. If instruction is needed, it must be supplied from some other source. There is not much variability in texts relative to this aspect of instruction; only rarely does text material recognize a need for it in any way.

4. The Identification of Higher-Decade Addition Combinations

Teaching material in arithmetic which omits or slights specific identification of the higher-decade addition combinations assumes either: (a) that a mastery of the primary combinations will carry over to the higher decades, as knowing $5 + 7$ is sufficient preparation for $25 + 7$, etc., or (b) that the teacher can take care of the higher-decade additions without aid or help from the text. Contrariwise, teaching material which stresses specific identification of the higher decade addition combinations is built upon the theory that when transfer is not known to exist, it is reasonable for the teaching material, as well as for the teacher, to provide instruction on these essential facts.

If all useful combinations to $39 + 9$ (a desirable limit to the demands of column addition) are considered, there is a total of 270 combinations properly described as 'higher-decade combinations of importance.' The median of eight texts supplies instruction on 48 higher-decade combinations. To many, 48 combinations will seem a relatively narrow base of specific instruction. But for the presence of one text, which provided relatively thorough teaching, the median of the eight texts would have been much less (the text which provides the next greatest amount of instruction identifies but 61 of the higher-decade combinations used in column addition, and of these 61, only

3 deal with any addends above 19). In general, the verifiable data support the assertion that the methodology underlying typical texts is based on the assumption that all combinations above those involving 19 can be left to inferential thinking; most texts, in fact, assume that higher-decade additions of any type need not be considered as in need of instruction at all.

a. Further Analysis of Higher-Decade Combinations.—No apology need be offered for pushing the analysis of the facts relative to the higher decade combinations a bit further. It is probable that a clear report of present practice against a background of the total function involved will be more effective than a mere verbal discussion of the problem.

When considering higher-decade combinations, it is useful to distinguish between those which occur in column addition of reasonable difficulty and those which occur only in carrying in multiplication. For example, $24 + 9$ would occur in a column addition of reasonable difficulty; whereas $56 + 5$ would hardly occur in such a column addition, but would occur in carrying in multiplication, as in the case of 88×7, perhaps in the form of $7 \times 8 + 5$. The combination $56 + 9$ would not occur, either in reasonable column addition or in carrying in multiplication. It is assumed that if any higher-decade combinations are taught, those which are used in addition and in carrying in multiplication should be chosen as instructional material.

There are eighty higher-decade combinations above $39 + 9$ which are not needed in column addition but are used in carrying in multiplication. The sparsity of specific instruction on these eighty combinations is even more noticeable than it is for the higher decade combinations used in column addition. *The average text gives instructional attention to but four of these eighty higher-decade combinations used in carrying in multiplication.*

If all the useful higher-decade combinations used in column addition and all combinations used when carrying in multiplication are considered, there are three hundred fifty higher decade addition combinations which a majority of teachers would agree to be essential for computation. Obviously, pupils must use them. Their instruction is based on some describable theory. The texts vary in their methodology of teaching—all the way from no instruction at all to specific identification of nearly one-half of the total number. Most texts are constructed on the theory that no attention need be paid to the teaching

of higher decade combinations. In contrast to this practice of the majority, a few texts supply direct instruction of varying amounts. Experimental data are wanting on the amount of instruction needed on these combinations. Probably all combinations need not be taught. The child can generalize and does indulge in inferential thinking. Whether, in the end, neglect of instruction on a liberal sampling of the combinations (which is typical of present practice) will prove defensible is a speculation.

The higher-decade combinations which the majority would agree to be useful are not on the same level of difficulty. Even in the absence of experimental proof, it is clear that 22 + 2 is not so difficult as 29 + 7. Bridging is involved in one instance but not in the other. If only some of all useful higher-decade additions are to be identified, which ones should they be? A rough division of difficulty can be made with the presence or absence of bridging as the criterion. Table VII shows the number of combinations involving bridging and not involving bridging actually useful in addition and in carrying in multiplication, and the specific instruction provided upon these two classes of higher-decade combinations in the average textbook. It will be noted that the average of eight texts does not emphasize instruction on the presumably harder section of the combinations.

TABLE VII.—SUMMARY OF PROVISION FOR HIGHER-DECADE ADDITION FACTS APPEARING IN THE INSTRUCTIONAL MATERIAL OF EIGHT CURRENT ARITHMETIC TEXTS (All Grades)

Addition Facts Needed For	Addition Without Bridging	Addition With Bridging	Multiplication Without Bridging	Multiplication With Bridging	Total
Number needed........	135	135	52	28	350
Number provided (average of eight texts)	19	19	1	1	30

Table VII indicates that skill in higher-decade addition is expected from a very narrow base of instruction. Two types of instruction on the higher-decade combinations may be distinguished: (1) direct and specific instruction, and (2) incidental instruction in connection with explanations of process difficulties. In a few instances, the explanation of some procedure contained certain combinations and their an-

swers otherwise omitted. In this way rather tardy identification of combinations may be secured. A full listing of such incidental identification, though not reported here, gives practically the same picture relative to care and neglect as is given in the foregoing tables. It cannot be judged that, although specific instruction may be slighted, all is well because sufficient attention is paid in connection with other types of instruction. Such incidental practice (as a matter of fact) is very slight and in no way changes the picture of the situation presented in Table VII.

Variations in the eight texts studied are impressive. Some texts do not accept the theory basic to the majority. In these much more specific instruction is provided.

5. Summary on the Teaching of Addition Combinations

A perusal of Tables V, VI, and VII will show the reader almost at a glance that the responsibility assumed by representative texts relative to the primary and higher decade addition combinations with which a child must do his work is open to very serious question. It is obvious to the writer, at least, that a teacher using even the average text must not be content to teach the text as written; for if this is all she does, many important facts are left either to good luck or to rough trial-and-error learning. It would seem that precise knowledge relative to what a text does and does not do, like that presented in these tables, is essential to its successful teaching. Such precise information should be made available by textbook writers.

6. Instruction on the Units of Skill Used in Addition

a. Combinations versus Procedures.—Probably no one attempting an analysis of addition would omit reference to the addition combinations themselves. Any user of any instructional material would presumably benefit by an exact knowledge of what that material does by way of identification of the combinations. When, however, attempts are made to analyze the *procedures* of addition, it is evident that equally competent persons may produce quite different analyses. Any maker of learning analyses in arithmetic has to draw all too heavily on his own judgment and opinions, and in making an analysis he will be impressed with the paucity of experimental data upon which to base his work. However, it is clear that every bit of teaching is wit-

tingly or unwittingly based upon some assumptions relative to the learning involved in the material taught.

The following pages report compressed digests of the teaching of addition procedures. The data are based on the analysis presented in Table IV. A different analysis might have produced quite a different set of data. It is felt, however, that careful thought will show that data based on any analysis which is at all penetrating in its attempts to get at the actual learning difficulties met by children and which does not assume huge amounts of transfer and inferential thinking would in the main agree with the data in Tables VIII, IX, and X.

In Grade III, as is suggested, a child 'knows' addition when he has learned the facts used in addition and when he knows what to do with these facts. The knowing what to do may be broken into skills which, when operating smoothly and in the right order, lead to successful results.

Table VIII shows the average amounts of instruction provided in eight texts for various units of skill in addition of whole numbers. For a description of the units of skill, see Table IV.

Ideally we should know not only whether a text teaches a given skill or not, but also how effective each bit of instruction really is. Qualitative analyses of teaching power of text material are not yet available. An approach may be made, however, to the measurement of effectiveness by considering the factor of repetition. A single impression is hardly sufficient except under optimal conditions. Some repetition may be assumed as useful, either in oral or printed instruction. Table VIII reports not only the technical presence or absence of instruction on units of skill but also attempts at least partial qualitative estimation of merit by further reporting the amount of repetition specifically provided. Table VIII is a summary of an original table reporting in detail the data called for in the case of each text. Table VIII is read as follows: the numbers in the body of the table report the number of distinct repetitions of instruction on each unit of skill involved.

A careful reading of Table VIII presents a rather pessimistic picture. The reader will be impressed with the variations in the amounts of teaching material. Some units of skill are accorded much repetition; others, presumably of importance, are given scant attention.

It need not be asserted that the worker responsible for this table had any knowledge of the intentions of textbook writers; what the

TABLE VIII.—PROVISION FOR INSTRUCTION ON UNITS OF SKILL INVOLVED IN ADDITION OF WHOLE NUMBERS IN GRADE III FOUND BY AVERAGING EIGHT CURRENT TEXTBOOKS

Unit of Skill (See Table IV)	Average Number Times Taught in Eight Texts	Unit of Skill (See Table IV)	Average Number Times Taught in Eight Texts
1	2	26	1
2	3	27	27
3	41	28	15
4	5	29	5
5	9	30	1
6	10	31	1
7	8	32	0
8	29	33	5
9	9	34	2
10	37	35	1
11	7	36	5
12	8	37	72
13	25	38	28
14	24	39	10
15	4	40	25
16	13	41	14
17	16	42	0
18	33	43	8
19	15	44	1
20	3		
21	0*	No. of Units Taught	41
22	3		
23	1	Av. No. of Times a Unit Is Taught	13
24	28		
25	3		

*In this table '0' means less than one.

printed page does and does not carry by way of instructional material is ascertainable on a relatively impersonal basis. Superficial score cards may be arranged to prove a given set of materials superior to any other, but as soon as trivial superficialities are passed and critical analyses are used, the differences reported are real differences in the materials judged.

Assuming, then, the general adequacy of the analyses used, the following data on variation between texts are noteworthy. The average of eight texts, Table VIII shows, deals with 41 units of skill, meaning that some combination of at least four texts deals with 41 of them. The 'least' text, however, deals with 26; the median text with 32, and the 'most' text with 39 units of skill. Texts vary approxi-

mately in average number of repetitions per unit of skill from 8 for the 'least' text, through 13 for the 'average' text, to 38 for the 'most' text.

b. Instruction above Grade III.—An important methodological problem deals with the need for reteaching in grades beyond the one in which initial instruction appears. For example, long division is customarily taught in Grade IV. What provisions of instruction upon long division should be made for the pupils in Grades V or VI who need, not more drill, but more sheer information (instruction) upon this process? The problem of remedial teaching is a pressing one. Among its solutions are: (a) no text provision at all, the remedial teaching being supplied by the teacher or a fourth-grade text being obtained by the pupil; (b) some remedial instruction provided in the texts used in the upper grades, but of such a nature that the teacher's responsibility is clearly assumed; (c) rather full reteaching units provided in the text to facilitate pupil responsibility for relearning for those pupils able to do so, and material for coöperative work between pupil and teacher for more seriously handicapped pupils. Those acquainted with the practicalities of actual classroom work realize that these three solutions, or modifications of them, impose quite different teaching situations upon the class, the individual pupil, and the teacher.

Table IX reports the amount of instruction on addition of whole numbers provided in text material above Grade III. It deals, therefore, with remedial teaching. Table IX is similar in construction to Table VIII and is read in the same way.

It is obvious that texts assume some responsibility of instruction in addition beyond Grade III, where the first instruction is found. But in fairness to users of texts it should be said that, with the exception of a few texts, if children need instruction in addition in the fourth grade or later, they must depend upon other sources than their texts for such teaching. The range between texts in this matter of remedial instruction is from no instruction to rather complete instruction in each grade.

The practice of presenting in texts little or no teaching material beyond the grade in which initial instruction occurs is based upon such theories as: (1) Children profit so well by initial instruction that a need for reteaching (as well as reviewing in drill) in later years is rare and provision for it is unnecessary. (2) Children who need

TABLE IX.—PROVISION FOR INSTRUCTION ON UNITS OF SKILL INVOLVED IN ADDITION OF WHOLE NUMBERS ABOVE GRADE III FOUND BY AVERAGING EIGHT CURRENT TEXTBOOKS

Unit of Skill (See Table IV)	Average Number Times Taught in Eight Texts	Unit of Skill (See Table IV)	Average Number Times Taught in Eight Texts
1	0*	26	0
2	0	27	3
3	9	28	6
4	1	29	2
5	1	30	0
6	1	31	2
7	1	32	0
8	3	33	3
9	6	34	3
10	15	35	1
11	4	36	1
12	2	37	9
13	15	38	4
14	18	39	5
15	4	40	3
16	32	41	6
17	43	42	0
18	3	43	5
19	13	44	1
20	2		
		No. of Units Taught	36
21	0		
22	2	Av. No. of Times a Unit Is Taught	6
23	0		
24	4		
25	1		

*In this table '0' means less than one.

reteaching are so dull that they cannot profit by instructional material. They must be taught by word of mouth. (3) Children in the third grade can read well enough to profit by further instructional material in their own texts, but in later grades, in case further study of this process is needed, their ability to read has so degenerated that they cannot then profit by reading instructional material; that is, they can profit by reading materials in the third grade, but not in higher grades. (4) In the case of three-book series, enough reteaching is available in a sufficient number of instances. Thus a study of addition is possible for fourth-grade pupils, but not for fifth-grade pupils. A restudy of long division would not be available for fifth-grade pupils, since its first teaching appears in the fourth grade. In three-book

series, fourth-grade pupils have access to third-grade material, sixth-grade pupils to fifth-grade material, and eighth-grade pupils to seventh-grade material. A comparable provision for fifth-grade and seventh-grade pupils is rarely, if ever, made in three-book series.

Instructional material which does provide explanations of addition in grades above the third is based upon such theories as: (1) Be initial instruction as effective as possible, a significant fraction of pupils will profit by a reconsideration of the ideas of addition as well as by further drill upon it. Hence reteaching material in texts will prove useful. (2) Such reteaching material may well be in texts for pupil study, since reading ability increases grade by grade; and if instructional material properly appears in a third-grade text, it even more properly appears in later grades where reading ability is greater. (3) If a reconsideration of addition is needed in a grade beyond the third, absence of proper instructional material in the text practically forces the pupil to depend upon his teacher. But as children mature, one major aim is to practice self-reliance and self-direction in learning. To facilitate this aim, material should be available to allow the child at least to attempt his own learning through his own reading abilities. (4) Unless reteaching material is present in the upper-grade texts, the assumption must be made that an upper-grade teacher can fairly be assumed to be an expert in lower-grade content and methods; but with the increasing specialization of the teacher's work, this assumption may not be entirely desirable.

7. Instruction in the Abilities Used in Addition

Units of skill may be combined into larger hierarchies, which have been named 'abilities.' Tables are available showing what abilities are taught and with what units of skill each ability is taught, but to save space they are not presented. The summary presented in Table X will serve instead. In reading Table X the following points should be kept in mind: (1) The lettered abilities appearing here are those listed in Table IV. (2) The columns headed Units of Skill show how many unit skills are used to make the ability. Hence, how fully or how scantily an ability has been taught may be inferred by comparing the figures under the Units of Skill column with those in the average for the texts. (3) The total of the units of skill in this table (75) does not correspond with the total (44) reported in Table IV because certain units appear in more than one ability.

TABLE X.—SUMMARY OF PROVISION FOR SPECIFIC TEACHING OF ABILITIES IN ADDITION OF WHOLE NUMBERS, ACCORDING TO EIGHT CURRENT ARITHMETIC TEXTS IN GRADE THREE

The Abilities (See Table IV)	Number of Units of Skill in the Ability	Average Number of Units Taught in Eight Texts
a	6	3
b	6	2
c	3	0
d	5	1
e	2	1
f*	5	3
f†	5	2
g	4	2
h	6	3
i	7	4
j	5	1
k	3	1
l	9	4
m	2	1
n	3	1
o	3	1
p	2	1
q	3	1
Totals	75	32

* Combinations as such.

† Combinations in example with other combinations.

A reading of Table X will impress one with a very important consideration. Typical teaching of an ability is often very uneven; that is, some of the units of skill involved are taught, but others of perhaps equal or greater importance are omitted. A hurried examination of instructional material might give the impression that a certain ability was taught, when a more penetrating analysis of the same material shows that really only part of it has been dealt with. The original data upon which Table X is based show that the texts studied vary greatly in completeness of instruction. The amounts of 'mutilation' range from practically no instruction at all (because of great terseness or omission of many vital aspects) to rather complete instruction with few, if any, of the basic units of skill omitted. Theoretically, all texts teach addition; pedagogically, they vary greatly in their service to children and teachers.

8. Massed versus Distributed Instruction in Addition

The primary concerns in this portion of the discussion are: (1) the ideas of a process as a hierarchy—not as a unitary thing on the one hand or a mere collection on the other, (2) the idea of competent learning analyses as guides to the preparation of material or to the appraisal of instructional material, and (3) the idea of impersonal text analysis based on objective data.

It has been stressed that the analyses of text material are not to be deemed final appraisals of superiorities or deficiencies, because many aspects of good instructional material are not dealt with at all, and because instructional material is only one aspect of total teaching equipment—drill, problem, and test material are of great importance also. A further aspect of instructional material, now to be considered, concerns the serious learning problem of massed versus distributed learning periods. Data on this are pertinent to the problem of method. Various possibilities of massed versus distributed learning periods are present. When a process in arithmetic is viewed as a general ability composed of lesser ones which must be practiced, some describable theory relative to massed versus distributed teaching must be put into operation. Shall all or a major portion of a process be explained and followed by practice on the total skill, or shall instruction proceed in piecemeal fashion, with practice on sections of the total process interspersed? Actual classroom technique may lie at either of two extremes (see the accompanying schema) or between the two.

POSSIBLE DISTRIBUTIONS OF INSTRUCTION OF A TOTAL ABILITY VIEWED AS A HIERARCHY COMPOSED OF FACTORS A, B, C, D, ... n

Massed Instruction:	*Distributed Instruction:*
Teach, explain, identify, learn:	Teach A
A, B, C, D, ... n	Drill A
Follow by:	Teach B
Drill on total ability.	Drill B
	Drill A and B together
	Teach C
	Drill C
	Drill A, B, and C together, and so on.

A quantitative study of the practices relative to massed versus distributed instruction in textbooks reveals definite variations in prac-

tice. Those who were responsible for the instructional material and its organization seem not always to have been clear what theory they were applying, since inconsistencies are evident. A measure of the amount of distribution of instruction may be obtained by ascertaining the numbers of distinct instruction units provided, when to be 'distinct' means to have drill material between two instructional units. Table XI shows the average practice for the eight current texts used in this study.

TABLE XI.—NUMBER OF DISTINCT AND SEPARATE INSTRUCTIONAL UNITS IN ADDITION OF WHOLE NUMBERS IN EIGHT CURRENT ARITHMETIC TEXTS FOR GRADE III

Major Divisions of Addition	Average of Eight Texts
Basic Addition Facts	5
Higher Decade Addition	2
Column Addition, Single Column	2
Column Addition, Two or More Columns	5
Generalized Instruction	1
Checking	1
Total	16

The average—sixteen distinct teaching units for addition—obscures the existence of important variations in practice. These variations are so great as to prompt several questions: First, is there sufficient disagreement in acceptable theories of learning to warrant differences in distribution of instruction in current texts in the ratio of 3 to 1? Second, what considerations, other than those dealing with effective learning, may have provoked these differences in practice? Third, what effects do differences in the distribution of instructional units have upon the actual classroom situations in which pupils and teachers together attack the learning of addition?

The experimental data bearing upon this problem of massed versus distributed learning are very scant. It cannot now be proved that one best distribution exists. Two facts seem evident. It is unlikely that the somewhat mixed evidence as to the merits of 'part versus whole' learning of poetry can be safely carried over to arithmetic. The 'whole' of a process in arithmetic is perhaps more comparable to several poems than to a single one. It is further evident from an examination of the data that, as a general tendency, those texts which provide relatively modest amounts of instruction also tend to mass the instruction given.

9. The General Status of Instruction in Addition

Table XII reports a digest of crucial data relative to instruction in addition. To these data should be added the fact of serious disagreement relative to the fundamental theory which should support the minutiae of instruction on this process. Basic instruction in texts varies on every point. It matters little whether one makes a relatively elaborate analysis of some narrow aspect of addition or whether he assembles accurate data dealing with the high points, or generalities, of instruction; it will always be found that those who have assumed the responsibility for text construction differ seriously in their actual practice and hence presumably in their basic theory of good teaching. A difference in practice based on defensible differences of consciously accepted theories is one thing; a difference due to apparent inattention to the problems involved is quite another. It can easily be seen that the keeping of instruction separate from drill and problem work sets up an arbitrary distinction which is unreal. Few would contend, however, that thoughtlessness and inadvertency in instruction can or should be condoned. Excellent drill or extensive drill is, after all, the companion of instruction, not its substitute.

Table XII.—Summary of the Provision for Instructional Material on Addition of Whole Numbers in Eight Current Arithmetic Texts

Grades	Material	Average of Eight Texts
III	Instructional Units	15
	Number Combinations (Total 100 basic, 350 higher decade)	74
	Units of Skill (Total 44)	41
	Average Number Repetitions per Unit Taught	13
	Abilities (Total 18)	17*
IV to VII	Instructional Units	7
	Number Combinations	70
	Units of Skill	36
	Average Number Repetitions per Unit Taught	6
	Abilities (Total, 18)	3*

*The figures on Abilities show only the presence of some instructional material on abilities. The presence or absence of any constituent elements or units of skill in instruction on an ability is not shown, but may be read from Table X.

It might prove of interest to the reader to consider the following question: If a defensible and adequate list of theorems of human learning were made and all proper interests of arithmetic as a branch

of mathematics noted before instructional material was constructed, and then such material were competently made, in what ways would an analysis of such material differ from the analysis of actual material as reported in Tables V to XII?

10. The Problem of Qualitative Merit, or Effectiveness

Up to this point the discussion of the teaching of addition has been in terms of quantity. It is thought that such a mode of consideration is legitimate in a total view of method. Thus, a study of data on each text (deleted here to avoid direct comparisons) reveals almost astounding differences in the instructional material presented in representative textbooks, so that we are led to declare that a textbook may be a textbook before the eyes of the law but hardly before the eyes of a pupil attempting to learn from it.

Yet it must have already occurred to the reader that certain qualitative measures are also to be desired. The arithmetic teacher or supervisor must face not only the questions of whether an item is taught at all and whether it is taught several times, but also the question how well and how skillfully it is taught. It would be possible for a quantitative analysis to show that two given sets of material—two textbooks for instance—were exactly alike as far as such analyses as those we have reported are concerned, even if they were quite dissimilar as dynamic learning situations. It is obvious that quantitative analyses of the types here presented do not reveal differences which may, and probably do, exist relative to crucial factors of learning. For example, one treatment may be interesting and another boresome and musty. One may be so organized that the youngster likes to concern himself with it, while the other may be forbidding. One may be couched in a context well within the reading ability of the learner, while another may be written with a reading difficulty far beyond the reading capacity of the pupil. One may lead the learner through thought-provoking questions; another may be flat composition made up wholly of declarative sentences. One may use some imagination in typesetting and illustrative material to aid and abet attention; another may be set in such a prim and prosy fashion that it neglects fundamental facts or is too proud to use a single technique which the psychology of advertising has found to be effective in teaching the pupils' parents in the game of competitive buying and selling.

Unfortunately reliable analyses of the qualitative aspects of instructional material are not yet available. Perhaps in the future they will be. However, we can at least be sure that if quantitative analysis shows that a given skill or ability or number fact is not taught at all, there need be no question raised relative to the qualitative merit of the instruction thereon. We can also be sure that teaching material is actually based on varying and significantly different assumptions of a qualitative nature.

11. A Few Applications to Method in General

Methods of teaching in general may well be considered in connection with objective and impersonal data such as are presented in Tables IV to XII.

The discussion of addition, it is hoped, will serve to make clear a point of view in method of teaching arithmetic which is of apparent importance. This is, in summary:

1. Instruction in whole numbers should have as its theoretical basis competent analyses of the processes taught, in terms of learning difficulties, or steps, or units.
2. These analyses can view each process as a hierarchy composed of lesser abilities with the abilities in turn composed of units of skill.
3. Instructional material of any form should then be constructed and appraised in terms of its respect for the learning analyses.
4. Omissions of treatment of abilities or of units of skill should be based on known facts of transfer or of the inferential powers of children which would justify omissions. In the lack of such information, wholesale omissions are unwarranted since they assume powers of children to learn without instruction, and such assumptions are unsupported by the known facts.
5. Drill is not a substitute for instruction. Both are needed, and needed in superior quality.

This point of view applies to informal teaching as well as to printed instructional material. Our discussion has drawn illustrations from printed material, not to give the impression that the point of view relative to method here taken is limited to text material, but because text material best serves the purposes of illustration.

A study of such data as shown in Tables IV to XII throws some light for example, on such questions as these:

1. Should instructional material cover rather thoroughly the number facts, skills, and abilities involved, and as reported by an analysis of the content, or are meager samplings of these facts, skills, and abilities sufficient?
2. Should teaching be done at one time, or should there be several exposures?
3. What variations in oral teaching should be made in connection with learning instruments, which themselves vary in the success with which they assume the responsibility of spreading before teacher and pupil the elements of a process to be learned?
4. Upon what describable and definable theory of text-construction is any given learning material built?
5. Is excellence in drill a justification for weakness in instruction? Or, are instruction and drill active partners, each of which serves in proportion to its effectiveness and worth?

With such analytical data as have been presented, the reader can easily answer for himself these four questions and many others of similar nature which will occur to him.

VIII. Analyses of Instructional Material in Subtraction, Multiplication and Division

Economy of space forbids the treatment here of subtraction, multiplication, and division. Detailed analyses for these processes are at hand which show in the case of the same eight texts analyzed that average textbook instructional material in those three processes is somewhat inferior to that in addition, when the greater complexity of these processes is considered. It is also plain that the variations among the texts are as great, and in some cases greater, than those existing in instructional material dealing with addition—in fact, in most texts the efficiency of instruction is greatest with addition, whence it progressively declines.

IX. Analyses of Instructional Units in Common Fractions

It will be interesting and profitable to push the analyses of the instructional units beyond the field of whole numbers.

An advance in control over fractions would be made if there were generally available competent analyses of the several processes in fractions. Further advance would be made if instruction in fractions proceeded in the light of such analyses, omitting direct treatment of any phase only when experimental data on learning justified the omission.

It is possible here only to present a specimen analysis and data on representative instruction based on such an analysis. "The Meaning of Fractions" is chosen for the illustration with no implication that it is more important than other such aspects as "The Addition of Fractions," or "The Multiplication and Division of Fractions." The discussion of fractions that follows is based upon the work of Malcolm P. Price.

1. The Meaning of Fractions

Just as with other phases of arithmetic, the total ability that we may name 'the meaning of common fractions' may be analyzed into a series of unit skills. A description of these unit skills is given in Table XIII where the meaning of fractions is depicted as an organization of important ideas, facts, and judgments which are distinctive from computational ability (such as addition or subtraction of fractions).

When attempting to analyze instruction in common fractions, one is immediately faced with the problem of limiting the field of analysis. Obviously no teacher or text can be discredited for not dealing with a fraction such as 45/75. There is, perhaps, more reason for penalizing one who does so. On the other hand, it is just as obvious that any teacher or text that does not teach the meaning of the fraction $\frac{1}{3}$ is justly open to adverse criticism. So far there are no scientific data showing just where the line should be drawn—just which fractional parts must be taught and just which need not be taught. An analysis of textbooks shows that there is a decided disagreement on the part of textbook authors. Some texts teach fractional parts up to, and including, 8ths; others carry the instruction to 24ths, and even to 75ths. To draw a line between the fractions which should be taught and those which should not be taught, yet not be unfair to any textbook, it was decided to limit this analysis of instruction to those fractional parts up to and including 6ths.

It is also highly important in the teaching of fractional parts that not only fractional parts of units but also fractional parts of groups should be explained. The concept of one-third of a unit is quite different from the concept of one-third of a group. Life demands both; therefore, instruction must be given in both.

The inclusion of Skill IV (transference from words to numbers) can be defended because life demands that ability on the part of the adult. Therefore, the pupil should receive instruction on it.

TABLE XIII.—AN ANALYSIS OF THE MEANING OF FRACTIONS

Unit Skill	Unit-Skill Number	Illustration
I. Fractional Parts		
1. Halves a. Of groups	1	1/2 of 6
b. Of units	2	1/2 of 1
2. Thirds a. Of groups	3	1/3 of 9
b. Of units	4	1/3 of 1
3. Fourths a. Of groups	5	1/4 of 8
b. Of units	6	1/4 of 1
4. Fifths a. Of groups	7	1/5 of 10
b. Of units	8	1/5 of 1
5. Sixths a. Of groups	9	1/6 of 18
b. Of units	10	1/6 of 1
II. Comparative size of Fractional Parts	11	Which is larger, ½ or ⅓?
III. Meaning of Terms of Fraction	12	The numerator and denominator are the terms of a fraction.
1. Numerator	13	In the fraction ⅓, 1 is the numerator.
2. Denominator	14	In the fraction ⅓, 3 is the denominator.
IV. Transference from Words to Numbers	15	Three-fourths is written ¾.
V. Transference from Numbers to Words	16	⅓ is read "one-third."
VI. The Meaning of a Fraction	17	The fraction ½ means that the whole is divided into two equal parts, and that one of the equal parts is being taken or considered.
VII. Kinds of Fractions		
1. Proper fractions	18	A proper fraction is one whose numerator is less than its denominator, as 3/5.
2. Improper fractions	19	An improper fraction is one whose numerator is equal to or greater than its denominator, as 4/4 or 5/3.

TABLE XIII (CONT'D).—AN ANALYSIS OF THE MEANING OF FRACTIONS

Unit Skill	Unit-Skill Number	Illustration
3. Mixed numbers	20	Mixed numbers are numbers composed of an integer and a proper fraction, as 2½, 3¼.
4. Equivalent fractions	21	Equivalent fractions are fractions having the same value, as 1/2, 2/4, 3/6.
VIII. Reduction		
1. Reduction to lowest terms		
a. Type 1	22	2/6 = 1/3
b. Type 2	23	7/5 = 1 2/5
c. Type 3	24	6/4 = 1 2/4 = 1 1/2
d. Type 4	25	3/3 = 1
2. Reduction to higher terms		
a. By specific cases	26	$\frac{3 \times 2}{4 \times 2} = 6/8.$
b. Statement of a general rule	27	To reduce a fraction to higher terms, multiply both numerator and denominator by the same number.
3. Reduction of an improper fraction to a whole or mixed number		
a. By specific cases	28	3/2 = 1½, 5/3 = 1⅔.
b. Statement of a general rule	29	To reduce an improper fraction to a whole or mixed number, divide the numerator by the denominator.
4. Reduction of a mixed number to an improper fraction		
a. By specific cases	30	1½ = 3/2.
b. Statement of a general rule	31	To reduce a mixed number to an improper fraction, multiply the integer by the denominator of the fraction, add the numerator of the fraction, and place the sum over the denominator.
IX. A fraction as an expression of division	32	⅔ means 2 divided by 3.
X. The expression of the remainder in division as a fraction	33	$8\overline{)25}$ = 3⅛

The importance of Skill VI (the meaning of a fraction) may not be apparent at first. If instruction is given in fractional parts up to and including 6ths, and during this time Skill VI is developed, the child should be able to grasp the meaning of any fraction regardless of the figures used.

2. The Total Amount of Instruction

A rather gross measure of sheer amount of instruction may serve as an introduction to a consideration of the teaching of the meaning of fractions. Such a measure is given in Table XIV.

From an examination of Table XIV it is evident at once that the meaning of fractions is considered a matter of importance since it is allotted considerable space. Variations in sheer amount of material are impressive. One text gives only 179 square inches to this instruction, while another gives 579 square inches.

Another interesting fact brought out in this study, but not shown in the table, is that three texts use roughly one-half as many figures and symbols as they do words. Other texts use only about one-sixth as many figures and symbols as they do words. It is safe to say that the former relationship is the more desirable.

Objection may very well be raised on the score that the measure of amount of instruction in terms of square inches of material is too gross a measure and such grossness of measure may very well involve serious inaccuracy of impression. It is to be noted, however, that other measures of amount of instruction are included in Table XIV. No violent disagreements exist between the several measures used.

TABLE XIV. — AVERAGE INSTRUCTIONAL CONTENT DEALING WITH THE MEANING OF FRACTIONS IN EIGHT SERIES OF ARITHMETIC TEXTS

Square Inches of Total Area	Square Inches of Pictorial Matter	Number of Words	Number of Figures and Symbols
318	38	2121	707

3. Instruction on the Several Units of Skill

Analysis of eight current texts shows that the median text provides instruction on each of the unit skills in the meaning of frac-

tions listed in Table XIII except Nos. 9, 24, 31, and 33. Furthermore, Unit Skills 1, 2, 12, 13, 14, 20, 22, 25, 28, and 30 were taught by all the texts, and all the skills were taught by at least two texts. Of course, no attempt is made to pass judgment on the relative effectiveness of instruction. Even the amount of repetition of instruction must be omitted here. It is obvious, then, that the foregoing tables fail to give much needed information.

4. Summary of Instruction in Common Fractions

The preceding analysis is suggestive only. Space forbids the insertion of analysis of the four processes using fractions and reports of instructional material in current texts. In general, verifiable data lead one to assert that average instruction in fractions is quite inferior to average instruction in whole numbers. Perhaps the social utility of fractions is so little that instruction here is not worth serious effort. It is also clear that texts vary significantly in the adequacy of their instruction.

The discrepancies between the different texts may raise the question as to just what the fundamental theories of learning are which should govern the construction of situations intended primarily for instructional purposes. It is highly probable that if instructional material committed to the printed page shows upon even semi-rigorous analysis to be rather carelessly built, oral instruction provided by typical classroom teachers is also incomplete and ragged. It is not necessary to insist that printed instructional material should provide complete instruction, leaving the teacher little to do but to guide the pupil in his use of the text, in order to urge the point that any practice of teaching which falls much behind adequate theory does not gain but rather loses by this fact. In whatever form instruction is presented, it should be worked out in the full light of adequate analysis of the topic taught. That does not seem to be typical practice in the field of fractions, any more than in the field of whole numbers. Improving the content of instructional material appears to be one point toward which advance attacks may be sagaciously directed if fractions are to be considered a major section of an arithmetic curriculum.

X. Analyses of Instruction in Percentage

Courses of study agree that considerable attention should be paid to percentage as a process and to several of the social and business

uses of percentage. It is not the province of method in arithmetic to inquire into the worth of aims, but that the teaching of at least some aspects of percentage has a vigorous rationale seems evident. The relative difficulty of percentage for pupils to whom it is taught is not now known. Whether it is harder for upper-grade children than long division is for fourth-grade children, for example, is not known. Few published data are available bearing on the relative difficulty of the various elements which in combinations make the total subject. No report has been made on the amounts of practice needed by children of various levels of ability or intensities of interest. Buswell, who by common consent has been given the responsibility for reporting researches in arithmetic, has to date listed but few studies of even a semi-scientific nature, and few of these can be considered as fundamental.

There are several ways of teaching percentage. Each of these has famous protagonists, but experimental study of the many problems inherent in the teaching of percentage is conspicuously lacking. In a strict scientific sense, very little is known about the methods of teaching percentage. Perhaps as good a way as any to begin a deliberate attempt to get percentage under better control is to find out the facts relative to the present status of its teaching, its drill, and the success or failure of this teaching. Elsewhere in this Yearbook is reported a partial analytical study of the accomplishment of typical pupils by Edwards. Toward the end of this chapter are reported data showing typical drill provisions in percentage. The present discussion deals with attempts to build a quantitative picture of some crucial aspects of the teaching of percentage as teaching. It goes without saying that one error study, one quantitative analysis of instructional material, and one orderly account of drill provisions are but a beginning on the total work to be done.

1. Methods of Analyzing Instructional Material in Percentage

The data presented in Tables XV-XVI are the work of Miss Nano E. Mahoney.[14] The method used in collecting the material for study was to identify carefully all instructional material in representative textbooks on the subject of percentage as a process. The present report does not include instruction on the applications of

[14]Photographic reproductions of all instructional material used in this report are on file at the Education Library, State University of Iowa, Iowa City, Iowa.

percentage. All texts in arithmetic purport to teach percentage as well as to provide practice upon it. At present, provisions for instruction printed in textbooks seem a defensible object of study because we have very little supplementary material in this subject and few if any courses of study contain teaching material not freely duplicated in current textbooks.

a. An Analysis of the General Ability—It is useful to think of percentage as a process in terms of elements rather than as one total thing. The analyses which follow use an apparently rather satisfactory list of elements. Some consolidation of the original tables has been necessary for space consideration, but a study of the next tables will impress the reader with several important aspects of instructional material.

Percentage has been studied under these main divisions: (1) ideas of percentage, (2) ideas of parts and wholes, and (3) changing decimals to equivalent percents.

2. Instruction in Ideas of Percentage

Table XV presents a list of specifics under the general head "Ideas of Percentage." In the body of this table verifiable data bearing on the status of present-day instruction of these ideas appear. These data are based on a study of actual instructional material in which study of an impersonal and unbiased record of the facts was the sole aim. The numbers in the body of the table denote the number of specific identifications occurring. It thus shows not only whether or not an item appeared in instructional material but indirectly denotes amounts of emphasis on any item taught.

Table XV contains several minor omissions. For example, the idea that the percent sign stands for a decimal point was identified once in one text, but in no others. That the sign % may be thought of as a short way of writing fractions with a denominator of 100 % being $\frac{}{100}$, for short, was identified in one text, but in no others. The omissions in the table falsify in the sense that no notice is given to inclusion of a very few scattered identifications, appearing in a single text.

Some of the items or ideas appearing in this table may appear either trivial or actually wrong to the reader. It is to be remembered, however, that the instructional materials studied are the attempts to build a body of meaningful thought about percentage.

TABLE XV.—PROVISION FOR INSTRUCTION IN TWENTY-THREE IDEAS OF PERCENTAGE IN TEN SERIES OF ARITHMETIC TEXTS*
(Figures show average times idea is presented)

1.	Sign = %	2
2.	% means percent	34
3.	Hundredths means %	2
4.	% means hundredths	5
5.	"% of" means hundredths times	19
6.	"Percent of" = fraction times	2
7.	"Of" means times	5
8.	% may be translated to a decimal	1
9.	% may be translated to a common fraction	1
10.	Percentage = the fraction, $\frac{}{100}$	1
11.	% = decimal fraction (with denominator 100)	1
12.	Percentage is the application of decimal fractions	1
13.	"What % of" means "what fraction of"	5
14.	Percent written in words†	1
15.	Base defined	1
16.	Percentage defined	1
17.	Amount defined	1
18.	Difference defined	1
19.	Rate defined	1
20.	Estimating results and answers (as 19% = approximately 20% = 1/5)	1
21.	Hundredths = Percentage	1
22.	% stands for two decimal places	1
23.	A fraction can stand for %	1
	Number of Ideas Taught	23
	Average Number of Presentations per Idea Taught	3

*This study is based on ten texts rather than eight texts used in the report on whole numbers. This increase in number of texts used was due to the fact that two series of arithmetics appeared on the market after the work on whole numbers was completed, but in time for inclusion in the study of percentage.

†No credit was given unless both the number and the word "percent" were written out.

a. An Estimate of Current Instruction on Ideas of Percentage.—Typical instructional material does mention most, if not all, of the items classified as 'Ideas of Percentage,' but it provides for little repetition. The theory seems to be: Assuming that children need to have but few repetitions in instruction in order to profit by such instruction, a satisfactory teaching method is to call attention to an idea but a few times. A more subtle aspect of the theory practiced, but an important one nevertheless, is: An idea may be presented satisfactorily in but a few forms. Variety of statement, which is a function of many repetitions, is not needed in instruction on percentage.

Variations in texts on the matter of repetition or emphasis is marked. While the central tendency of repetition per item, or ele-

ment, was 3, some texts provided far less than this and other texts provided much more. It will be noted that as a general rule texts provide significantly less teaching in the field of percentage than in the case of processes presented in the lower grades. With few exceptions, the higher up in the grades instructional material in texts is studied, the more terse and compressed becomes the instructional material. It is quite probable that texts are constructed on the theory that pupils develop in their abilities to profit by encyclopedic instruction much faster than the psychological facts of mental development justify. Or, stating the matter in another way, instructional material in the lower grades seems to be constructed with much more care than is corresponding material for the upper grades. The effects of this decrease in instructional adequacy of texts as the higher grades are reached may suggest to many one cause for relatively unsatisfactory work, especially in the case of percentage.

3. Instruction in Ideas of Wholes and Parts

The numbers in the body of Table XVI refer to the average number of specific identifications in the ten texts.

In Item 13, it is to be noted that the same fraction was not used in all cases. In no text was an unusual fraction given. One-half

TABLE XVI.—PROVISION FOR INSTRUCTION IN IDEAS OF WHOLES AND PARTS
(Average in Ten Arithmetic Texts)

1. 100 = one whole	2
2. Less than 100% is less than one whole	1
3. More than 100% is more than one whole	1
4. 100 times the number equals whole amount or 100% of number	1
5. 100 hundredths = one whole	2
6. 100% = whole	1
7. 200% = twice whole	1
8. 300% = 3 times whole	1
9. 400% = 4 times whole	1
10. 100% = 100/100	1
11. 100% = 1	1
12. 1 = 100%	1
13. Fraction of whole = equivalent % (½ of whole = 50%, ¼ of whole = 25%)	2
14. 50% more than number = 150% of number	1
15. 10% less than number = 90% of number	1
16. 4/4, 6/6, etc. = 1 or whole or 100% of anything	1
Number of ideas taught	16
Average mention per item	1

was the most usual fraction to appear. It is noteworthy that the average mention was but once. Even the text providing the greatest emphasis gave an emphasis of but four repetitions per item on the average.

4. Instruction in Changing Decimals to Percents

The material in Table XVII is presented on the same plan as that in Table XVI.

TABLE XVII.—PROVISION FOR INSTRUCTION IN CHANGING DECIMALS TO EQUIVALENT PERCENTS
(Average in Ten Arithmetic Texts)

1. Two-place decimals, as .04 = 4%, .12 = 12%	10
2. Two-place decimals with fractions, as .12½ = 12½%	5
3. One-place decimals, as .8 = 80%	2
4. Three-place decimals, as 625 = 62½%, .005 = ½%	1
5. Three-place decimals, as .625 = 62.5%	3
6. Four-place decimals, as .0725 = 7.25%, .0725 = 7¼%	1
7. Decimal greater than 100%, as 1.75 = 175%	2
8. Decimal greater than 100%, as 1.5 = 150%	1
9. Rule for changing from decimal to %	2
10. Changing whole numbers to %, as 2 × n = 200% of n	2
11. Changing mixed numbers to %, as 1½ × n = 150% of n	1
12. Changing mixed numbers to %, as 1⅛ × n = 112½% of n	1
13. Changing mixed numbers to %, as 3/2 × n = 150% of n	1
Number of items taught	13
Average mention per item	2

An inspection of error in percentage relative to changing a one-place decimal to an equivalent percent is of note in connection with Item 3. Pupils have great difficulty in translating a decimal such as .8 to its equivalent, 80 percent. Instruction on either item is lacking in seven of the ten texts studied. The reader may find it profitable to study the difficulties presented in Chapter XII of Part II with the omissions in teaching presented in the tables of the present section.

It will also be noted that little specific instruction is provided on changing whole and mixed numbers to equivalent percent forms (Items 10-13).

5. Instruction in Other and Related Changes

It will not be possible to record here the verifiable data relative to items involved in the broad skills of (1) changing common frac-

tions to decimal forms, (2) changing percents to fraction forms, (3) changing percents to decimal forms, or (4) certain other relationships of a miscellaneous nature, such as certain ratios and addition and subtraction of percents. Suffice to say that a study of such skills and of typical instruction upon them yields impressions similar to those given in Table XVII for instruction upon the changing of decimals to equivalent percents.

6. Instruction in the Three Cases of Percentage

a. Instruction in Case One.—The main factors of instruction in the first case of percentage are reported in Table XVIII. The num-

TABLE XVIII.—PROVISION FOR INSTRUCTION IN CASE ONE (TO FIND A PERCENT OF A NUMBER)
(Average in Ten Arithmetic Texts)

1. Where % is changed to a two-place decimal multiplier, as 35% = .35	
a. Answer completed	6
b. Method suggested	5
2. Where % is changed to a two-place decimal plus a fraction as a multiplier, as 2¼% = .02¼	
a. Answer completed	1
b. Method suggested	1
3. Where % is changed to a three-place decimal multiplier, as 3½ % = .035	
a. Answer completed	1
b. Method suggested	1
4. Where % is changed to a four-place decimal multiplier, as 7.46% = .0746	
a. Answer completed	1
b. Method suggested	1
5. Where % is changed to a common fraction, as 50% = ½	
a. Answer completed	2
b. Method suggested	1
6. Where % is changed to a common fraction, as 37½% = ⅜	
a. Answer completed	1
b. Method suggested	2
7. Where % greater than 100% or 100% is turned into a decimal multiplier, as 125% = 1.25	
a. Answer completed	2
b. Method suggested	2
8. Where % greater than 100% is turned into a fractional equivalent, as 133⅓% = 4/3	
a. Answer completed	1
9. Rule	2
Number of items taught	16
Average mentions per item	2

bers in the body of the table represent the number of specific repetitions provided.

b. Instruction in Cases Two and Three.—An analysis of the skills involved in Case Two and Case Three and a report of instruction upon them will show that typical teaching of them involves relatively little repetition on each aspect, but that most aspects are mentioned. Almost uniformly, however, attention to these cases is not as adequate as is the attention paid to Case One. While data for Case Two and Case Three are omitted, summaries of them are contained in the next table.

7. Summary of Instruction in Percentage

Using Table XVIII and similar tables not here reported, the summary of instruction in percentage shown in Table XIX may be made.

TABLE XIX.—SUMMARY OF PROVISION FOR INSTRUCTION IN PERCENTAGE
(Average in Ten Texts)

Main Heads	Number of Skills*	Average Text
Total	218	66
Percent of Elements Identified	....	30

*The total number of specific elements which were identified in the learning analysis upon which this investigation is based.

It may be noted here that no text identifies all the elements involved in the total process of percentage. When an element is not identified, the assumption must be either that the child will infer that element from others closely allied which are identified or that the teacher will supply such identification. In the absence of data informing us about the facts of inferential powers of children in the field of percentage or the facts relative to the contributions of teachers, it will seem to some readers that present instruction in percentage takes too much for granted. This opinion is further enforced by a perusal of the data on typical performances of pupils in the process as reported elsewhere in this Yearbook. To the writer, the disappointing results (in pupil performance) are directly traceable, in part at least, to the obviously inadequate instruction. Most of the instructional material studied had a method of teaching percentage

defensible from the standpoint of mathematics, but not from the standpoint of a methodology of learning. Most of the texts were probably satisfactory for adults, but probably not satisfactory for children.

If, for the moment, percentage is considered as a complex of some 218 elements all woven into a hierarchy, and if it is further granted that huge amounts of facilitation exist between many of these elements, even then it will seem highly questionable to many if instruction based on but 66 elements (or about a third of the total) represents sufficient instructional material. Still, this is the average basis in ten current texts. Some texts, it is true, provide a much broader basis, identifying well over 50 percent of the elements, but, of course, other texts provide far less than the average. In addition to teaching from a relatively narrow base, most texts provide very little instructional emphasis on the elements they do teach. That the preparation of instructional material in percentage has yet to receive the serious attention of many textbook authors will be perfectly obvious to any reader who will go beyond the present report and study the instructional material itself. Here, then, is one section of arithmetic standing in need of radical improvement.

8. The Specific Material Used in Instruction in Percentage

The following data deal with the actual numbers used in instructional material. What decimals, whole numbers, and mixed numbers are actually changed to percentage equivalents in instructional material? It is curious to note that there is not a single decimal which all texts change to its percentage equivalent. For example, only one text actually tells the pupils that .00125 of N equals $\frac{1}{8}\%$ of N. But a single text identifies .01 of N as 1% of N. All texts identify several decimal-percent equivalents, but there is little agreement as to which decimals should be specifically identified. Decimals used in instruction to show percentage equivalents range from .00125 to 4.5. One hundred and twenty different decimals are used, but only three of these are identified in five of the ten texts studied. These are .20 of N equals 20% of N, .25 of N equals 25% of N, and .75 of N equals 75% of N. These three identifications represent the greatest agreement between the texts studied. The average of ten texts identifies 25 percent-decimal equivalents. The range is from 4 to 65.

Data relative to the actual quantities used in instruction on decimal-percent, fraction-percent, and percent-fraction relationships

amply demonstrate the wide variations which exist in instructional material in the choice of the actual numbers used in such material. While it is probably true that there is no list in which all would agree each number should appear in instructional material, it is more than possible that there are many numbers so frequently used that they could well appear in all texts. Their use is doubtless more important than the almost sacred apples and pies that regularly appear in all textbook explanations of fractions. Fractional apples and pies become the common heritage of our youth, but in the case of the numbers to use in teaching the changing from fraction and decimal forms to percent forms, no agreement is found in just the places where common experience might prove of real value.

9. The Technical Vocabulary of Percentage

It is difficult to present the technical vocabulary of percentage as a process in distinction from arithmetic in general or the social uses of percentage (which do not come within the scope of this study). However, there are one hundred forty-two words which may be considered as forming a reasonable technical vocabulary, including some which are not strictly percentage-process words. Here, at least, is found some agreement. All texts used the following words of a technical vocabulary: *decimal, fraction, hundredths, percentage, percent,* but no others were used by all ten texts. Fifty-four of the 142 in our list were used by not more than two texts. There are only 35 words used by fifty percent of the texts. The average text uses 55 technical words in its instructional material in percentage. The range in vocabulary is from 32 to 79. Whether to use but few words of a technical nature is a wise economy or not, and whether a rich vocabulary is a burden or a help to the learner of percentage need not be discussed here. The point to be emphasized now is that in the teaching of percentage there is little or no agreement relative to the extent or nature of the vocabulary with which the child must do his thinking. Perhaps there should be little agreement, since there is practically no dependable experimental evidence bearing on the problem.

10. The Gross Amount of Instruction in Percentage

Scantness of instruction is, of course, highly correlated with small amount of space and paucity of explanatory material. Fullness of

instruction is correlated with large amount of space and fullness of discussion. Table XX reports gross but useful data.

TABLE XX.—GROSS AMOUNTS OF INSTRUCTIONAL CONTENT DEALING WITH PERCENTAGE
(Average of Ten Texts)

Total Area, (in square inches)	Pictorial Matter, (in square inches)	Number of Words in Discussion	Number of Figures and Symbols
353	5	2286	1466

The variations from the central tendency reported in Table XX are large. One of the texts studied uses above seven times as much area and six times as many words as another and about three times as much area and as many words as the average. If all texts are of equal merit, it can literally be said that 'every which way' is a good way to teach percentage. It would be of interest to compare the number of words a text consumes in teaching percentage with the number of words an experienced teacher will use in accomplishing the same objective. One text provides the number of words a teacher, speaking quite slowly, would speak in about 12 minutes. The average of ten texts provides the number of words a teacher would speak in about 25 minutes. No teacher would agree to do a good job of teaching with 25 minutes of oral presentation allowed, but the equivalent of 25 minutes oral presentation is the central tendency of instructional material in percentage in current texts. It is suggested that the whole problem of instructional materials in percentage be carefully reviewed by those responsible for them in order to make very sure that present provisions are satisfactory.

11. The General Adequacy of Instruction in Percentage

If crucial data are summarized from various tables already presented, the reader can judge for himself the adequacy of instruction in percentage in representative textbooks. Such a summary is presented in Table XXI. This table probably overstates the adequacy of instruction, because each single identification is recorded, without any estimate as to how well the elements were identified. The adequacy of current instruction can certainly be no better than sug-

gested in the table; it is probably distinctly worse, especially with respect to instruction in the main ideas and procedures (first row). It is quite possible than any percent larger than twenty-five in the table represents fairly adequate identification. The variations from the averages reported are significantly large in both directions in the case of individual texts.

TABLE XXI.—PERCENT OF ELEMENTS OF IMPORTANT ASPECTS OF PERCENTAGE SPECIFICALLY IDENTIFIED IN INSTRUCTIONAL MATERIAL
(Average of Ten Texts)

Instruction in Ideas and Procedures (Table XV)	30
Identification of Decimal-Percent Equivalents (Table XVII)	14
Identification of Fraction-Percent Equivalents*	20
Identification of Percent-Fraction Equivalents*	23
Identification of Percent-Decimal Equivalents*	17

*Not reported in other tables.

XI. SUMMARY ON INSTRUCTIONAL MATERIALS

Instruction as such is an integral part of method. The preceding sections of this chapter (VI-X) contain several points of view relative to instruction which seem of prime importance. These pages have presented typical learning analyses of arithmetical processes and have reported verifiable data on the present status of instructional material.

It is obvious that many aspects of method have received no attention whatever. Thus, vital considerations of the quality of instruction must in due course supplement studies of its quantity alone. It is further obvious that the data in the foregoing pages which deal with printed instructional material are but partial in nature. The data printed here represent about one-fifth of the total data carefully obtained to support the points of view presented. A full report on status of instruction should deal also with the actual contributions of the teacher—with the contributions which she is asked to make, or assumed to make, or which she might make. Further, a full report on instruction should include the contributions to actual learning situations made by courses of study. Many of these, especially the newer ones like the Denver Course, contain within their pages commendable method. No data are yet available, however, to show just how much of a course of study written for teachers functions for pupils. Of course, no data are yet available to show how much text material

functions for pupils, but it at least is in the pupils' hands and can be made the material for contact between pupil and teacher as no courses of study or method books can be made.

The attitude toward method in arithmetical instruction which has been presented in the foregoing pages is, in short:

1. Instruction should be based on a full analysis of the topics taught. Such analysis should be both a learning analysis and a mathematical analysis.

2. The construction of learning material (printed or oral, in method books or in courses of study, in the hands of the teacher or in the hands of the pupils) and the appraisal of such material should be largely influenced by the analyses of the processes concerned.

3. It is useful, when striving for improvement, to know the present status of the teaching situation. Some knowledge of the present status can be gained by an impersonal and objective study of typical printed instructional material.

XII. Method in Drill: Some General Considerations

We deal next with a few of many vital issues relative to effective drill.

Practice is but one of several prime factors in teaching and learning. Its importance is axiomatic, and no space need be given to arguments for it. Our present need is not an increased faith in practice, but increased skill in constructing good drill situations and more information bearing on the relative merits of various alternatives used in applying the principle of drill to concrete learning enterprises.

1. Two Purposes of Drill

For the discussion here, two major purposes of drill may be mentioned: (1) drill to build a skill, and (2) drill to maintain a skill. This division is somewhat arbitrary, since in most cases drill to build a given skill may help indirectly to maintain other previously learned ones. But in the practical situation the evidence does not warrant much confidence in a dependence for maintenance of x on incidental practice of x while learning y. Subtle interrelationships between multiplication and division, for example, might be worked out in drill material, so that when practicing division sufficient practice on multiplication would come as an extra booty, but such

refinements do not now exist. In our present stage of control over forgetting, it seems best to use multiplication for maintaining skill in multiplication and to use division for maintaining skill in division. There exists, then, in the classroom, the problem of maintaining previously learned skills as well as teaching new ones and applying all of them to problem situations.

The discussion which follows is concerned mostly with certain considerations of drill to maintain skills after they have been learned. Of course, much drill serves the purposes of learning in contrast to maintaining. Such learning drill should, of course, follow adequate explanation, since even learning drill based on inadequate or faulty instruction may all too often be simply practice in error, and that has no known virtues whatever. Sheer drill work, either to learn or to maintain, has little social value in itself. But the solution of a problem in arithmetic requires accurate computation. Skill in computation must be built into the neuro-muscular mechanisms of the worker if his good intentions to solve a problem accurately are to be followed by satisfactory performance. That drill as such contributes to skill is assumed in this discussion. It is more desirable to assume the benefits of drill properly built than the benefits of any type of drill providing there are large amounts of it. In thinking about the effects of drill, one should insist that quantity is never the sole consideration. The quality of the drill which can be accurately described in terms of 'specifications' is perhaps of major importance.

But few systematic discussions relative to the maintaining of skills as a problem of great practical import are available. One of these[15] (which need not be repeated here) attempts to lay down a list of specifics relative to effective maintenance programs. Some data reported in the present Yearbook deal with three of these specifications, namely: the alternative of mixed versus isolated drill material, the alternative of standardized versus unstandardized drill material, and the alternative of distributed versus undistributed practice on the items contained in any body of drill material.

It is also useful to study current provisions for drill. It is possible that these provisions do observe faithfully all, or a major part, of the best theory of drill. If so, all that need be said about drill is that present practices relative to it accord with available theory and

[15] *Third Yearbook of the Department of Superintendence, National Education Association*, 1925, pp. 63-91.

ideals, and the practical worker may consider this aspect of method to be in satisfactory condition. Summaries of verifiable facts relative to what current material provides in the way of drill material in the case of whole numbers, fractions, and percentages, may, however, show that much improvement in actual drill situations is still possible.

2. Typical Alternatives in Drill

When one attempts to make concrete decisions about types of material to use in programs of maintenance, he is impressed with the number of choices he must make. For example (1) He may provide review drills frequently or infrequently. (2) He may make the maintenance program systematic, in the sense of being the object of endeavor, say, every Friday or every other Friday, or he may provide reviews whenever the spirit moves. (3) He may make the review work oral or written or some combination of oral and written work. (4) He may organize drill so that at any one time the pupil confines his attention to a single process or he may construct mixed drills. (5) He may arrange examples within a drill unit in order of difficulty or he may pay little heed to the specification of difficulty of individual examples. (6) He may provide meaningful standards with each drill unit, or no standards, or some compromise statement relative to goals of accomplishment. (7) He may give much drill on certain processes and slight others or he may carefully attend to proper emphasis on all processes he desires maintained. (8) He may confine his maintenance drill program to computation only or he may more or less systematically review the ideas, principles, and thought aspects of the processes. (9) He may organize his drill in but one or two forms, running the risk of the doubtful effects of monotony of presentation, or he may invest time and effort in presenting review work in various ways, on the assumption that variety of organization is a useful principle of drill construction. (10) He may organize his drill methods to provide a flexibility of content to function in relation to individual needs or he may use the principle of 'one for all, and all for one,' in the sense that any one drill unit should be used by all pupils and that all pupils should use every one of the available drill units. (11) He may organize drill in such a fashion that all pupils use a minimal array of materials and that other materials be available for individual work in relation to individual needs. (12) He may set up situations in which testing—maintaining—remedial units form a

feasible pattern of endeavor meaningful to both teacher and pupil or he may never take time to think through to the classroom actuality the possibilities of dynamic unity of testing—maintaining—remedial programs, in which case he would insist that his drill is 'informal,' whereas another would judge it to be mere careless hodge-podge. (13) He may use drill material carefully constructed in anticipation of possible needs or he may construct it under the pressure of immediate emergency.

It is easily seen that when a pupil works upon maintenance drill material he addresses himself to specific drill material at the moment, and that every bit of such drill material is built, consciously or unconsciously, on decisions with respect to many alternatives of drill construction. That much drill work is patently haphazard, and that much other drill material has only a superficial varnish of presumably desirable characteristics is only too obvious to those who study the provisions for drill at the present time. We may take for granted that in the long run the differences between good and poor drill material will be settled, not by argument or *a priori* reasoning, but by experimentation.

It will be profitable to consider three of the many typical alternatives in drill construction:

1. Shall drill material contain distributed practice on all the number combinations coming in the scope of the material, or shall drill material be written without respect to the amounts of practice it contains on each combination involved?
2. Shall drill material for purposes of maintenance be in mixed or isolated form?
3. Shall drill material provide standards of accomplishment so that pupils work with a knowledge of results or shall knowledge of results be confined to test situations?

These three are representative of many questions which every teacher answers every time he provides situations for review in arithmetic. They may be discussed from various angles, three of which are: first, their meaning; second, their theory; and third, experimental data contributing to answers of them.

3. Should Drill Be Distributed or Non-Distributed?

a. The Problem.—The problem here is vital to drill construction. If quantity of drill is the main factor, it is not necessary to build

drill material with care that every number combination will appear a calculated number of times. But if quality is important, drill should be so built that each number combination appears with a calculated frequency. The issue here may be expressed as 'distributed versus non-distributed' drill.

Distributed drill is that type of drill which provides a (presumably sagacious) number of practices to each combination involved in the process being drilled upon. If the actual practice in fifty consecutive addition examples contains as much, or more, practice on the harder combinations as upon the easier, it is to that extent well distributed. If each of the harder combinations is practiced a reasonable number of times, it is to that extent well distributed. Non-distributed drill would be illustrated by a set of fifty addition examples in which some combinations did not appear at all or only a very few times. Or in the case of carrying, if fifty examples practiced the carrying of 1 sixty-five times, the carrying of 2 seventeen times, and the carrying of 3 two times, the carrying practice would be open to criticism in that it piled up practice on carrying 1 and slighted the carrying of 2 and 3. Whether any set of material distributes its practice on the combinations involved properly or poorly cannot be determined by casual inspection.

Dr. Eva M. Luse studied the effects of two such sets of drill material approximately equal in quantity, but differing in that one set contained well distributed drill and the other did not. This drill material was given to 600 fifth-grade pupils equally divided into two groups on the basis of general arithmetical ability.[16] These two groups were given fifty consecutive drill periods of fifteen minutes each. One group used drill material carefully constructed as to the distribution of practice in addition, subtraction, multiplication, and division of whole numbers. The other group used material slightly in excess as to sheer amount, but so built that certain combinations were slighted. All other conditions were held constant. The appearance of the drill sheets was such that even experienced teachers and superintendents, when asked to select the body of drill which they, upon a half hour's study, judged to be the better, chose the poorly distributed drill as

[16] Space forbids a complete description of the administration of Dr. Luse's investigation. For a complete report of this fundamental research see Luse, Eva M., *Transfer within Narrow Mental Functions (A Study of the Effects of Distributed vs. Non-distributed Drill in Arithmetic)*. College of Education Library. State University of Iowa.

often as the other. In constructing the drill all possible practice was studied and controlled. For example, the addition practice concealed in checking addition examples, the combinations used in carrying in multiplication, all addition practice in carrying in multiplication used in long division and in checking long division were controlled.

b. Addition.—An inventory of all addition used by both groups appears in the next tables.

Table XXII shows the practice given to the group which used well-distributed drill. Table XXIII shows the practice in the material which was rather poorly distributed. In both tables the units' digits only are reported. In a complete analysis, the distribution in terms of higher decades would also appear.

TABLE XXII.—A CREDITABLE DISTRIBUTION OF PRACTICE

Number Added	Number Added To										
	0	1	2	3	4	5	6	7	8	9	Total
0	33	102	70	84	89	79	43	46	43	36	625
1	47	78	101	92	91	86	80	72	80	76	803
2	26	86	85	85	78	72	77	90	68	47	714
3	35	62	67	34	58	60	50	49	54	46	515
4	31	83	55	89	68	73	60	34	52	35	580
5	34	57	73	60	70	51	53	42	41	27	508
6	36	74	73	57	79	45	60	55	48	41	568
7	23	62	39	34	50	56	46	43	44	35	432
8	41	81	56	63	58	60	57	51	48	53	568
9	35	39	58	36	41	52	36	40	56	42	435
Total....	341	724	677	634	682	634	562	522	534	438	5748

These tables are read as follows: In Table XXII zero was added to zero 33 times; zero was added to one 102 times; zero was added to two 70 times; and so on. In Table XXII good distribution is illustrated by the fact that 9+9 received 42 practices, whereas in Table XXIII poor distribution is illustrated by the fact that 9 + 9 received

only 11 practices. It is important to note that the drill material in Table XXIII amounted to 7,290 specific practices, or 1,542 more than the better distributed drill reported in Table XXII.

c. Other Operations.—The arrangements for subtraction, multiplication, and division were similar to those for addition.

d. The Issue.—Dr. Luse had a clean-cut issue. Here are 600 fifth-grade pupils who since the second grade have had approximately

TABLE XXIII.—A DISTRIBUTION OF PRACTICE ILLUSTRATING LACK OF EMPHASIS UPON DIFFICULT COMBINATIONS

Number Added	Number Added To										
	0	1	2	3	4	5	6	7	8	9	Total
0	131	242	211	97	87	64	74	47	65	51	1069
1	112	245	217	187	154	92	108	75	60	49	1299
2	76	220	179	122	67	59	42	31	18	14	828
3	58	155	115	96	72	25	30	15	16	21	603
4	62	143	158	104	72	49	34	22	19	10	673
5	67	191	168	117	75	68	34	21	3	17	761
6	69	208	112	90	57	30	32	17	13	12	640
7	67	124	117	66	42	21	20	18	12	9	496
8	60	173	137	50	51	17	20	23	16	4	551
9	60	106	63	51	31	14	17	9	8	11	370
Total....	762	1807	1477	980	708	439	411	278	230	198	7290

equal experience with the four fundamental processes involving whole numbers. They are now divided into two equal groups and for twelve and one-half hours the experience of one group varies from that of the other in the matter of frequency with which the number combinations are present in all drill work, with the pupils using poorly distributed drill receiving over 1,500 more gross practices.

e. The Testing.—After the fifty consecutive drill periods, the two groups were given series of tests. One series of tests was given a few days after the drill periods and another three months later. Each series was so built that one part contained combinations upon

which both groups had received much practice; another part contained many combinations that one group had practiced liberally but the other not so much; and a third part contained combinations upon which one of the groups had received relatively little practice. This made it possible not only to look for gross mean differences, but also to study the flow of the size of difference as examples based on disparity of practive relatively increased.

Further, each group was divided into fifths on the basis of performance on tests given at the beginning of the investigation. This made it possible to inquire into the relative effects of distributed versus non-distributed drill on different levels of general arithmetical competence.

f. The Outcome.[17]—The more pertinent of Dr. Luse's findings are quoted from her study:

> 1. Both groups made a decided gain from the fifty periods of drill. The gain for the distributed drill was from 19.6 to 53.7 percent in attempts and from 31.1 to 84.8 percent in rights. The gain from the haphazard type of drills was from 11.2 to 39.8 percent in attempts and from 13.3 to 60.8 percent in rights.
>
> 2. The distributed drill gave an excess over the non-distributed drill in examples solved correctly of 17.7 percent in addition, 18.8 percent in subtraction, 35 percent in multiplication, and 23.9 percent in division.
>
> 3. The same relative differences in gain from the distributed and non-distributed drill held for the different levels of ability as for the whole group.
>
> 4. The residuum after the summer vacation in actual number of examples was greater for the distributed drill group.
>
> 5. In the comparisons made there were 126 opportunities for either form of drill to yield results higher, equal to, or lower than the other form of drill. The distributed drill excelled in 120 and the non-distributed in 6 of these opportunities. Of the 120 in which the distributed drill excelled, 75 showed statistically significant differences (differences of the means varying from 3 to 12.9 times the P. E. of the difference). Of the 6 cases in which the non-distributed drill excelled, no one of the differences was statistically significant.

[17] These data were previously reported in the *Second Yearbook of the National Council of Teachers of Mathematics*, pp. 52 ff.

Most readers will agree that Dr. Luse's research virtually closes the issue of distributed versus non-distributed drill, not, of course, by establishing the particular distribution used by Dr. Luse, but by establishing the more general theorem of careful rather than careless distribution.

The present tendency of writers of drill material to pay far more heed than formerly to the matter of distribution of practice on all number combinations is to be commended.[13] Unfortunately, this tendency, while evident in the case of whole numbers, is still hardly apparent in the case of much material in fractions and percentage.

4. Should Drill Be of the Isolated or Mixed Form?

It is perfectly possible to present review units in such a way that each of a series will deal with but one process, as one unit with addition, another with subtraction, and so on. Exactly the same material may be organized so that each of several units will contain a few examples on addition, a few on subtraction, and so on.

Eight examples in each process: addition, subtraction, multiplication, and division are organized into four drill units of the isolated drill type as suggested below:

UNIT I	UNIT II	UNIT III	UNIT IV
8 Examples All Addition	8 Examples All Subtraction	8 Examples All Multiplication	8 Examples All Division

The same examples would be organized in units of the mixed drill type as suggested below:

UNIT I	UNIT II	UNIT III	UNIT IV
2 Ex. Addition 2 Ex. Subtraction 2 Ex. Multiplication 2 Ex. Division	2 Ex. Addition 2 Ex. Subtraction 2 Ex. Multiplication 2 Ex. Division	2 Ex. Addition 2 Ex. Subtraction 2 Ex. Multiplication 2 Ex. Division	2 Ex. Addition 2 Ex. Subtraction 2 Ex. Multiplication 2 Ex. Division

[13]For the facts relative to distribution of drill in representative textbooks and attempts at objective technique for appraising them see: Lutes, O. S., and Samuelson, Agnes, *A Method of Rating the Drill Provisions in Arithmetic Textbooks*, (Monograph in Education No. 3, State University of Iowa). Discussions on methods of rating text and drill material are omitted here since a forthcoming Yearbook is to treat the textbook and its selection.

Here the word 'test' implies that the results of pupils' work on the material so described are to be marked or graded. Marking and grading implies comparison or reference to some standard. Quite naturally the word 'test' has come to be usefully attached to materials possessing some type of standards. But as soon as standards are suggested, comparable conditions of work are important. Hence, rather by general consent, 'tests' are bodies of material with prescribed conditions of work and standards supplied for comparison of results. Such organizations of material have certainly won their way. They are of advantage to everyone concerned. Hence 'tests' are popular and deservedly so. With their popularity come abuses. The main abuse is that of claiming for a body of material the name of test when the material itself does not possess in any adequate way the main characteristics of tests. Such abuses are frequent and should not be countenanced by a professional body.

c. *Time Standards.*—Time standards are charter members of the whole test and standardization organization. Scores are not comparable if one group takes ten minutes and another twenty. But many time standards in current use mislead the user because they are either futile or represent very primitive attempts in the use of this technique. Several major types of inadequacies may be listed. It is to be remembered that these types appear in a great variety of forms.

(1) Perhaps the most insincere of all the capitalization of the time-standard idea is the heading of a body of drill material with some variety of the following: "See how many of the next examples you can work in 10 minutes." The silliness of this is obvious. After the child has 'seen,' there is no answer to the real question: How many *should* I be able to work in ten minutes?

(2) An improvement, but by no means a superior provision, is some form of the following: "All pupils should be able to work five of the examples below in 10 minutes." Any such single standard is weak. If the slow child can, or should be able to, do the work in the time suggested, the time allotment is so easy for the superior child that it either disgusts him or leads him to think that he should be well pleased with performance greatly below his ability. On the other hand, if the time allowed is a real task for the average or the superior child, it is an impossible or unfair task for the slow child. Further, much of the material to which such time standards are improperly attached does, as a matter of fact, leave quite unclear which

five examples should be done in the time allowed. If any five, then all examples should be approximately of equal difficulty. Whether they are or not is rarely, if ever, determined.

(3) Other directions have as their pith the statement: "Practice on this material until you can do all 20 examples correctly in 20 minutes." While 100 percent perfect work is a goal which sounds well in lectures, anyone knows that, especially with young children, to get a large number of examples of any difficulty all correct at the same time involves excessive practice on some examples which are gotten correct time after time and are still repeated because one or a few examples contained error. It is true that most teachers have too much sense to insist on this direction being carried out, but authors should have enough sense, too, not to ask for it.

d. Performance Standards.—Three major abuses of performance standards permeate arithmetic at present.

(1) A single standard only is provided. An illustration of this is: "The standard for this test is 7 examples right in 10 minutes." There is a limited use for such a standard when the average performance of a group is alone desired, in which case the direction should be written to mean: "The *class* should average 7 examples right in 10 minutes." But single standards are not so written. It is obvious that a single standard fits only the mythical 'average pupil' in a group. For some it is too easy and may well set up habits of laziness and half effort; for others it is too hard and is therefore quite undesirable.

(2) The second abuse of performance standards is that they are the product of guesses rather than of *bona fide* experimentation. Few, if any, authors have a divine inspiration when they guess at performance standards. Less harmful than guesses, but inadequate nevertheless, are those based on too restricted a sampling of pupil performance. Test technique in general is now sufficiently stabilized to enable anyone to standardize material properly if he really wants to go to the trouble and expense of doing so. Probably few will go to such trouble and expense, however, until a critical buying public demands it.

(3) Abuses, or perhaps too hurried study of the progress-chart idea, lie at the cause of a third inadequacy. Much material now is so arranged and labelled that a child may keep his own record on a series of tests. It follows, of course, that a teacher may similarly

keep a record of the class performance. The theory is that such a record provides the child or class with the facts about growth in power and that this awareness of success (or failure) during the year, when made a systematic enterprise, introduces a valuable motivation and also tends to make arithmetic a total growing experience rather than a mere collection of rather unrelated enterprises.[10] The main idea is feasible and squares with a reputable psychology of interest and effort. But because the idea is good it does not follow that any particular scheme for working it will be good, or even indifferent, in effect. Two main dangers exist.

(a) Difficulty in administration. When the child is asked to keep a system of records over a period of time, certain practical classroom conditions are to be considered. Unless the technique itself makes ample provision for economical, neat, and quick record-keeping, the result is a good start, then laxity, then forgetfulness. Many readers of this discussion will agree that it is difficult for adults to keep records when data are inserted in the record only occasionally. When the child is left with only the suggestion to keep a record but with no record book or card, ultimate failure may be predicted. In fact, typical record-keeping, unless the method involved is very carefully thought through, becomes like many New Year's resolutions which have enthusiastic beginnings but badly frayed endings.

(b) When scores on a series of tests are to be recorded, the record made may give a picture of progress (or its absence), but the picture may be so distorted that little real good should be expected. Only when the drill units or tests involved are so standardized that every score made represents the true measure of the pupil's gain or loss, can one expect really meaningful results. It is in failure properly to standardize such related units that an obvious abuse exists. This is clear from an example, which illustrates a common misuse of data.

Suppose that the pupil is provided with a series of ten units, each of ten examples. These units are to be taken through the year and a record of progress made as a systematic enterprise. Further, suppose that the child's score shall be the number of examples he gets correct. For any test on which he gets 7 examples correct he will record a 7 on his record card. Now if the tests are all equal in difficulty, a score of 7 on one test will be comparable to a score of 7 on another. As a matter of precision, it is better to say that if all tests are equal in difficulty for the pupil at the time when he takes the different tests, and if the flow of

[10] See Part II, Chapter XI, for experimental data supporting this theory.

difficulty in all tests is the same so that the first quarter of any test is of equal difficulty with the first quarter of any other, then the scores are comparable and the record will show what it pretends to show—progress as it is made. But what are the facts? A set of 10 examples in the addition of whole numbers at the beginning of the year should be considered equal in difficulty to a set of 10 examples in the division of fractions placed in the middle of the year only when an adequate amount of experimentation has been indulged in to demonstrate such equality. What actually happens is that the tests used vary in difficulty, and the scores made are a function of variability of difficulty of the material as well as a function of the growth of the child's power over arithmetic. He seems to get a record of progress, but it is a fictitious one unless equality of the material is provided. Everyone familiar with the construction of test items knows that to construct a series of drill units all of the same difficulty, each on a different process, and all of them coming at different times in the year, while not impossible theoretically, is so practically. A score of seven, then, means something quite unknown about the child's progress, plus some differences in the difficulty of the materials, but how much of either is a mystery. The graph goes up and the child thinks he is gaining; he may have hit upon an easy unit. The graph goes down and he thinks he is losing, but as a matter of fact he may be gaining, for the unit may have been even harder than his loss of score suggests.

The assumption that one can build drills for use with such a scoring device in connection with the progress chart idea is, then, a bit presumptuous; yet one of the sets of arithmetic materials widely used to-day so fits exercises together not only for a year, but for several years; and further, fits together drill exercises, some with 5 examples, some with 6 examples, some with 7 examples, and so on, and complicates the matter yet further by having some units deal with a single process and others deal with several processes in mixed order.

The idea of a record of progress is theoretically sound but abuses of it have great probability of doing as much harm as good because the scores on the record are a fusion of variable difficulty of material as well as a measure of the pupil's power. It is possible by other techniques so to organize a series of drills that the scores made will reflect growth of power with substantial reliability.

e. What May Be Done.—When a series of units or tests is properly standardized, inequalities in the difficulty of the units may be ironed out by having a specified, though varying, number of examples correct stand for a given rating. This number can be determined by statistical treatment of experimental data. Thus, in the ex-

periment reported in Chapter XI the standardized material used had been properly made. In it a rating of 5 meant the same thing on every test—the number of examples that the middle ten percent of an adequate sampling of pupils had actually done correctly when working the drills under usual classroom conditions. This, it is repeated, is quite a different thing from letting a rating of five represent the *same* number of examples correct on all drills, irrespective of differences in their difficulty.

To summarize: In selecting or constructing drill material a choice must be made between standardized and unstandardized material. Technical terms, such as 'standards,' 'test,' and 'diagnostic' should have definite meanings and not be used in careless and misleading ways. At the present time much material formerly called simply 'drill material' is miscalled by names which suggest that the material possesses certain characteristics which it does not possess.

6. Drill Material and the Supervisor's Needs

It is quite possible that many workers in the field of arithmetic, especially supervisors who must keep in mind the progress of many classes, will desire to consider the possibilities of the alternatives of drill organization just discussed from the standpoint of their bearing upon supervisory techniques as well as their bearing upon learning and skill.

That drill material which is unstandardized has very limited use as a means of supplying data crucial to objective types of supervision is evident. Should exactly the same material have added to it thoroughly dependable standards, then data based upon the ratings of pupils and upon class averages are furnished as a sort of extra booty, which may form the basis of a perpetual inventory. Such data yield information useful in supervision. Data based on standardized drills coming frequently throughout the year yield information at times when it can best be used to affect the life of the classroom. Such data gathered from regular drill periods are obtained at no extra cost in time on the part of the pupil and negligible amounts of time on the part of the teacher.

From the supervisory standpoint there are administrative advantages in mixed versus isolated drills. A maintenance program based on the use of isolated drills will be able to review any one topic rather infrequently. This necessitates long periods of no practice

on any process and probably increases the amounts of remedial work needed, since forgetting between drills may cause weakness. Long periods between drills on any one process also involve the postponing of specific remedial work long past the times when such work is most advantageous, since it is axiomatic that the less time that elapses between a need for remedial work and its provision, the better for all concerned. Mixed drill provides information upon strengths and weaknesses on all processes much more frequently, and hence gets actual breakdowns in skill and remedial work when needed much closer together. While the advantages to the pupil of standardized mixed drills over unstandardized isolated drills have strong experimental support, other supervisory advantages should of course affect the supervisor in his choice of types of drill material.

7. Summary

In this section (XII) which deals with certain aspects of drill material no attempt has been made to cover in a systematic fashion all the problems involved in a complete discussion of the topic. It has seemed best to refer to substantial data bearing on a few important issues and postpone discussion of the total field until later.

It seems clear that convincing experimental evidence exists in support of certain desirable characteristics of drill material used for maintenance purposes. These characteristics are: (1) well-distributed practice, (2) a mixed type of drill organization, and (3) the use of *bona fide* standards of accomplishment as a means of providing both individuals and groups with dependable information about success or failure.

Many of those responsible for the teaching of arithmetic will be impressed with two facts: (1) that alternative drill materials which appear on the surface to be practically identical may in truth be quite different in actual potency, and (2) that the construction of drill material of merit requires some measure of expert knowledge and much patience and accuracy.

XIII. Analyses of Provisions for Drill

Drill is the phase of arithmetic that has probably received considerably more attention than have companion phases such as instruction, applications of computational ability to problem situations, remedial programs, and the like. Until very recently methods of

teaching arithmetic had proceeded little farther than considerations of quantity of drill. The supplying of more drill was, and in a measure still is, the chief therapeutic treatment for mathematical ills. One does not proceed very far in a study of the learning problems involved to be impressed with the fact that the quantity of drill material is but one of many aspects with which method in arithmetic must be concerned.

In the pages which follow, verifiable data relative to actual provisions for drill are presented. The reasons for this sort of textbook analysis have already been set forth.

1. Drill in Whole Numbers

a. Previous Work.—There have appeared in the literature several analyses of the drill in whole numbers provided in typical textbooks and in material such as pads, workbooks, and the like. Many committees responsible for appraising textbooks in arithmetic have made thorough studies of drill provisions in texts considered by them. Sufficient material has already appeared to make the well-informed reader familiar with many, if not all, of the major problems relating to the appraisal of drill in whole numbers. It should also be said that during the past few years, the drill material appearing on the market has, in general, shown definite evidences of better quality. Proper distribution of practice, for example, practically neglected even ten years ago, is at least claimed for all the newer drill materials.

b. Rationale of Rating Drill.—Perhaps the most ambitious attempt to describe drill material in whole numbers in a quantitative way is the work of Lutes and Samuelson. In their monograph[20] a definite series of credits and demerits was set up, each of which was directed toward a presumably important aspect of drill material. By this device (or one similar to it), it becomes possible to give objective scores to the drill material in any set of books or pads rated.

A brief discussion of the general rationale of rating drill provisions will serve our purposes here.

> Vague verbal descriptions of the merits of drill material in contrast to objective and verifiable analysis is the general pitfall in estimating drill material. Other pitfalls are outgrowths of this chief one. Certain naïve errors in estimating drill material deserve consideration.

[20]Lutes, O. S. and Samuelson, Agnes; *A Method for Rating the Drill Provisions in Arithmetic Textbooks.* College of Education, University of Iowa, Iowa City, Iowa, 1926.

1. Counting examples. A rough measure of the differences in drill material in textbooks may be gotten by counting the number of examples on any process. Thus, one text may have 400 examples on the addition of whole numbers, another 600 examples, and a third 800 examples, from which it might be uncritically inferred that the text with the 800 examples is the best of the three as far as drill on whole numbers is concerned. But from the standpoint of actual classroom conditions no such conclusion is warranted. The text with the 400 examples may give, as a matter of fact, far more practice on the harder combinations than does the text with the 800 examples. It may be quite demonstrable that 600 examples are all that any teacher can find time for, and that an additional 200 or 500 examples makes no real difference in the classroom. Four hundred examples may be all that are needed.

This estimating of drill provisions by counting examples has a long history behind it, but it has always been bad practice. It assumes that one example is equal to another, or at least that an average of many would be equivalent to a similar average, and hence the sheer number will reveal important difference. But a count will not even reveal actual quantity and is hopelessly incompetent to reveal other important characteristics. An example like (a) contains eleven practices, while an example like (b) contains six practices. Thus, 100 examples like the first contain more practice than do 180 examples of the second type. Further, counting examples leads us far astray in comparing amounts of practice provided for different processes. A beautiful example of this misinformation is in the case of addition versus subtraction. A typical drill example in addition might well be that shown at (c), a typical one in subtraction that shown at (d).

(a)	(b)
437	38
218	21
693	8
416	72
1764	139

(c)	(d)
378	30172
90	12091
217	18081
354	

It may be pointed out that if a given text has 600 examples in addition, averaging in difficulty as illustrated above, and 600 examples in subtraction averaging in difficulty the typical example here, equal practice is not provided. It would seem that the text in question does not slight subtraction. But notice that the typical addition example really provides ten and the typical subtraction example five practices. Multiplying each by 600, we get 6,000 practices in addition and 3,000 practices in subtraction. Estimation by counting examples would have led us to infer that subtraction had not been slighted. Anyone who will take the time to compare the amounts of practice given to addition and to subtraction in practically any set of drill material will be impressed with the futility of estimating drill provisions by counts of examples.

2. Failure to consider difficulty. Any scheme of estimating the worth of drill material which neglects to derive precise information relative to the nature of the skills actually used is very sure to mislead. It can easily be shown that in such a simple skill as carrying, two sets of drill material alike in quantity are unlike in the experiences given to carrying. A fair illustration would be: One text provides 256 practices in carrying 1,

87 practices in carrying 2, and 9 practices in carrying 3. If we find many errors of carrying 1 when some other number should be carried we can partly blame the material, for it has surely set up a preferred response for the carrying of 1. Another text of even less bulk might prove quite superior because it assiduously avoided stressing one carrying situation and neglecting others equally useful.

A pertinent example of misleading information of the sort we are describing may be found in long division, wherein a text might provide huge amounts of drill in long division and also (with apparent design) carefully avoid the use of those divisors computation with which does not happen to be taken care of by the rules provided in the text. Of course, the divisors thus avoided are quite as likely as any others to happen in random division work.

3. A third pitfall in estimating provision for drill is the neglect of those features of drill material, other than quantity, which we may safely assume possess definite value in provoking learning. Suppose a text does have 800 examples, but it can be shown that 150 of them are so hard that not one child in five ever works them correctly and that another 100 are so easy that they possess little stimulation for even slow pupils, should it be concluded that the text provides 800 examples or 550? Certainly, something should be known about the difficulty as well as about the amount of drill material. Again are fifty examples for review all in one place as effective a defense against forgetting as thirty-five presented in seven groups coming at twenty-page intervals? Perhaps enough has been said to indicate the answer.

c. *A Sample Objective Rating.*—Table XXIV presents an objective rating of the drill material on whole numbers in the third-grade and fourth-grade sections of typical textbooks. It will be noted that quantitative ratings are given to several aspects of drill. These are based on the work of Lutes and Samuelson,[21] with additions to include certain texts not available when their monograph was printed.

Several observations may be made on Table XXIV.

1. The ratings are based on the assignment of definite credits to different general aspects of drill.[22] Such amounts are, of course, purely arbitrary. They may be changed by any one. More important is the fact that all of the values assigned are also based on actual counts of practice. The main idea is: Having agreed on what aspects of drill to judge, the whole scheme becomes impersonal and can be repeated by another worker. Definite and verifiable facts are reported on analysis sheets and simple calculations made from them.

[21] *Op. cit.*

[22] The details of awarding credit need not be reported here since they are fully described in the original work of Lutes and Samuelson.

TABLE XIV.—RATINGS BY THE LUTES-SAMUELSON PLAN ON THE DRILL PROVISIONS IN GRADES III AND IV OF SEVEN CURRENT ARITHMETIC TEXTS

	Perfect Rating	Text Earning Smallest Score	Average of Seven Texts	Text Earning Largest Score
ADDITION				
Distribution	250	62	96	166
Bulk	125	62	83	85
4th Quadrant	75	15	62	75
Carrying	50	10	33	50
Totals	500	149	274	376
SUBTRACTION				
Distribution	250	−45	− 1	116
Bulk	125	−20	17	5
Harder 45	75	39	69	75
Unseen Minuends	50	40	39	40
Totals	500	14	123	236
MULTIPLICATION				
Distribution	250	114	117	172
Bulk	125	72	93	90
4th Quadrant	75	5	59	75
Carrying	50	20	41	50
Totals	500	211	311	387
DIVISION				
Distribution	250	93	113	156
Bulk	125	82	82	85
4th Quadrant	75	15	32	75
Carrying	50	....	17	50
Totals	500	190	244	366
Four Processes Totals	2000	564	951	1365
DRILL UNIT CREDITS				
Placement of Drills	70	....	8	56
Order of Difficulty	140	....	16	112
Use of Standards	280	....	32	224
Mixed Drills	140	....	16	112
Totals	630	....	72	504
Final Grand Totals	2630	564	1023	1869

2. The aspect of drill labeled "distribution" deals with the relative emphasis placed on the various combinations making up the total process. When certain combinations are slighted, demerits are assigned.

3. The aspect of drill labeled "bulk" deals with sheer amount. On the basis of decisions as to how much practice should be provided for each combination and on what the amount of practice is on each combination beyond which practice becomes wasteful, too little or too much practice receives demerits. Thus, if 9+9 were practiced but 5 times, rather severe demerits would be assigned. While on the other hand if, say, 1,000 practices were provided on 1+1, demerits would also be assigned, as obviously a case of waste of time. The limits set for too little and too much were generous.

4. The aspect labeled "fourth quadrant" deals with the practice given to the section of combinations which is known to be the hardest. The aspect in subtraction labeled "unseen minuends" is used to provide for practice when the actual combination is not seen. For borrowing, a 'seen' practice on 7—5 appears in this example:

$$\begin{array}{r} 657 \\ 335 \\ \hline 322 \end{array}$$

while an 'unseen' practice on 7—5 appears in this example:

$$\begin{array}{r} 283 \\ 159 \\ \hline 124 \end{array}$$

It was considered defensible to hold that drill had better be built so that all combinations receive practice in both 'seen' and 'unseen' situations. If the 'equal addition' method is used in subtraction, the same plan would hold, but the actual counting would, of course, vary.

5. Certain credits are allowed when such factors as are listed at the bottom of the table are present in drill material. In other sections of this chapter experimental evidence is presented which will defend this awarding of credit for the "Use of Standards" and for "Mixed Drills." It is presumably defensible also to award credit for the other two aspects. A drill unit whose examples are in "order of difficulty" may be assumed to provide a better drill situation than one in which examples are arranged at random. Drill material which is scattered through the book has less chance of being slighted than similar material not so placed, hence credit for "Placement."

No contention is made that the reader should agree with the amount of credit assigned to each aspect of drill, if any seem unreasonable to him. It is hoped, however, that the reader will see in Table XXIV a radically different method of estimating the merits of

drill material than those based on mere counting of examples or those based on verbal description of the provisions.

Many questions which a thoughtful reader will raise relative to the whole point of view of objective and impersonal rating of drill material are discussed fully in the monograph of Lutes and Samuelson.

2. Drill in Common Fractions and Small Mixed Numbers

The previous sections on analyses of textbooks in arithmetic have shown that pupils studying operations with whole numbers in different textbooks have distinctly different experiences. It is felt that a similar systematic treatment of fractions would next be in place in this discussion. Hence a study of drill in the addition of fractions is presented.

a. The Purpose and Scope of the Study.[23]—The purpose of this study is: (1) to show the amount, the nature, and the distribution of the practice provided in each of the grades in which addition as a process in common fractions is practiced; (2) to indicate to persons in charge of instruction in arithmetic where supplementary and remedial materials may be necessary with respect to the addition of common fractions and small mixed numbers. This will enable those interested in the work for a particular grade to study the data with reference to the grade with which they are concerned. The procedure has the additional advantage of making it possible to draw comparisons from grade to grade, thus determining the differences in the amounts and the nature of the practice provided for the various grades.

For the purposes of this study nine widely known series of textbooks were selected. The selection was determined by the widespread use of each series. These series give a fairly clear picture of the situations children are encountering in their textbooks at the present time.

The study includes all practice which could be classified as: (1) drill for the building of skills, (2) drill for the maintenance of skills, and (3) the incidental drill which is present in varying amounts in all textbooks. This procedure removes all grounds for differences of opinion as to just where the line should be drawn between the different classes of drill materials in the various textbooks, and it presents a complete picture of just what practice is present in each text.

b. The Materials Analyzed.—The directions and instructions given by the authors of the textbooks were carefully followed through-

[23]This study is the work of Ross H. Beall.

out each series of textbooks, so that the results for a given series reflect the intention of the authors and not the judgment of the writer. Only that material was included in which it was clearly evident that the use of one of the processes in common fractions was intended.

c. The Limitations of the Study.—Since all textbooks were analyzed exactly as printed, it is evident that the study has certain limitations, which are quite beyond control. There is no guarantee that the pupils use all of the practice provided in a given series of textbooks, as teachers frequently do omit portions of the practice materials given in the textbooks. On the other hand, the total learning experience with fractions for any given pupil may be greater than that reported for the textbooks he uses, because of drill that may be supplied as supplementary to the drill provided for the purpose of building skills or that may be supplied to supplement the maintenance program, thus bridging over periods when little or no practice would have occurred if the textbook had been followed as printed. No effort was made to determine the relative difficulty of the examples; that is, the analyses do not show the examples which may be so difficult that the average pupil cannot work them or so easy that they bore him needlessly to be forced to work them. The effectiveness with which the textbooks enlist the interest of the pupil in an adequate mastery of the fundamental processes in fractions has not been measured or taken into account. The study has been confined solely to the analysis of the practice provided in the process of addition of proper fractions and mixed numbers, taking the materials as they were printed.

The subsequent discussion considers for the addition of common fractions: the number of practices provided for each grade by each series of arithmetics; the method used in making the analyses of the practice; the nature of the practice provided for each grade; the distribution of the practice for each series; and summary.

d. The Number and Distribution of Practices.—The number of practices provided by these texts as they appear for each grade is indicated by Table XXV in terms of averages.

The variation in the total number of practices provided by the nine series of textbooks is not shown by Table XXV. The least number of practices provided is about one-fifth the greatest number. The second lowest series has one-half the practices of the second highest series.

TABLE XXV.—THE AVERAGE NUMBER OF PRACTICES IN ADDITION OF FRACTIONS AND MIXED NUMBERS PROVIDED BY NINE SERIES OF ARITHMETIC TEXTBOOKS

Grade	Number of Practices	Percent in Each Grade
IV	136	18.42
V	294	41.48
VI	130	21.55
VII	67	10.97
VIII	43	7.53
Total	670	99.95

It is evident from these data that authors of textbooks show little agreement in the number of practices in the addition of common fractions that should be provided by a series of textbooks. Since it is not known how much practice should be provided, it is fair to assume that serious underlearning may exist in some instances and that in other instances wasteful overpractice may exist. If 1,000 practices are actually required for a satisfactory mastery of the skills involved in the addition of fractions, then it may not be necessary to look farther for at least one of the causes for the poor results frequently secured from pupils, since many texts provide even less than half of this amount. This situation may also throw some light on the necessity for the elaborate remedial programs that are frequently necessary. On the other hand, if 200 practices are all that are required, then textbooks which require considerably more practice are diverting the pupil's time and energy from other subjects or other aspects of arithmetic. Either situation is undesirable. The one produces unsatisfactory results or involves a more elaborate remedial program than should be necessary. The other is wasteful of valuable time and energy that could be directed into other channels.

There seems to be considerable agreement as to which grade should receive the most practice in addition of fractions. Six of the nine texts provide the most practice in the sixth grade, two texts most in the fifth grade, and one text most in the fourth grade.[24] This

[24]Considerable amounts of practice, which are often bound at the back of textbooks, should be scrutinized with some care, when evaluating such practice for its effectiveness in learning. This material, while it is actually present in the textbooks, may be of very little value when the dynamics of learning in actual classroom situations are considered. Unless the authors refer constantly to such material at appropriate intervals in the textbooks, it is apt to be neglected and hence to be of little value.

tendency to place a large share of the practice in the fifth grade is also indicated by Table XXVII. Seven of the texts provide more than 30 percent of their practice in that grade.

TABLE XXVI.—PERCENTAGE OF TOTAL PRACTICE IN ADDITION OF FRACTIONS PROVIDED IN GRADE FIVE

Amount	Number of Texts
More than 20% of total	9
More than 30% of total	7
More than 40% of total	5
More than 50% of total	2
More than 60% of total	1

It is safe to assume that a good text should provide sufficient *practice for first teaching.* That all of these texts were built on that assumption is apparent from the large percentage of the total practice which is provided in the fifth grade. Doubtless most of the discrepancies appearing here are caused by authors not agreeing on the grade in which all or most of the first learning in addition of fractions should occur. There is also a clear disagreement in the amount of practice needed in the first teaching.

A problem which is just as important as that of first teaching is the *problem of maintenance.* After the first teaching of the skills, what provision should be made to see that skills stay learned? Concerning maintenance, there is apparently an even greater disagreement on the part of textbook authors than there was concerning first teaching. Judging from the amount of practice succeeding the first teaching, some authors not only feel that there is a need for a maintenance program, but also believe that the needed practice for this purpose should be incorporated in the textbooks. Other authors apparently see no need for incorporating sufficient practice in the text to maintain previously learned skills.

It is apparent from Column 2 of Table XXV that there are wide differences of opinion between authors in respect to the distribution of the total practice from grade to grade.[25] Column 2 may be thought of as the average of opinion of textbook authors as to the

[25] The distribution of practice within the grade should be considered. Texts vary in the 'bunching' or 'spreading' of drill on a given process within a grade. Data on this aspect are too space consuming for presentation here.

proper grade by grade distribution of practice. The widest differences in grade distribution are found in Grades VII and VIII. One text gives 3.5 percent of the total practice in these grades, another better than 31 percent of it. The average percent (Column 2) for these grades is 18.5.

e. Analyses of Unit Skills in Fractions.—The analyses presented in this study are based upon the unit skills which are involved in the fundamental processes in common fractions and mixed numbers. Separate charts upon which the analyses are based were constructed for each process.[26]

In constructing the analysis it was attempted to secure a chart that would be broad enough to deal competently with the main difficulties and short enough to be economical to use. More complex analyses would have been possible and desirable if instructional or teaching materials were being analyzed. Since this study is concerned only with drill material, a scheme of analysis which presents the facts concerning the more crucial unit skills will indicate the general nature of these materials and still be reasonably simple. A complete analysis of the more than 36,000 examples and problems which have been analyzed during the course of this study would hardly have been possible.

'Unit skills' are defined as the simplest skills which when fused together form a total ability in the given arithmetical process under consideration. It is comparable to the term 'unitary ability,' as used by Thorndike. Unit skills as defined here differ from 'types of situations' as used by Brueckner[27] in that types of situations involve more than one of the unit skills. A given type of situation involves a given combination of unit skills.

These analyses might have been based upon types of situations if it were known when examples are sufficiently alike to be classified as belonging to one type and when examples are sufficiently unlike to be reckoned as representative of different types. At present, 'types of situations' are not defined sufficiently to make their use feasible in a study such as this one. However, the general effect upon the reader would have been much the same, as either method of analysis

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[26] All analyses used in this study are adopted with modifications from *Problems in the Teaching of Arithmetic*, by F. B. Knight, G. M. Ruch, and E. M. Luse, Iowa Supply Co., Iowa City, Iowa, 1925.

[27] Brueckner, L. J., "A technique for analyzing distribution of drill in fractions." *Jour. of Educ. Method,* 7: May, 1928, 352-358.

would reveal which unit skills are given the greater emphasis and which are given little or slight attention.

Table XXVII presents the unit skills used as the basis for the analyses of examples and problems in the addition of common fractions and mixed numbers. In all, nineteen unit skills are included in the table. Other investigators might have chosen a somewhat different group of unit skills. For the purposes of this study, however, it is thought that these analyses present the facts with respect to drill

TABLE XXVII.—UNIT SKILLS IN ADDITION OF FRACTIONS

Unit Skills	Skill Number	Illustration*
I. The Form of the Example		
A. Numbers written in figures		
1. Addition indicated by signs	1	1/2 + 1/3
2. Column addition	2	2 1/2 3 1/4
3. Addition indicated by words	3	Verbal problems and the use of such words as add, plus, find the sum of
B. Numbers written with words		
1. Addition indicated by signs	4	One half + one third
2. Addition indicated by words	5	One half plus one third
II. Procedure		
A. Nature of the addends		
1. Proper fractions	6	1/2 + 1/3
2. Proper fraction and whole number	7	1/2 + 5
3. Proper fraction and mixed number	8	1/2 + 5 1/4
4. All mixed numbers	9	3 1/4 + 2 1/5
5. Mixed number and whole number	10	2 3/4 + 4
6. Proper fraction, whole number, mixed number	11	1/2 + 6 + 3 1/4
B. Number of addends		
1. Two addends	12	1/3 + 1/4
2. More than two addends	13	1/2 + 1/3 + 2/4
C. Addition of similar fractions	14	1/6 + 5/6
D. Reduction to similar fractions		
1. Common denominator present	15	2/3 + 1/6
2. Common denominator not present	16	2/3 + 3/4
E. Analysis of the sum		
1. Irreducible	17	1/3 + 1/3 = 2/3
2. Reducible	18	1/4 + 1/4 = 1/2
3. Carrying of units from fraction	19	2 3/4 3 2/3 6 5/12

*Each illustration contains more than the unit skill it illustrates.

materials in sufficient detail to justify valid conclusions, and that the true picture of the situations pupils are encountering in the classroom is not distorted.

f. Provision for Practice on the Unit Skills.—The data in Table XXVIII show the frequency with which the various unit skills appeared in the examples and problems allotted by the average textbook to each of the grades from IV to VIII. No effort is made to report the frequency with which the various unit skills appear in given combinations. The data show how the amount of the practice varies in the different grades.

TABLE XXVIII.—PROVISION FOR PRACTICE ON THE UNIT SKILLS IN ADDITION OF FRACTIONS IN GRADES IV TO VIII
(Average of Nine Series of Textbooks)

Unit Skill Number	Average Times Skill is Used					
	Grade					
	IV	V	VI	VII	VIII	IV-VIII
1	34	88	45	26	18	212
2	71	103	43	25	12	265
3	27	90	41	15	11	185
4	1	1	—	—	1	1
5	1	1	1	1	1	4
6	42	137	54	23	14	272
7	8	5	3	2	1	18
8	17	23	21	4	6	71
9	46	110	43	27	18	244
10	18	16	8	8	4	55
11	2	2	1	1	1	7
12	91	176	87	34	27	415
13	45	118	43	33	15	255
14	77	64	31	7	7	186
15	55	142	59	31	17	304
16	3	81	40	27	19	171
17	44	81	36	15	10	187
18	92	213	94	51	32	482
19	54	109	48	31	21	262

A study of Table XXIX raises the question: How many practices should a textbook provide for a given unit skill for adequate mastery of it. Experimental evidence on this point is lacking. It is clear, however, that sufficient practice for any unit skill must include prac-

tice for the initial learning period and also for the maintenance of the unit skill after it has been acquired.

Table XXX shows clearly that the practice for Skill 12 is distributed among the five grades by two texts in very different ways, though both of them provide practice on it in all these grades. The

TABLE XXIX.—RANGE IN PROVISION FOR PRACTICE ON UNIT SKILL NUMBER TWELVE

	Series Providing Practice on Least Number of Skills						Series Providing Practice on Greatest Number of Skills					
Grade	IV	V	VI	VII	VIII	IV-VIII	IV	V	VI	VII	VIII	IV-VIII
Number of Practices per Grade	4	14	76	3	18	115	80	182	164	32	17	475
Percent of Total Practices per Grade	3.5	12.2	66.1	2.6	15.6	....	16.8	38.3	34.5	6.7	3.5	....

discrepancy between the total number of practices on Skill 12 in these two series of arithmetics (115 and 475) is so great that it is clear that there is either serious underlearning in one instance or wasteful overlearning in the other. Another obvious difference is the disagreement as to when the majority of the practice should come. One text provides the maximal amount of practice in Grade VI and the other in Grade V, with almost as much in Grade VI. Another striking discrepancy concerns the percent of practice in Grades VII and VIII.

Again, with respect to the form of the example, one text tends to emphasize only Skills 1 and 2, while another text tends to emphasize not only Skills 1 and 2 but also the equation form of the example (Skill 3). The nature of the addends that appear in the examples and problems used in these two series of textbooks shows a wide variation in the emphasis given to the different combinations of addends. One text places the greatest emphasis upon the addition of mixed numbers, while the other places the greatest emphasis upon the addition of proper fractions.

One of the specifications for drill materials which has been drawn up elsewhere[28] stipulates that drill should be on the entire process. This specification is disregarded in the addition of fractions. The

[28] *Third Yearbook of the Department of Superintendence, National Education Association,* 1925, pp. 63-91.

scant practice provided by some texts on some unit skills suggests the possibility that examples involving these unit skills operate more as interference factors than as practice for either the building or maintaining of skills. It is doubtful that skills which are given only two, four, or five practices throughout the elementary-school course in arithmetic receive sufficient practice to provide for an adequate mastery of them. The very fact that they appear so seldom may easily cause them to be overlooked entirely, thus leaving the pupils deficient in their mastery.

g. Analysis by Types.—The foregoing tables report drill provisions in terms of unit skills. They fail to show in what combinations of unit skills the practice was cast. It would be quite easy to reproduce from the original tables reports of practice in terms of combinations of units of skill. Arithmetical examples are combinations of units of skill. Examples having the same combination of units of skills are called 'types.' Breuckner and others have made useful contributions about fractions by presenting analyses of fractions in terms of types (combinations of unit skills) and reporting provisions for drill in terms of types. In the following discussion, drill analysis is in terms of unit skills. Of course, counts of unit skills in isolation are inadequate. The associations, or groupings, of unit skills as they appear in examples should also be made.

In building drill examples or in analyzing drill material, if equal insight in the listing of unit skills on the one hand, or types on the other, is assumed, the final results will be almost exactly comparable, irrespective of the basis of analysis. A discussion of analysis by skills versus an analysis by types is not in place here. The relation between analysis based on the chart (page 254), which is in terms of unit skills, and a similar analysis in terms of types will be evident when it is seen that the analyses of the three examples given in terms of unit skills in the accompanying chart are easily translated into types by simply verbalizing the check marks in Columns I, II, and III. A 'type' is simply a given combination of unit skills.

h. Drill in Subtraction, Multiplication, and Division of Fractions.—Space forbids the inclusion of similar data on the subtraction, multiplication, and division of fractions. Such data have been prepared. On the whole they give for these processes a general view quite similar to that gained for the addition of fractions from a study of the preceding pages.

ADDITION OF FRACTIONS

I. Find the sum of $1\frac{1}{2}$ and $2\frac{1}{6}$.

II. $1\frac{1}{6}$ III. $\frac{2}{3} + \frac{1}{6} + \frac{1}{2}$

$3\frac{3}{4}$

$2\frac{1}{3}$

Name of text.
Page No.
Exercise No.
Example No. or problem No.

		I	II	III
I. As to form of stating the example.				
A. Numbers written with figures.				
1. Indicated addition	1			x
2. Column addition	2		x	
3. Addition indicated by words	3	x		
B. Numbers written in words.				
1. Transference from words to numbers	4			
II. As to procedure.				
A. Nature of addends.				
1. Use of proper fractions	5			x
2. Use of whole numbers	6			
3. Use of mixed numbers	7	x	x	
B. Consideration of number of addends.				
1. Use of more than two addends	8		x	x
C. Changing of fractions to similar fractions.				
1. Denominator apparent	9	x		x
2. Denominator not apparent	10		x	
D. Analysis of answer and reduction to its lowest terms.				
1. One reduction	11	x		
2. Two or more reductions	12		x	x

i. Crucial Questions about Fractions.—It is hoped that the preceding discussion on drill in the addition of fractions and small mixed numbers will stimulate thought by supervisors and teachers on such questions as the following: (1) Is the drill material in use built with care in respect to an analysis of the unit skills involved or in respect to types (combinations of unit skills)? In other words, does the drill material square with such an analysis as that presented in Table XXVII? (2) Are the amounts of drill, both gross and in respect to units of skill, used for learning and used for maintaining sufficient for their purposes? (3) Is the distribution of attention to addition among the several grades concerned with teaching and maintaining a process a justifiable distribution? Or does the classroom practice tolerate long periods of no drill which tend to permit undue amounts of forgetting? Such questions as these drive directly to important aspects of good method.

j. Summary on Fractions.—The average provision for drill on fractions is at present ragged and ill-proportioned. The variations in

the provisions in different textbooks are far wider than any known facts or reasonable presumptions concerning the needs of pupils would justify. While it is evident that some drill materials have been built with regard to analyses of the processes drilled upon, the general situation is far from satisfactory. The gathering of crucial data relative to drill in fractions offers a promising field to the investigator in arithmetic, since very little precise information concerning the actual requirements of children in the field of fractions is available.

3. Drill in Percentage

a. Percentage Is a Complicated Skill.—Few thorough analyses of percentage from the standpoint of its learning appear in the literature. Sections of such an analysis, the work of Miss Nano Mahoney, are presented in an earlier section of this chapter. While difference of opinion may quite properly exist relative to many aspects of percentage, it is assumed that all would agree that the following skills are basic to any reasonable mastery of this process:

(1) Changing values from decimal to percent form,
(2) Changing values from percent to decimal form,
(3) Changing values from fraction to percent form,
(4) Changing values from percent to fraction form.

In the next few pages are presented digests of data bearing on the kind and amounts of drill provisions supplied by representative materials on these four processes. At present there are but a very few supplementary drill devices in the field of percentage. In general, the actual experiences in percentage responded to by pupils in the sixth, seventh, and eighth grades are those contained in the textbooks they are using, plus such incidental material as each teacher constructs for his own classes.

b. Six Types of Difficulty.—It will prove useful to report the verifiable data on drill provisions in percentage in such a fashion that some knowledge beyond gross unanalyzed amounts may be obtained. Basing the analysis of difficulty mostly on the study of errors in percentage by Edwards and Knight (reported elsewhere in this Yearbook) the data on drill will be reported for six types of difficulty as follows—the order of presentation is based on the size of the percent involved:

Type A. Forms related to percents less than 1, such as: the decimal .004, the fraction $\frac{3}{200}$, or the percent $\frac{1}{2}\%$, or $\frac{1}{2}$ of 1%.

Type B. Values related to whole percents from 1% to 9% inclusive, such as: the decimal .04, the fraction $\frac{3}{100}$, and the percent 6%.

Type C. Values related to percents from 1% to 9%, inclusive, which include a fractional part of a percent, such as: the decimal .045, the fraction $\frac{1}{40}$, and the percent $4\frac{1}{2}$% or 4.5%.

Type D. Values equivalent to whole percents from 10% to 99%, inclusive, such as: the decimal .45, the fraction $\frac{1}{4}$, and the percent 65%.

Type E. Values equivalent to percents from 10% to 99% which include a fractional part of a percent, such as: the decimal .245, the fraction $\frac{1}{8}$, and the percent $12\frac{1}{2}$%, or 12.5%.

Type F. Values equivalent to 100% or more, such as: the decimal 1.5, the fraction $\frac{3}{2}$, or the percent 125%.

These six types of difficulty mask certain more finely graded types which probably will be used when advances over the teaching of percentage have been made. But for the present purposes these six types may push the distinctions far enough.

c. The Source of the Drill Data.—The reports which follow are based on counting all practice which would be made on the processes studied if the work (in eight representative textbooks) was done once. Of course, any drill material may be assigned for work more than once, but no estimates are available on this point. Examples and problems on interest and other applications of percentage were not used. The preliminary organization was made by Mrs. Clara Rice.

d. Changing Decimals to Percents.—This skill is used both in thinking about verbal problems when translation from one number system to another is useful and in the correct reporting of the answer of all problems involving Case II of percentage. Thus in working the example: "45 is what percent of 384?" the computation gives the decimal .125. This decimal form is changed to $12\frac{1}{2}$% in the proper answering of the example.

Table XXX shows the average amount of practice provided in the representative texts. The types of difficulty, A, B, C, etc., are those just described.

The reader may be so impressed with the scantness of drill reported in Table XXX that he will fear that an error in printing has been made. None has. As a general rule, the higher up in the grades one pushes exact analysis of drill as well as of instructional material, the

TABLE XXX.—PROVISIONS FOR PRACTICE IN CHANGING DECIMALS TO PERCENTS
(Average of Eight Textbooks for Grades VI, VII, and VIII)

Type	Grade VI	Type	Grade VII	Type	Grade VIII
A	1	A	3	A	2
B	7	B	5	B	2
C	4	C	5	C	3
D	24	D	16	D	3
E	18	E	14	E	8
F	7	F	13	F	6
G	5	G	6	G	2

clearer it will become that much more care is taken in the preparation of material for the lower than for the upper grades.

The variations in amount of practice given *in toto* and on each type of difficulty in the texts studied is obviously significant. Some texts provide several times as much practice as others. While it is extremely doubtful that any text is open to the criticism of giving too much practice, it seems evident that many give far too little.

e. Changing Percents to Decimals.—This skill is, perhaps, the one most frequently used in the bulk of percentage work. In solving examples and problems involving the use of Case I, percents are changed to decimal forms as shown in the example: "Find 24% of \$560." This example would ordinarily be computed as follows: \$560 $\times$ .24. Examples and problems involving the use of Case III also use this skill. Thus in the example: "If 789 is 54% of a number, what is 100% or the number itself?" the computation ordinarily used would be: $.54\overline{)789.}$

Table XXXI reports the amounts of practice provided in changing percents to decimals with no attention paid to what types of percents are involved; in other words, without respect to the proportion of practice given to the various levels of difficulty of the forms used. As in other drill provisions, the variations among texts are impressive.

TABLE XXXI.—AVERAGE OPPORTUNITIES FOR PRACTICE IN CHANGING PERCENTS TO EQUIVALENT DECIMALS PROVIDED IN EIGHT TEXTBOOKS

Grade	VI	VII	VIII	VI-VIII
Average	164	152	59	375

Table XXXII shows the material of Table XXXI classified into the six levels of difficulty previously mentioned. This table shows the distribution of practice by grades, but not by kind of situation (drill in contrast to verbal problem).

TABLE XXXII.—DISTRIBUTION OF OPPORTUNITIES FOR PRACTICE IN CHANGING PERCENTS TO DECIMALS CLASSIFIED INTO SIX TYPES OF DIFFICULTY

Types of Difficulty	Grade	Average of Eight Texts
A	VI	4
	VII	9
	VIII	6
B (Easy)	VI	25
	VII	25
	VIII	7
C (Hard)	VI	12
	VII	17
	VIII	7
D (Easy)	VI	53
	VII	49
	VIII	18
E (Hard)	VI	44
	VII	24
	VIII	8
F	VI	27
	VII	28
	VIII	12

In addition to the variations in distribution of practice on the various levels as reported in this table, the reader would be impressed with the wide divergence in the amounts of practice (not shown here) provided in the different textbooks. The text providing the most practice and the one providing the least are both among the newer texts. While there is no experimental work to show what amounts of practice pupils actually do need in order to learn how to change percents to decimals, it is highly improbable that there are many differentials in motivation techniques existing between the 'least' and the 'most' text which even up the differences in amounts of practice. Both may be quite wrong, but both can hardly be right.

It may be fruitful to compare the relative emphasis on various levels of difficulty. On the whole, Types B and D (not involving fractions of a percent) are significantly easier than Types C and E (involving fractions of a percent). Yet the average of eight texts gives but 64 percent as much practice to Types C and E (the harder) as is provided for Types B and D (the easier). The data show that the text providing the largest gross practice gives but 60 percent as

much practice to Types C and E as to the easier Types B and D. None of the texts studied gives as much practice to C and E as to B and D. These comparisons indicate none too careful thought about drill construction.

It may occur to the reader that Types B and D should receive more practice, in spite of the fact that they are easier, because they are so much more useful than levels C and E. That 4% is more useful in modern business than 4½% or that 12% is more useful than 12½% is a very doubtful assumption, however. The finesse of modern business is apparently requiring the use of fractional percents at least with sufficient frequency to render them a minimal essential. Presumably they should receive as much practice as easier forms.

f. Changing Fractions to Percents.—Examples of this skill are: "¾ of N equals what percent of N?" "⅖ of N equals what percent of N?" The skill of changing fractions to equivalent percent forms does not play a large part in the written computation of percentage. It is possible that this fact accounts for the little attention paid to it in practically every text studied. Several texts gave so little practice on this skill (some none at all) that it is clear that genuine skill was either taken for granted or assumed unworthy of time and effort. The facts relative to amounts of practice are presented in Table XXXIII.

Table XXXIII.—Average Opportunities for Practice in Changing Fractions to Equivalent Percents Provided in Eight Textbooks

Grade	VI	VII	VIII	VI-VIII
Average	25	26	8	59

g. Changing Percents to Fractions.—The main facts relative to opportunities for practice in this skill are presented in Table XXXIV.

Table XXXIV.—Average Opportunities for Practice in Changing Percents to Equivalent Fractions Provided in Eight Textbooks

Grade	VI	VII	VIII	VI-VIII
Average	16	20	9	45

h. The Fractions Used.—The original study by Mrs. Rice, upon which this report on drill in percentage is based, included an account of the frequency of practice on certain important numerical values. These were, stated as fractions: ⅛, ⅙, ¼, ⅓, ⅜, ½, ⅔, ¾, ⅚, and ⅞.

Space considerations forbid the spreading of the data on the practice received on these values in the cases of: (1) changing decimals

to percents, (2) changing percents to decimals, (3) changing fractions to percents, and (4) changing percents to fractions. Suffice to say that several texts give no systematic attention to these values at all. Practice is given in one situation but neglected in another. Other texts carry practice on these values through the grades rather doggedly and provide practice in the four types of changes just mentioned.

i. Summary on Drill in Percentage.—It is obvious that current material differs in the provision for drill in percentage as much as, if not more than, it differs in any aspect of arithmetic yet studied. Many of these differences are doubtless due to sheer inadvertencies. But the fact that serious workers in arithmetic, of presumably comparable ability, fail to agree even approximately on an important phase of arithmetic is probably due to the serious want of precise knowledge of the learning difficulties involved. Percentage as a school subject offers an almost virgin field for scientific investigation. It is hoped that the attention paid to percentage in this Yearbook will stimulate such investigation.

4. Summary of Method in Drill

In Section XIII I have attempted to outline several important problems in the field of drill by presenting reports based on verifiable data. It has seemed better to deal in detail with small parts of the total problem than to indulge in sermonic verbalisms about drill in general. One pressing need relative to method in arithmetic is not bigger and better discourses, but more information both precise and searching in its nature. This discussion has centered in textbook material because that is the material in the hands of the pupils which is very close to the heart of the problem. Although many general principles of drill have not been discussed at all, the main position taken here might perhaps be translated into these few general principles of drill:

1. It is useful to distinguish between drill for building a skill and drill for maintaining a skill after it is built.

2. Every bit of drill, be it oral drill or written, be it prepared by the teacher and written on the board or be it prepared by another and presented in a text or drill pad, be it prepared on the spur of the moment or built long before it is used, is nevertheless built according to certain specifications and is the working out of certain

theories of drill construction, even though these specifications and these theories may not be consciously in the mind of the builder at the time of work.

3. The actual characteristics of any method or material in drill represent in every instance choices from among several alternatives. Such choices are illustrated by the following phrases:

(a) Distributed vs. non-distributed drill
(b) Standardized vs. non-standardized material
(c) Mixed vs. isolated types of organization
(d) Drill in small amounts frequently or large amounts infrequently
(e) Examples in known order of difficulty or in unknown order of difficulty

4. The basis of choosing one such alternative instead of another should be adequate experimental evidence. Samples of such experimental work are reported. In the want of experimental evidence it is prudent to choose alternatives based on the inferences of good theory rather than the products of less dependable methods of choice.

5. As far as possible, data useful to the teacher and to the supervisor should come from drill work itself. This seems wise not only because of economy of time but because supervisory data based on drill material come during the school year, not at its end. Not only the pupil, but also his teacher, profits by the use of competently standardized drill material.

6. Precise and exacting knowledge of the nature of drill material used is a first principle of method.

7. Loose and superficial methods of appraising drill material should be abandoned.

8. The present status of drill material in the case of whole numbers, fractions, and percentage, as evidenced by the data here reported and referred to, suggests that a direction for immediate improvement lies in the insistence upon more carefully prepared material. Such improved material will in its construction be greatly influenced by analyses of the process drilled upon, and by experimental evidence and sound inferences concerning many specifications of organization in addition to the specification of sheer amount of drill.

XIV. In Review: Some Points of View on Method

A review of this chapter will impress the reader with its fragmentary nature. Nowhere is there to be found a clean-cut outline of all the essentials of method. Rather it is a collection of more or less separate problems, many of them followed farther in detail than may be of interest to the general reader. It is even more fragmentary than was at first planned, since certain comparisons and contrasts of present-day textbook practices originally planned were felt to be too blunt or too critical for use here. It has seemed best to devote the space allotted to detailed studies of a few considerations rather than to offer only general observations of the total field. Scientific knowledge is not now available for a necessary foundation for a general treatment of method unbiased by personal opinion and prejudice. A general treatment of the whole field of method unsupported by scientific information would only add one more general treatment to those already available.

It is possible, however, to suggest a few points of view of method that have general application. Many of these are tentative; many are unsupported by experimental evidence of sufficient amount and quality; none of them should be applied to actual learning situations without the many qualifications which any particular classroom situation properly places upon them.

1. *Reasonable care should be taken to create in the pupil an active, in contrast to a passive, or absorptive, attitude.*

The pupil should attack his work with a feeling that it is his work, his learning, his responsibility.

When the teacher carries the load of an explanation by his own oral presentation of a new topic and by his own use of the blackboard, the pupil is in a passive attitude. When the pupil attacks the same topic for himself in a focalized reading lesson, he is in an active attitude.

2. *The content of instruction and drill should be greatly influenced both by the mathematics of the process and also by an appreciation of its various difficulties for children and by the interrelations of these difficulties. The ability of a child to understand some aspects of a process through logical inference should not be taken for granted and should not be used in teaching until it has been shown to exist.*

Thus, logically the carrying of 2 can be inferred from explanations of the carrying of 1, but such a practice would assume a power

of inferential thinking not known to exist in young children. Again, the combinations 4×1 or 4×0 are mathematically similar to 4×2 or 4×3, but the actual difficulties of children warrant attention being paid to all of them.

3. *Both instruction and drill should approach a process or a computation from various angles. No important item should be presented in but a single setting. This principle of variety should be scrupulously observed. Settings should be included which the child will use.*

If a child practices a combination such as $4 + 5$ only in the forms $4 + 5$ and $\begin{array}{r}4\\ \underline{5}\end{array}$, he can hardly be expected to have as well rounded a grasp of this combination as he would gain by practice upon it in more varied forms, which might well include such forms as: $4 + \ldots\ldots = 9$; $\ldots\ldots$ and $5 = 9$; $9 = 4 + \ldots\ldots$; and $9 = \ldots\ldots$ and 5.

An explanation in but one setting is apt to be unduly affected by the particular setting. The same explanation presented in varied forms and with various settings has greater chance of becoming freed from any particular setting and thus be more easily extended to general use.

4. *It is not necessary to teach all that there is known about a process. The temptation to be logically complete often necessitates teaching too little about too many things instead of teaching enough about a few aspects of the topic.*

Before beginning instruction on any process the teacher should have well in mind the answer to the question: How much of this process shall I expect the pupil to learn?

It is better to stress a few ideas about insurance (for example) and get them genuinely appreciated by the pupil than it is to crowd a great many ideas about insurance into the brief time allotted to it, and develop as a result of this logically complete treatment only a confused blur in the pupil's mind.

5. *Drill and application should follow effective learning of the content. Drill undertaken before the ideas involved are mastered only piles up error, discomfort, and a resistive attitude.*

6. *Rarely, if ever, should aspects of a process be explained unless they are given practice and application.*

Sometimes, to avoid a stricken conscience lest some item in a course of study be omitted, the teacher hurries through a topic so rapidly that, although technically it has been taught, it has not been actually

learned by many of the pupils. It is as important to teach with care topics coming during the last of the school year as those coming at its beginning.

7. *Instruction and drill should be interspersed. It is doubtful pedagogy to explain large sections of a process and then drill upon large sections. It is better to explain a bit, then drill on it, explain a bit more and drill on it, and so on.*

8. *As far as possible the intention should be dominant to learn not for the moment but permanently. A realization of the reasonableness of the task at hand presumably increases the pupil's intention to learn permanently.*

9. *Definite provisions should be made for relearning as well as for the learner's reviewing.*

Most children, as well as most adults, are faulty learners. A reconsideration of the ideas involved, as well as drill upon the process involved, is an important item in remedial work. Clear explanations of processes first attacked even several years previous to the present grade should be readily accessible to the pupil. Thus, explanations of long division should be available for study by sixth-grade pupils.

10. *Genuine and legitimate motivation should permeate the daily work.*

In explanations the frequent appearance of questions, in contrast to a series of declarative sentences, is thought to increase the interest of the learner. The keeping of progress scores as a systematic enterprise is known to provide motivation to average and superior pupils. The presentation of work in problem settings by means of projects is commonly assumed to be a practical device to increase motivation. Adjusting the difficulty of the task to the ability of the pupil is a prime condition of motivation. Work which is too easy or too hard will not long engage the effort of most pupils.

11. *As far as possible, processes taught and drilled upon should be carried into those types of applications which children can respond to with understanding.*

Arithmetic, however, shares with many other school subjects a certain difficulty in making all of its topics particularly 'meaningful' to the child, in the best sense of the adjective.

12. *There is considerable doubt about the efficiency of rules and definitions as aids to genuine understanding and insight as well as skill.*

When rules are used, they had better be sets of directions instead of formal statements. Not that the statements in conventional form are incorrect, but that they are of questionable aid to the young learner.

13. *Logical niceties in rules and definitions often run counter to the needs of children and should be reserved for adults. Mathematical improprieties are not to be countenanced, but neither are mathematical pedantries.*

Conventional rules relative to the finding of areas and volumes, for instance, contain pedantries which only confuse and vex the child and the practical adult worker.

14. *All explanation and drill as well as problem work should respect individual differences in ability. While most of arithmetic will be learned with the learners in a class organization, much provision for individual differences within the class is both possible and highly desirable.*

15. *Knowing what a thing is not, as well as what it is, appears to be an important aspect of well-rounded knowledge. Present knowledge relative to interference factors should affect not only explanations but also drill work.*

16. *A critical attitude toward one's own work and the work of others is to be desired. An ability to find and correct errors is of high social utility as well as an essential to surety of computation. Errors should not be presented to children during the first stages of learning and never without directions stating that errors are present, but under proper conditions practice in critical work is to be desired.*

17. *The vocabulary burden of arithmetic presents a problem to which all teachers should be sensitive. Words not well understood should not be used in explanations, for such practice is simply explaining one mystery in terms of another. The minimal technical vocabulary which should be mastered warrants specific teaching in the same sense that any other aspect of arithmetic does.*

18. *Adequate instruction, in contrast to hurried or compressed instruction, saves both time and energy. A clear understanding of a new process at the outset has a distinct advantage over hurried teaching and premature drill and problem application.*

19. *The use of competent tests after learning, of inventory tests at the beginning of a semester's work, of diagnostic tests when break-*

downs in skill are noticed, and of remedial work assigned in the light of individual needs are integral parts of effective teaching. Remedial work consists of remedial instruction as well as of remedial drill.

20. *Instruction and drill should be associated with concrete experiences and be set in problem situations which are sincere and genuine.*

21. *Of several ways of doing the same process, one method may be significantly better than the others. Yet it is very doubtful if the inherent superiorities of one type of procedure, or method, over others is so great that a change of method alone will solve any serious teaching problem. In all probability, any method now commonly in use, if well taught, will give better results than another method poorly taught.*

It is probable that future research will reveal great improvements in all aspects of method. At present, however, the issues of effective method do not lie in such unsettled, controversies as: (1) in teaching addition and subtraction together, (2) in adding down instead of up, or adding down and then up if the work is important enough to check for accuracy, (3) in using one scheme of subtraction or another, (4) in writing the common denominators in one set way when adding unlike fractions, (5) in finding quotient figures in long division by one rule, (6) in logical hairsplitting about the exact statement of rules which few pupils will hear or read and fewer will profit by after hearing or seeing, or (7) in any of the other items in a long list of disputed matters.

The really important issues of effective learning lie quite outside such controversies. They concern such matters as: (1) clear, well-understood ideas of a process before drilling upon it, (2) assiduous breaking up of overt and subtle interference factors, (3) the careful construction of drill so that certain combinations are not forever bugbears to the child, (4) the presenting of work in such varieties of situations that the child is not lost the moment the problem is stated in a way he is not accustomed to, (5) genuine motivation of effort, (6) careful testing to reveal weak points, followed by prudent reteaching and drill before such weak points become chronic, (7) a judicious grading of work to varying abilities, (8) the facilitating of active attitudes of learning and work, (9) the transformation of arithmetic from a series of drudgelike tasks into a series of problems, inquiries, investigations, and challenges which become opportunities for conquest. These and other factors which are based on a human

and dynamic psychology of human learning afford the real foundations of effective method.

Finally, it must be noted that a more effective method must not be content to improve the present situation in instruction and drill only. In addition to these important tasks, advance must be made in the field of problem-solving. Perhaps the greatest challenge now presented to the psychology of learning is the need for more information about, and greater control over, the higher thought processes involved in the curriculum of elementary education.

GUIDE TO CHAPTER V

CHAPTER V

TESTING, DIAGNOSIS, AND REMEDIAL WORK IN ARITHMETIC

Charles E. Greene and G. T. Buswell

I. The Purpose of Tests in Arithmetic

1. Survey Tests

Standardized tests in arthmetic have been in general use in the public schools of the United States for the past fifteen years. Most of these tests have been of the survey type. Survey tests in arithmetic are still widely used; in fact, they probably constitute the chief measuring instruments now employed in arithmetic in public schools. The principle involved in survey testing is that of securing a sampling of pupil achievement in a wide range of subject matter on an objective basis. When such samplings are tabulated and compared with norms representing many school systems in various parts of the nation, such a measure has value. The survey test has made it possible for a given school system to compare the achievement of its pupils with the achievement of pupils in other school systems and with the so-called 'nation-wide norms' which have been obtained from the compilation of scores from various communities. The survey tests demonstrate the value of an objective comparison of the achievement of different school systems. General standards have been maintained through such comparisons. However, the value of the survey type of standardized tests is almost always general in nature. When a given school system ranks below standard attainment, as shown by general practice, it is probably time for further examination into the cause of the low standing. A common practice is simply to announce to a group of teachers and principals that their schools are below standard and that a follow-up testing program will be carried on at the end of the teaching period. This probably constitutes a misuse of the survey test, yet it has been one of the chief uses that has been made of the survey test in the past.

Probably once in four or five years is often enough for a school

system to apply survey tests in all classrooms, provided sufficient use is made in the meanwhile of diagnostic tests and of the preventive and remedial measures suggested by diagnostic tests. When a survey test is given two to four times a year, the mere examination of schools and the comparison of the scores of each school with norms tends to place pressure on teachers and pupils for securing better results without giving any clue or hint as to how such results should be obtained. The teacher works to prepare for the tests and in many cases uses the approaching test as an incentive to make pupils work. As a result, testing is likely to become distasteful to both pupils and teachers. Such a situation is unfortunate and undesirable, yet it exists in many school systems at the present time.

The survey test has some value and this should be recognized. It may show teachers and administrators that achievement based on a sampling of cases is above or below par. It does not in any way indicate in what respects the attainment is lacking. That must be done by the use of diagnostic tests.

On the other hand, when a program of teaching arithmetic under a given type of organization has been carried on for a period of years within a school system, the survey test can be given to show the success of the arithmetic organization. Results may be compared with norms and achievements elsewhere. One large school system in the eastern part of the United States uses a plan of giving a city-wide survey test in arithmetic every five years. Each five-year period of instruction may then be compared with each other five-year period. This is probably the most legitimate use made of the survey test in arithmetic.

2. Diagnostic Tests

A more detailed treatment of results than that usually made of survey-test scores is necessary before difficulties can be diagnosed. To make possible an analysis of difficulties is the purpose of the diagnostic test.

Diagnostic tests may be of the group or the individual type. Every person engaged in an instructional field, whether he be supervisor, curriculum-maker, or teacher, needs the information that can be furnished by group diagnosis. Until one knows what are the hard spots in learning arithmetic, and until he knows what difficulties are experienced in the various classes in a school, he is unable to address himself definitely to the problem of helping children learn

in the most efficient manner. When a teacher first begins his work with a new class, he is comparatively ignorant of the abilities of the pupils. The group diagnostic test will give him this information. When the curriculum-maker has finished the course in arithmetic and wishes to measure its success in actual use, the group diagnostic test will tell him the particular phases and parts of the course that are not under control. When the supervisor wishes to help teachers to give instruction where instruction is most needed, the group diagnostic test will furnish the information desired.

Calling a test 'diagnostic' does not make it so. Some school people make a separate tabulation of a few items on a survey test, point out disabilities so revealed, and call the process a diagnosis. To be truly diagnostic, test results must first reveal weaknesses in all significant types of exercises in specific processes and operations measured. To complete the diagnosis one should go farther and show why pupils have difficulties with such processes. Group diagnostic tests generally must be limited to cover smaller units of work than the survey test, so that the significant types of desirable abilities in a given process or operation may be measured in the time allotted for testing. Thus, the group diagnostic test may be confined to addition in integers, while the survey test may cover all four operations in fundamentals, fractions, decimals, percentage, and denominate numbers.

a. Tests Used Before Instruction.—Most measurement in the past has been for the purpose of checking what has been taught, and has been carried on after the work of the unit or semester has been finished. The pupils that form a new class in a given grade vary widely in arithmetical abilities. Some may be already skillful enough to be permitted to omit a portion of the work of a grade; others have not properly finished the work of the preceding grade. Differences among children in such a group are great and should receive early attention from the teacher. The group diagnostic test will not only show the level of ability of pupils in various operations, but will point out pupil weaknesses so that the teacher may organize his work in such a way as to give some time to those who are below the standard and omit deadening drill for those who are above.

The pre-instruction test is designed to indicate at the beginning of a teaching period what remedial work needs to be done on processes and operations that have already been taught. This may be needed

to find out what reteaching is needed with phases of the previous semester's work which will be used on the new topic about to be presented. The pre-instruction test should also discover what familiarity pupils already have with the new course which they are about to attempt. A pre-instruction test, properly analyzed, gives valuable and definite information relative to the readiness of pupils for new topics and accordingly saves time in teaching and learning. For example, tests on decimals in January, 1929, in the Denver Public Schools showed the following percentages of success with each example made by fifth-grade pupils who had never had decimals.

	Percents of Success			
Addition of Decimals—4 Examples........	40	61	62	75
Subtraction of Decimals—3 Examples.....	34	42	70	
Multiplication of Decimals—4 Examples....	16	17	21	56
Short Division of Decimals—2 Examples...	9	18		

b. Tests Used During Instruction.—A test may be given for diagnostic purposes at times other than at the beginning and close of a teaching period. A teacher may find need for a diagnosis during a teaching period. There may be some question as to whether the methods and content used in teaching are developing the knowledges and skills that are expected. Pupils may fail to grasp the process being taught and the teacher may need to know just where the difficulty lies. A teacher should therefore be in a position to carry on diagnosis at any time when the need arises. This will make possible greater achievement and will prevent pupils from going on in an inefficient way until the end of the teaching unit with no one knowing just what difficulties have been causing the trouble. Some teachers may wish to do such diagnosis orally and informally. Such procedure is likely to be unsystematic and superficial, with the result that significant weaknesses are overlooked. Better results will be secured by giving a properly standardized test covering the unit in question.

c. Tests Used After Instruction.—A consistent use of group diagnostic tests to check at the end of the teaching period or semester will show the extent to which learning has been effective. It will also make the teacher more skillful in observing pupils' abilities as he teaches and will help him to anticipate pupil difficulties. Teachers who use diagnostic tests soon become familiar with the types of abilities and skills necessary to learn arithmetic. They early discover

the processes that are particularly difficult and those that are relatively easy, so that by observation and insight in teaching they are able to make use of preventive measures which render unnecessary some remedial measures that otherwise would be necessary. For example, a teacher may learn by diagnostic testing that pupils able in most phases of multiplication have difficulty with an intermediate zero in the multiplier. Preventive work in teaching multiplication will accordingly place special stress on intermediate zeros in the multiplier. Diagnostic testing is aimed not only at finding deficiencies and organizing remedial instruction, but it is quite as valuable, and perhaps more valuable, for furnishing better insight to the teacher of arithmetic by acquainting him with what is involved in the learning process in arithmetic and what types of processes and operations offer greatest difficulty.

3. Relationship Between Individual and Group Tests

A complete testing program will involve both group and individual tests. While the group tests may be either of the survey or diagnostic type, the individual tests have no other purpose than that of diagnosis. Group tests will discover those pupils who have major difficulties in arithmetic. Since it is these pupils who constitute the greatest burden of teaching, their difficulties should be further diagnosed by means of detailed individual tests. Considered in this way, the individual tests are applicable to only a small percentage of the total number of pupils in a classroom. However, there are other considerations in the use of individual tests. Pupils who make many errors can be identified easily by means of their scores on the group tests, but pupils who make few errors will be rated as satisfactory on the group tests even though their methods of working the examples may be so crude and cumbersome that it takes them a long time to complete the test. Where a group test is a timed test, this latter type of pupil may be identified as the one who makes few errors but works slowly. These slow pupils frequently need as much individual attention as do those who make many errors, because the desired objective of teaching arithmetic is not alone the ability to work examples without error, but also the ability to work them quickly and by the most economical methods. Therefore, individual diagnostic tests are applicable to two groups of children: first, those who make many

errors in their work; and second, those who are slow but who may get correct answers.

There are also occasions when a third type of pupil should be tested individually. In order to interpret the data from the individual tests of pupils who experience great difficulties, one must have a body of data with which to compare these performances. Consequently, it is desirable to know not only the reactions which are made by pupils poor in arithmetic, but also for purposes of comparison, the reactions of those who are average or who are even excellent in the subject. Of course the data needed for this third purpose do not need to be accumulated by each teacher, but they should be supplied by the author of a diagnostic test in order that a teacher may know what to expect from various types of pupils.

A good diagnostic test has two principal functions: first, the discovery of errors; second, the discovery of causes of the errors. It is also important that the examples included in the diagnostic test be so graded as to difficulty and type that the pupil will be able to proceed from the very simplest of examples at the beginning of the test to the most difficult types of examples which are expected at a given level of maturity. Furthermore, all of the important intermediate types should be represented in the test.

To a considerable extent, the discovery of errors may be carried on by means of a group test. This is particularly true where a test is so constructed that the various types of examples in the process being tested appear in some known order. This type of test is illustrated in a recent article by Brueckner.[1] From such a test one may make an extended analysis of the errors in a given process and can show the frequency of each type of error. In some cases, one may even go so far as to show the cause of the error, though in doing this caution must be used.

Because of the danger of error in inferring from a test paper that a given answer resulted from a given cause, the individual method of diagnosis has been used. The individual diagnostic test is based upon the notion that improvements in arithmetic will result only from the removal of the causes of error. Consequently, it is highly important that the causes be discovered with certainty. One or two concrete illustrations may indicate the difficulty of trying

[1] Brueckner, Leo J., "Analysis of errors in fractions." *Elementary School Journal*, 28: 1928, 760-770.

to infer the cause of an error from the answer as given on a test paper.

In the example, 86 minus 4, a third-grade pupil secured as his answer 81 instead of 82. If this example had occurred in a test and the teacher were trying to discover the cause of the error from the answer on the paper, he would doubtless infer that the cause of the mistake was lack of knowledge of the combination, six minus four. What the boy actually did was as follows: "Six and four are ten. Ten and eight are 18. Turn it the other way makes 81." He then wrote 81 as his answer. Obviously, the case is much more serious than it would have been if he had simply made a mistake in the combination, for he showed by his work that he was entirely unfamiliar with the principles of subtraction. More drill on subtraction will not help this pupil until he has had definite instruction in regard to the processes involved.

Another illustration may be drawn from a fifth-grade pupil who was subtracting 36 from 42 and gave as his answer 14. Judging simply from the answer on a test paper, one would probably say that in addition to the error in the first combination the pupil had forgotten to borrow and had for this reason subtracted 3 from 4, getting 1 in the tens' place. What the pupil actually did say was this: "Thirty-two to forty-two is ten and four more (32 to 36) is fourteen." In such an example as this, simply observing the answer on a test paper does not help the teacher to understand the difficulty. Nothing short of a detailed individual diagnosis in which the teacher observes the mental processes of the pupil as he works would throw light on the real difficulties that are involved.

A third example may be drawn from a sixth-grade pupil engaged in multiplication. In multiplying 35897 by 2, she secured the answer 76244, instead of 71794. An individual diagnosis of this pupil produced the following statement: "Seven 2's are 14. Seven 9's are 63, and 1 is 64. Seven 8's are 56, and 6 are 62. Five 2's are 10, and 6 are 16. Three 2's are 6 and 1 is 7." She began by using seven as the multiplier instead of two, and for the next two products she continued to use seven as a multiplier. Then she used the proper multiplier, which was two, although she reversed the form of expression.

One can scarcely overemphasize the importance of discovering the mental processes which lie back of pupils' answers in arithmetic. Intelligent teaching can proceed only from an analysis of the pupils' methods of work. Consequently, when serious difficulties are encountered by pupils, the only final solution to the trouble is a detailed analysis of how the difficulties were produced, followed by an attempt to improve the pupils' work by some change in the methods involved.

Furthermore, one must not overlook the fact that in many cases pupils secure correct answers by methods which are very crude and

cumbersome and which should be replaced by more efficient methods of thinking. For example, the writer has observed pupils who, in multiplying nine by nine, counted nine nines on their fingers. They secured the correct answer, but the process was extremely slow. Pupils who rely on this method find it utterly inadequate in complex examples, simply because while they are carrying on one single operation they forget the sequence of procedure for the example as a whole. This difficulty may be illustrated again by the very common practice of splitting numbers in adding. For example, in adding in column addition, the subtotal in a certain case was 37. The next digit to be added was 9. Not knowing the combination 37+9, the pupil proceeded as follows: "Thirty-seven and 3 or 40, and 3 more are 43, and 3 more are 46." This practice of splitting numbers not only slows down the entire operation, but it also introduces an element of confusion which may cause errors.

Individual diagnostic tests, then, must be thought of as applicable not simply to children who have major difficulties in arithmetic, but also to children who may get correct answers but who get them by methods which are crude and unwieldly.

4. The Relation of Tests to Teaching

What has been said in the preceding discussion has stressed the idea that tests and test results should be connected closely to the teaching and learning process. This connection makes it possible to secure specific information for the aid of teachers and pupils in their work in the mechanics of arithmetic.

The survey test has generally been given on the initiative of the superintendent or principal to measure results in one school or school system as compared with those found in another. Quite often the test is not expected by either teachers or pupils, and its content may have very little bearing on the things which are being done in the classroom. The type of test which is most significantly related to teaching is the diagnostic test as previously described. Group diagnostic tests should probably be of two sorts. Teachers should have available for use in the classroom at any time diagnostic tests on short units. When a teacher has spent some time in teaching a given process, such as long division, it should be possible for him to give a diagnostic test in that operation and find out whether the process has been sufficiently mastered by the group so that further drill may be

carried on simply through the maintenance program and intensive work on the topic discontinued. Such tests would not be given simultaneously throughout a city school system or a given building. They would rather be given by the teacher at whatever times seem fitted to show the extent to which pupils are skilled in a given process and to indicate types of exercises that need further drill and emphasis. Although the teacher is chiefly concerned in giving these unit tests for a group diagnosis, the supervisor is also interested in this matter. The supervisor may take an active part in deciding upon the time at which such unit tests shall be given, in studying the results of the test, and in working out the future procedure in connection with this topic in arithmetic and with the one which is to be presented next.

As indicated already, probably tests should be given frequently on the initiative of teachers at the end of short units of teaching in arithmetic. However, it is possible and desirable, from many points of view, to set up a system of testing on a city-wide basis for the sake of measuring results of courses of study, textbooks, and systems of supervision. When a city school system has a highly unified program of instruction under an adequate course of study in arithmetic, with adequate textbook facilities and supervision from the administration offices and principals, it is both possible and desirable to carry on a system of diagnostic testing. If the teachers have had sufficient opportunity to participate in this program, there is no reason why there may not be a sympathetic understanding among all concerned, so that a situation exists in which pupils, teachers, principals, supervisors, and administration authorities are seeking the same end; namely, to develop skill in the arithmetical process in the most effective and economical way.

Under the direction of the research department, the superintendent or the supervisors, teachers, and pupils should know their standing in each step in the learning process of the operation which the test measures. They should also know what is involved in each step of this learning process.

When such diagnosis has been made of each class, children may be grouped within the class according to their deficiencies, and the way is made easier for remedial work.

II. The Material of Tests

Whatever purpose the tester may have in mind, whether he wishes to make a survey of a school, a group diagnosis, or an individual

diagnosis, his selection of a test should depend largely on the subject matter that the test contains. No test is better than the examples or problems that compose it. A careful examination of arithmetic tests will disclose in some of them examples that tend to invalidate the tests themselves. Some of the first standardized tests made in fundamentals and integers contained only examples of the maximal difficulty in which their makers thought it necessary for pupils to develop skills. According to many recent courses of study and the opinions of authorities on the subject, such tests were made up entirely of material which is more difficult than elementary-school pupils need to learn. Because these difficult examples were included and less complicated ones omitted, the tests failed to measure the significant abilities of pupils. Some tests contain examples prepared in forms which are not frequently taught nor frequently used. The equation form in fundamentals and integers and fractions might be cited as an example. Other tests contain an unduly large proportion of a few types of examples to the neglect of other significant types. The selection of material in a test is of the greatest importance for its validity, for getting a real measure of the thing one sets out to measure.

1. The Selection of Material

a. For the Survey Tests.—As indicated earlier in this chapter, the purpose of the survey test is to measure the abilities of pupils in certain phases of arithmetic by taking samples of those abilities. One survey test may cover fundamentals and integers, another fractions and decimals, and a third other mechanical processes, or a single survey test may attempt to cover the whole field of mechanical processes in the elementary school.

Since the survey test is necessarily limited in time and material, it is likely to have a better selection of material if it covers a relatively short range of subject matter. A survey test in fundamentals in integers thirty minutes in length may cover practically all of the significant types of examples. Another for fractions and decimals may likewise cover the significant skills in those processes. But it is highly important for the sake of validity and reliability that the survey test contain examples that are frequently taught and used and that are typical of other examples closely related in the skills involved. Before anyone chooses a survey test, he should take the test

himself and see just what is involved in each example, and unless he is very familiar with the course of study used in the school which he is testing or unless he has a definite reason for making a certain type of survey, he should compare the examples in the test with the material that is taught in the school to be surveyed.

b. For Group Diagnostic Tests.—The selection of material in testing for group diagnosis should depend on the use to be made of the diagnosis. If the test is to be made of pupils' abilities in significant types of examples at the beginning of the semester or year or at the end of the semester or year, some of the more recent survey tests that have been carefully prepared may serve the purpose. For example: a survey test of thirty minutes in fundamentals and integers in which the significant types of examples that can be included in thirty minutes are used will prove to be a fairly satisfactory measure for group diagnosis so far as the selection of material to be tested is concerned. If the group test, however, is to be used to check the work on a short unit, such as subtraction of fractions or division of decimals, the survey test is inadequate. A unit test is of value to the teacher chiefly when it indicates in detail the phases of the operation or process tested in which children fail and succeed. It might be said that a survey test at its best is almost certain to be too brief for the teacher's use in measuring the work of short units of instruction, but that it may be adequate for group diagnosis in the work covering a year or for the measurement of the maintenance of skills previously learned. Thus it should be noted that group diagnosis may be carried on for two definite purposes: first, to find out definitely whether operations and processes are well understood and have been adequately learned—this is the place for the test on short units; and second, to secure measurement of the maintenance of skills previously learned, which may be done once or twice a year or semester, at the beginning and end.

c. For Individual Diagnostic Tests.—The material for the individual diagnostic test should be rather detailed. The test should generally cover a single process or a part of a process. It should begin with a very simple example in the process and continue with examples that increase in difficulty by short steps. For example, if the process were carrying in addition, the first few exercises would probably be very easy examples in carrying, varying from one to the other only slightly in difficulty. This would make it possible to

secure measurement of an individual pupil's ability in carrying in several easy examples and to trace his difficulties accurately because his results would show just where they occurred. Following the first set of easy examples in carrying would come a set of examples in which the carrying was more involved and difficult. Steps in difficulty would still be small. The content of the individual diagnostic test should contain examples that are significant from two points of view; first those that are significant because they form steps in the learning process; second, those that are significant because they have been recognized as such in the best modern courses of study, textbooks, and drill materials.

2. The Amount of Material

a. For Survey Tests.—The survey test must be given in a short period of time. It will probably cover the bare essentials so far as significant types of examples in an operation are concerned. All tests should be long enough to cover adequately the field to be measured, but short enough so that the administering and scoring will not be burdensome to teachers. The problems of administering, scoring, tabulating, and evaluating the results of tests for an entire city school system become complicated when long tests are used. Probably such results are not used to the fullest possibilities when long involved tests are employed.

b. For Diagnostic Tests.—A test need not be longer for diagnostic purposes than for a survey, but the range of material used should be much less. A comparatively narrow function must be used for diagnosis, so that all significant types in a process may be included. Diagnostic tests should be given at more frequent intervals. A large amount of material and a comparatively long time may be used in the aggregate by giving diagnostic tests several times a year.

In a given test the amount of material used should depend upon the size of the class and to some extent upon the ability of the class in the functions measured (see the discussion farther on under the treatment of 'reliability'). For classes of thirty or more, one example of each type, after a process has been taught, will give a fairly reliable measure.

3. The Arrangement and Organization of Tests

a. Of Survey Tests.—Probably the best arrangement for standardized tests in arithmetic is to separate the test into parts according to operations. For example, in a test in fundamentals in integers, the test might be divided into four parts: Part I, Addition; Part II, Subtraction; Part III, Multiplication; Part IV, Division. Part IV might be subdivided into short division and long division. It is also desirable to time each part separately, so that those pupils who are slow will have relatively as much time in division as they have in addition.

b. Of Diagnostic Tests.—If the survey test is used for group diagnosis, it should be arranged by operations, with each operation timed separately, as just noted. It should further be arranged according to steps in difficulty. The simplest type of example to be measured should come first. The next example should be a slight variation and increase in difficulty. The third example should represent the next step, and so on. The test should thus be graded according to steps in learning difficulty, so that when the records of the group and the individuals composing the group are examined, it may be clearly seen just at what point a high percentage of error begins to occur and the teacher may thus know exactly where to begin with remedial treatment. This arrangement may or may not be exactly according to difficulty as shown on the results of the test; that is, it is conceivable that for some reason pupils may solve more of certain complicated examples than some of the simpler forms. This particularly is true if the different forms in which the examples given are changed, such as add $\begin{array}{r}3\\ \underline{7}\end{array}$ or $3+7=?$. The equation form for the fundamental operations generally proves more difficult than the vertical, or column, form. The zero difficulties may offer more difficulty than carrying with large numbers. These facts should not influence the test-maker to rearrange his test in the order of actual difficulty on the results of the test. He should rather arrange it in order of steps in learning, so that it may be seen just where in a process a breakdown occurs.

III. Important Characteristics of Good Tests

1. Degree of Difficulty

The chief points to be kept in mind with respect to the degree of difficulty proper in the case of survey tests and in the case of group diagnostic tests may be readily inferred from what has been said with respect to the general nature of the contentual make-up of those types of tests in the foregoing discussion of the selection of material, the amount of material to be employed, and its arrangement and organization. A few words may be added here, however, with respect to the problem of degree of difficulty in the case of individual diagnostic tests.

One who would construct diagnostic tests should first of all be familiar with the subject-matter field concerned. High coefficients of correlation for reliability and validity will not prove finally the value of a measuring instrument in the teaching of arithmetic. The diagnosis of abilities requires that all the factors, or at least the chief factors, involved in performing an arithmetical task be either controlled or measured. Difficulties in an arithmetical process are chiefly due to three factors: (1) the structure of an example, (2) the complexities and variations from type that develop in a process owing to increased length of an example, and (3) the combinations involved.

The factors of difficulty are the same whether the test is intended for group or individual diagnosis, though in the latter, smaller steps in difficulty and more variations in types of examples are used. The factors of difficulty used for diagnosis are generally and chiefly the first two just mentioned, structure and length of examples. The third factor, combinations involved, cannot well be ignored, however. The addition of a set of eights, nines, and sevens in a column of seven numbers is a different problem from the addition of a column of seven numbers made up of threes, fours, and fives. In one series of diagnostic tests recently published, the makers have used the same combinations in short and long examples in a given process so that the combination difficulties are the same and hence controlled. Other makers vary the examples not only according to structure and length, but also according to combinations. The latter procedure has the advantage of securing measurement on all types of a difficulty in the same test. The former has the advantage of getting a less complicated measure with regard to structure and length of the examples. Combinations may be taken care of elsewhere. The use of the three

factors mentioned is illustrated in Table I. The examples in the first row are alike in (1) structure and (2) length. They differ in (3) combinations, first in cases where reduction is not required and second where reduction is required. The examples without reduction are of four kinds, in terms of the combinations involved: in the first case the same denominators are used; in the second case the least common denominator is visible; in the third case the least common denominator is found by multiplying the denominators of the two fractions; and in the fourth case the least common denominator must be found by multiplying the denominator of the first

TABLE I.—STEPS IN DIFFICULTY IN SUBTRACTION OF FRACTIONS

SIMPLE FRACTIONS

No Reduction					Reduction		
5/7	1/3	1/3	5/6	3/4	2/3	2/4	2/4
2/7	1/6	1/4	4/9	1/4	1/6	1/3	1/6
3/7	1/6	1/12	7/18	1/2	1/2	1/6	1/3

MIXED NUMBERS

No Reduction					Reduction		
9 3/4	8 1/2	6 5/6	6 4/5	7 7/10	35 7/12	43 2/3	42 5/8
7 1/2	5 1/3	4 3/8	2 1/5	4 1/5	18 1/35	7 1/6	10 1/6
2 1/4	3 1/6	2 11/24	4 3/5	3 1/2	17 1/4	36 1/2	32 1/2

3-Figure Numbers		Other Difficulties		
421 3/4	543 5/6	9 1/2	24 1/4	8 1/2
297 9/16	306 1/12	9 1/2	7 1/4	5
124 3/16	237 3/4	0	17	3 1/2

CARRYING OR BORROWING IN SUBTRACTION OF FRACTIONS

5	4	10 1/9	5 3/8	10 1/6	7 1/5	6 1/4	4 2/8	15 1/12	3 1/2
3 5/8	1 2/10	7 6/9	3 7/8	8 1/3	4 7/10	3 1/3	1 2/3	12 3/4	2 3/4
1 3/8	2 4/5	2 4/9	1 1/2	1 5/6	2 1/2	2 11/12	1 7/12	2 1/3	3/4

fraction by three and the second by two. In the second row are mixed numbers. Those in the first set are alike in structure and length, but require different combinations for subtraction. The other examples on the page illustrate other difficulties produced by the three factors mentioned.

Individual diagnostic tests are often used for group measurement. When groups are small, individual diagnostic tests are required because of the unreliability of small group measurement on a single example.

2. Reliability

a. Of Survey Tests.—The question of the reliability of a survey test is fairly simple. A good thirty-minute test would contain examples for a given grade of the types generally taught and approved by best courses of study, textbooks, and arithmetic authorities, and should produce a fairly reliable total score. Unreliability or inaccuracy of measurement is caused by inadequate sampling, inadequate or ambiguous directions, subjectivity in scoring, unfamiliar forms of examples, and any other conditions that distract pupils from doing a definite, well understood task. The reliability of a survey test may be judged fairly accurately by analysis of its content and structural make-up.[2] If it contains the types of examples taught in the school to be measured, if it emphasizes the types of examples that are emphasized in the local schools, if the forms of its examples are the forms taught in the local schools, if there is plenty of space for working the examples, and if the test is properly timed and properly administered, one may be sure of a fairly satisfactory reliability for a given grade.

b. Of Group Diagnostic Tests.—The problem of reliability in a group diagnostic test is of a different sort. If the better type of survey test is used for group diagnosis, the total score may be reliable, as just noted, but the total score is not the point at issue in reliability in the group diagnostic test.

If the group diagnostic test deals with but one operation, then the total score will be the first indication of ability or of lack of it for any given pupil or for the class. If the test is made up of parts each dealing with only one operation or a closely related group of skills, then the total score of each part will give some notion of the ability of a class and of a pupil in the operation or process so measured.

What one wishes chiefly to know is whether the response of the class on an individual example in the test is a reliable measure. The reliability of the percentage of error and success on a given example will depend first on how large that percentage is, and second on how

[2] It is well known that similar inspection of a test may be made to judge validity (the extent to which a test measures what it purports to measure). Reliability, one phase of validity, may be judged from much of the same data used for the latter. The method of self correlation for reliability is omitted from discussion because of its technical nature. Those interested in self correlation, which is the more accurate method, will find an excellent treatment in the *Objective or New Type Examination*, by G. M. Ruch.

many cases there are in the group from which the percentage is computed. If 200 pupils in Grade IV are given a test in multiplication in integers, the percentage of the 200 who miss a given example will be a reliable measure of the ability of the group as a whole with that particular example. But when an individual pupil misses such an example, the measurement of his ability on it is not reliable, because one trial does not indicate whether he will always miss the example or not. One should probably use at least twenty examples of a given type to secure an accurate individual measure of a pupil's ability on a given example. But what is sought in the group diagnostic test is not a more accurate measure of an individual pupil's ability to work a given example, but rather the reliability of the abilities of the group in a given example. A large group gives a more reliable measure for such purpose than a small group. A class of only ten pupils is too small for the use of a group diagnostic test unless the abilities run very high or very low. In other words, there are two factors that affect the reliability of a percentage of error, the number of errors made and the number of pupils attempting the example.

Table II has been prepared to indicate to the teacher the sizes of classes and percentages of error that are necessary to give a reasonably reliable measure when the percentage of success or error has been computed. When the probable error is less than five percent, the measurement may be considered sufficiently reliable for most purposes; when greater than five, the percent missing or succeeding has relatively little value as an accurate measure.

This table is significant in dealing with the results of group tests. Classes are of various sizes. There is always danger of teachers making the assumption that the percent of success of a class on an individual example is equally reliable for any class and for any percent. Table II shows clearly that results for classes of fewer than thirty pupils are unreliable, unless the percent of success is high. Of course, percents of error or success may be significant even when they are within ten points of perfect reliability when the error is so large that drill for the entire class should be given on the type of example measured. The table may also be used to determine the probable error of an individual pupil's work on a given number of examples. In such a case N equals the number of examples in the test, and S equals the percent of success. Thus, it may be seen that the performance of

TABLE II.—THE RELIABILITY OF THE PERCENTAGE OF FAILURE OR SUCCESS ON A GIVEN EXAMPLE IN GROUPS OF N PUPILS

S	F \ N	10	15	20	25	30	35	40	50	70	100	200	500	1000
98	2	3.0	2.4	2.1	1.9	1.7	1.6	1.5	1.3	1.1	0.9	0.7	0.4	0.1
94	6	5.1	4.1	3.6	3.2	2.9	2.7	2.5	2.2	1.9	1.6	1.1	0.7	0.5
90	10	6.4	5.2	4.5	4.0	3.7	3.4	3.2	2.8	2.5	2.1	1.4	0.9	0.6
86	14	7.4	6.0	5.2	4.7	4.2	3.9	3.7	3.3	2.8	2.4	1.6	1.1	0.7
82	18	8.2	6.7	5.8	5.2	4.7	4.4	4.1	3.6	3.1	2.6	1.8	1.1	0.9
78	22	8.8	7.3	6.3	5.6	5.1	4.7	4.4	3.9	3.3	2.8	2.0	1.2	0.9
74	26	9.4	7.6	6.6	5.9	5.4	5.0	4.7	4.2	3.5	2.9	2.1	1.3	0.9
70	30	9.8	8.0	6.9	6.2	5.7	5.2	4.9	4.4	3.7	3.1	2.2	1.4	1.0
66	34	10.1	8.2	7.1	6.3	5.8	5.4	5.0	4.5	3.8	3.2	2.2	1.4	1.0
60	40	10.4	8.5	7.4	6.6	6.0	5.6	5.2	4.7	3.9	3.3	2.4	1.4	1.0
50	50	10.7	8.7	7.6	6.7	6.1	5.7	5.3	4.8	4.0	3.4	2.4	1.5	1.1

The table may be illustrated as follows: If in a class of 50, 49 pupils succeeded with a given example (S = 98% success or F = 2% failure), then the chances are even that the measure is not more inaccurate than 1.3%. Again with 60% success (40% F) in the same class the chances are one to one that the measure (60%) may vary as much as 4.7% either plus or minus.

The table holds good only for typical pupils. It may also be used to measure the probable error of an individual score, or percent of success or error, in which case N would equal the number of examples in a test.

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an individual on a small number of examples gives an unreliable measure unless the percent of success is very high (or very low).

As indicated previously, a test which is too short is both invalid and inaccurate. One of the things that makes for reliability is the length of the test. Ideally a test should be several times as long as most tests now in general use. The individual diagnostic test as compared with the survey test attempts to get a measure on a single arithmetical ability or skill, whereas the survey test attempts to get a single measure on a large number of skills. The assumption may be made that it requires almost as large a number of test items to secure a reliable measure on a narrow skill function as it does to secure an equally reliable measure on a large number of functions. If this is true, the individual skill or ability to be measured would have to be repeated long enough to equal in point of time the survey test. Practically this is impossible. G. M. Ruch reports[3] that for narrow functions approaching unit skills perhaps twenty to twenty-five or

[3] *Second Yearbook of the National Council of Teachers of Mathematics*, p. 26.

more examples and three to five or more minutes are required to secure the sampling necessary for reliable results.

c. Of Individual Diagnostic Tests.—The constructing of a reliable test for survey, group diagnosis, and individual diagnosis constitutes three separate problems. In the survey test one is concerned with the reliability of the total score for each individual. In the group diagnostic test one is concerned with the reliability of each type of arithmetical skill for the class as a group. As has been shown already, two of the chief requisites for reliability are: first, a large enough number of cases to constitute a sampling; and second, a narrow enough function under measurement.

The problem of reliability in the individual diagnostic test is, therefore, much more complicated than is the case with the survey test or with the group diagnostic test. The measure of an individual type of skill must be made with a sufficient number of examples of the type and carried on for a sufficient length of time to secure a reliable measure of the individual pupil's performance. This may be done by including a variety of slight variations of a given type of skill instead of attempting to give an exact duplicate of one type of example.

The following group of examples in a well-known individual diagnostic test will illustrate:

$\frac{6}{7} \times \frac{1}{2}$	$\frac{8}{25} \times \frac{3}{16}$	$\frac{6}{35} \times \frac{5}{18}$
$\frac{6}{5} \times \frac{5}{14}$	$\frac{5}{9} \times \frac{24}{25}$	$\frac{1}{4} \times \frac{4}{9}$
$\frac{22}{9} \times \frac{21}{44}$	$\frac{2}{3} \times \frac{9}{2}$	$\frac{1}{12} \times \frac{8}{19}$

3. Norms

There is confusion in the minds of some school people as to the meaning of norms and standards. By 'norms' one refers to the average performance of one group or several groups on a given test. By 'standards' one means the point of achievement beyond which it is uneconomical to go. A standard in accuracy is frequently set at 100 percent. The standard in a number of examples of a given type to be solved in two minutes may be set at ten; the actual norm for the number of examples solved in two minutes may be five for the same

class in which the standard of ten has been set. The average performance, or norm, is scarcely ever as high as the standard should be set.

a. For Survey Tests.—The value of a norm depends on the use to which it is to be put. If the norm is in a survey test which has been standardized and used widely over the entire country, then an administrator may use it to learn whether his school ranks as high in the test as other schools where the test has been used. Such a comparison by means of a nation-wide norm is of value in a community where evidence is needed to convince the board of education, teachers, or citizens that the local school system is above or below the average performance. However, this type of norm usually has little actual significance.[4] When one wishes to make further interpretations, he will probably find upon investigation that the various schools participating in establishing the norm may have had different time assignments for the study of arithmetic. The pupils who took the test may have varied widely in mental age, and different courses of study and textbooks may have been used. In short, there is not much basis for making a careful comparison, except that all of the communities were represented by pupils of the same school grade. It is generally much more to the point to compare the performance of a given class with the performance of other classes of similar intelligence working with the same course of study or the same textbook.

The achievement, or performance, in a given grade in a test for a city will yield a norm that will be useful for the various schools in that city and for other schools using the same course of study. A norm made up from a group of schools similar in type can be used to compare with the achievement in each of the schools concerned. Sometimes a number of cities agree to exchange results on a given test, and if the facts are known about course-of-study requirements, time allotments, and other pertinent data, interesting comparisons of achievement may be made.

Probably the norm most familiar to teachers is the median. A good many teachers understand the significance of the 25 percentile and the 75 percentile, and practically all teachers understand the meaning of the highest 25 percent, the lowest 25 percent, and the middle 50 per cent. Since the median and the percentiles are readily

[4] For a more complete discussion see F. B. Knight, "Crucial questions on arithmetic testing." *Chicago Schools Journal*, September, 1928.

understood, it is easy to take the next step and present norms in the form of percentile rank of scores. This seems to be a significant and practicable type of norm. All teachers know the significance of a score which is better than ninety or any other designated percent of the cases. The percentile rank can always be thought of in relation to the median, the 50 percentile. In some school systems teachers make use of the total percentile curve.

b. For Diagnostic Tests.—For a diagnostic test the use of norms is quite different. In the first place, the major results of giving a diagnostic test are never expressed in terms of a total score. From some of the tests a total score may be obtained, but it is generally of minor significance. Consequently, a norm which consists of the typical score on a diagnostic test would be of no particular value. The function of the diagnostic test is essentially different from that of a survey test. In the diagnostic test the teacher wants to know much more than whether the class or the individual is above or below the norm expected. He wants to know why the pupil is doing the kind of work that he does. It is the intimate analysis which is made of the pupil's processes of work which constitutes the real contribution of the individual diagnostic test. Therefore, the norms which are usable are those which show the common kinds of practice or common habits of work in doing arithmetic. These are generally expressed in terms of frequency tables where the most common difficulty is listed first, the next most common difficulty second, and so on. The complete list constitutes a catalogue of habits of work which may be used by another teacher in identifying the mental processes of individual pupils.

The completeness of this catalogue of habits is one of the chief criteria by which to appraise any individual diagnostic test. The author of such a test is obligated to determine, from the application of his test to many pupils individually, the complete variety of responses which may be expected from a particular operation. Two illustrations may be given of such a frequency table of difficulties encountered in arithmetic. The first of these is shown in Table III which is reproduced from a study made by Brueckner.[5] The data presented there were drawn from an analysis of the answers in a diagnostic test for decimals. This test was given as a group test,

[5] Brueckner, Leo J., "Analysis of difficulties in decimals." *Elementary School Journal,* 29: September, 1928.

and the list of difficulties was derived from the answers which the pupils gave on their test papers. While it is subject to some of the errors of inference mentioned earlier, the reader should note that it is possible to construct test materials in which the probability of more specific and accurate inference is increased.

A second illustration, drawn from an individual diagnostic test for the process of multiplication of whole numbers, is given in Table IV, which was based upon the work of 329 pupils.[6]

TABLE III.—DIFFICULTIES IN THE MULTIPLICATION OF DECIMALS (BRUECKNER)

	Frequency
1. Difficulties Basic to any Multiplication:	
a. Errors in multiplication (facts)	365
b. Difficulties in carrying	38
c. Added instead of multiplying	36
d. Errors in addition	34
e. Multiplier or multiplicand copied as answer	17
f. Inability to multiply by zero	11
2. Difficulties Peculiar to Decimal Situations:	
a. Placement of decimal point:	
(1) Misplacing of decimal point	631
(2) Omission of decimal point	119
b. Zero difficulties:	
(1) Failure to prefix zero	87
(2) Prefixing of unnecessary zero	38
(3) Annexing of unnecessary zero	19
(4) Failure to annex zero	10
c. Inability to express common fractions as decimals:	
(1) Inability to multiply decimals and fractions	62
(2) Answers written in fraction form	52
d. Multiplied whole numbers and added decimals	13
e. Multiplied whole numbers and decimals separately	2
3. Other Difficulties:	
a. Mathematical:	
(1) Misplacing of zero	10
(2) Miscellaneous	20
b. Non-mathematical:	
(1) No attempt	168
(2) Unknown	52
(3) Work incomplete	24
(4) Carelessness	6
Total	1,814

[6] Buswell, G. T., with the coöperation of John, Lenore. *Diagnostic Studies in Arithmetic*, p. 138. (Supplementary Educational Monographs, No. 30. Department of Education, University of Chicago, 1926.)

TABLE IV.—FREQUENCY OF BAD HABITS IN MULTIPLICATION (BUSWELL AND JOHN)

No.	Habit	Grade				Total
		III	IV	V	VI	
1	Errors in multiplication combinations	36	59	60	41	196
2	Error in adding the carried number	6	40	58	45	149
3	Wrote rows of zeros	2	33	40	34	109
4	Errors in addition	5	31	41	21	98
5	Carried a wrong number	5	28	40	22	95
6	Used multiplicand as multiplier	18	33	23	15	89
7	Forgot to carry	10	30	27	22	89
8	Error in single zero combinations, zero as multiplier	11	20	23	27	81
9	Errors due to zero in multiplier	5	26	30	17	78
10	Used wrong process	18	22	16	10	66
11	Counted to carry	4	20	28	9	61
12	Omitted digit in multiplier	1	15	20	16	52
13	Wrote carried number	8	16	14	9	47
14	Omitted digit in multiplicand	2	17	12	12	43
15	Errors due to zero in multiplicand	4	14	15	9	42
16	Counted to get multiplication combinations	15	11	9	5	40
17	Error in position of partial products	0	15	15	9	39
18	Error in single zero combinations, zero as multiplicand	7	13	11	8	39
19	Confused products when multiplier had two or more digits	1	13	9	9	32
20	Repeated part of table	3	11	11	6	31
21	Multiplied by adding	6	11	8	4	29
22	Did not multiply a digit in multiplicand	5	9	7	7	28
23	Derived unknown combination from another	3	11	6	6	26
24	Errors in reading	6	5	11	3	25
25	Omitted digit in product	0	5	7	5	17
26	Error in writing product	2	4	8	2	16
27	Error in carrying into zero	1	6	7	1	15
28	Illegible figures	0	3	5	7	15
29	Forgot to add partial products	0	3	7	2	12
30	Split multiplier	0	1	6	4	11
31	Wrote wrong digit of product	0	3	4	2	9
32	Multiplied by same digit twice	1	1	3	2	7
33	Reversed digits in product	1	1	2	2	6
34	Wrote tables	0	0	4	1	5
35	Used multiplicand or multiplier as product	1	1	1	1	4
36	Multiplied carrier number	2	1	0	1	4
37	Used digit in product twice	0	1	2	0	3
38	Added carrier number twice	0	1	1	0	2
39	Carried when there was nothing to carry	0	0	1	0	1
40	Began at left side	1	0	0	0	1
41	Multiplied partial products	0	1	0	0	1
	Total number of subjects	47	98	102	82	329

The norms on survey tests are chiefly administrative devices; the norms on diagnostic tests are chiefly teaching devices. The difference is of fundamental importance. Individual diagnostic tests are of value in proportion to the light which they throw on the intimate

processes of the pupil as he works his problems in arithmetic. Their whole purpose is to analyze the pupil's mental processes so that the teacher may redirect his thinking where it is less efficient than it should be. Therefore, a good diagnostic test is one which reveals the many details of the pupil's method of work and which makes possible a type of teaching which goes to the very basis of the difficulties which the pupil encounters.

4. Characteristics of Good Individual Diagnostic Tests

Another way in which the individual diagnostic test differs from the survey test is in regard to the use of time limits. There is no point in imposing time limits in an individual diagnostic test. The pupil should be allowed entire freedom to work the examples in just the way that he would normally work them, whether this is rapidly or slowly. The purpose of the diagnostic test is always to reveal how the pupil works rather than how much he does. Where the test is given individually, the rate of work will depend somewhat upon the amount of detailed analysis which the teacher is able to make of the pupil's processes.

From what has been said it will be clear that a good individual diagnostic test should possess these characteristics: (1) It should be based upon a graded set of examples in which the various types of difficulty which occur in given arithmetical processes appear singly and in order from the easiest to the most complex. (2) It should be accompanied by a catalog of habits of work arranged in order of frequency, with which the teacher can compare and check the procedure of the pupil being tested. This frequency table of habits should be based upon the work of children of the same grade as that of the child being tested. Furthermore, a supplementary manual should be available to give sufficient illustrations for each of the habits of work listed in the frequency table that the teacher will be perfectly clear as to the nature of the mental process which is characterized by each category in the catalogue of habits. (3) An individual diagnostic test needs no time limits and needs no norms expressed as total scores, but in their place it should have a clear and detailed description of the mental processes which characterize the work of the pupil.

There is one further contribution which a good diagnostic test, either group or individual, may make to the work of the class. If

the catalogue of habits supplied with the test is based upon an adequate number of cases, the habits which occur with greatest frequency may be selected for the basis of general teaching, since these are the sources of error for the majority of pupils. To illustrate, in the process of addition in practically every elementary-school grade counting is one of the most common undesirable habits. Counting is a primitive method of adding, and ought to be replaced by the automatic responses for each number combination involved in addition. Since the percentage of children who count is so large, a teacher is safe in assuming that general instruction in regard to the disadvantages of counting and the superiority of adding by direct combinations may be given to the class as a whole. If one would select the first ten habits of work from the frequency tables which are provided with individual diagnostic tests, one would have a list of the most common causes of errors in the process concerned. Teaching might proceed, therefore, by a direct attack upon these common causes of error and by an attempt to substitute better methods of work before more drill is given.

5. Administration and Scoring of Tests

Extended discussion of the administering, scoring, and timing of arithmetic tests under standardized procedure is omitted from this chapter because these matters have been so adequately treated elsewhere. A word may be said, however, about the value and accuracy of scoring by pupils. It is common practice for pupils to exchange papers and score them in informal tests in arithmetic and spelling. Experiments in pupil-scoring of both standardized and unstandardized tests have indicated that, if carefully supervised, the scoring of pupils is as accurate as the scoring of teachers; its accuracy is much increased if the unreliable scorers among the pupils are detected and their papers carefully reviewed by the teacher.

IV. How Group Difficulties Occur in Teaching and Learning

Although great individual differences manifest themselves in arithmetical abilities, it should always be borne in mind that in an ordinary class most of the pupils will have similar abilities in any given trait. In spite of the over-lapping among grades, probably from 50 to 75 percent of the pupils in a given class are reasonably similar in their arithmetical abilities and disabilities. This is true

partly because they have similar abilities to begin with and partly because they have been under the same instruction in the subject. It is possible in every class to find a group of pupils who have the same troubles in arithmetic. In a few instances it has been found that an entire class is incompetent in some one ability or skill. Teachers have long taken advantage of this fact, although not always intelligently. It is more economical in time to give group instruction than to give individual instruction. There are several reasons why groups as a whole, or a large part of a group, have the same difficulties in learning arithmetic. A discussion of some of these reasons follows.

1. Failure of the Teacher to Know What Is Involved in the Process

One of the first requisites in teaching arithmetic is that the teacher be familiar with the learning process in arithmetic. What is the child's idea of number when he begins his elementary-school career? What are the associations between his knowledge of numbers and the new things that he is to learn? Fortunately much work has been done and some progress made in working out the steps in learning the mechanical processes in arithmetic. But probably even the best arithmetic teachers in the past have assumed that some things are easily understood by a child which in fact are difficult to understand. Children have been required to take long steps from an easy, known, and well-understood process to a difficult and distantly related process. If teachers had understood all the steps in learning involved, the intermediate steps between the things known and the things to be learned would have been taught and easily understood. When a teacher takes the position that a pupil who has learned to do long division ought to be able to solve any example in long division, and that the only way to secure skill in long division is to give more drill generally on that operation, this teacher has no right to expect uniformly satisfactory results. The first requirement of intelligent instruction is that the teacher be familiar with the steps of learning involved.

Textbooks and courses of study differ greatly in the provisions they afford for clearly presenting arithmetical processes and operations. In spite of the explanations given in textbooks and many times in spite of the work of the teacher, new operations first presented to a class may remain vague and indistinct in the minds of

the pupils. In such cases, as in all other cases where deficiencies lie in the instruction given, this calls for group diagnosis and group instruction.

2. Inadequate Initial Learning

A method of teaching in arithmetic that is frequently used is somewhat as follows: The teacher presents a new process to the pupils without adequately tying it up or relating it to previously known and understood processes. Most new processes are easy for pupils to understand if the steps in learning have been observed as indicated in the foregoing paragraph. But instead of making the new processes understandable by relating them to previously learned processes, many teachers attempt to fix the processes in the minds of pupils only by blind, unanalyzed drill; so practice is begun on new processes and continued intensively until all, or most of the pupils can do the process with some facility.

When the teacher judges that enough time has been spent on this subdivision of the course or when the work on the process in the textbook has been covered, the class goes on to a new process, many times without any further drill on the process just studied except as it happens to be used in the new process. In other words, the understanding of a process is not secured by relating it to what is previously learned, but rather by drilling on it. In many cases, not enough drill is given for proper initial learning, but only enough to give an understanding of the process. This is making use of the drill for the wrong purpose and failing to do first teaching as it should be done. This results in disabilities that affect entire groups.

3. Change of Emphasis in Teaching

Teachers vary from year to year in the emphasis they place on different processes and in their methods of teaching. A teacher is likely to emphasize one process one year and, because of changed interest, neglect it the following year. The pupils may benefit or may suffer from these changes in emphasis. This is a matter that can not be controlled entirely by course of study requirements or textbooks. Teachers should be allowed some leeway in modifying methods and in placing emphasis. Unless some system of group testing and group diagnosis is employed, there is no easy way to detect these changes in emphasis, and pupils may go forward to the next grade efficient in one set of skills this year and in another set of skills next year.

4. Inadequate Maintenance Program

As has been indicated before, it is necessary not only to drill a new process until a child understands it, but also to continue the drill until a habit is formed. After habit formation has taken place and the teacher finds it necessary to go on to a second process, the first process may or may not be used in the second. It is well known that mechanical skills, regardless of how highly they may be developed, are soon lost unless practice in them is maintained. It is therefore extremely important that a maintenance program of drill be laid out and that group measurement and diagnosis be continued to locate the needs for remedial work and to give a measure to show whether the maintenance program is adequate. The group diagnostic test is an excellent device for making diagnosis of difficulties at any time. This makes it possible to carry on remedial work whenever needed to offset chronic repetition of error.

5. Inadequate or Poorly Organized Practice

Only in rare instances do textbooks furnish enough or well organized drill material in arithmetic. Most pupils in the elementary schools in the United States are working under a system of inadequate practice.

First, the practice is likely to be poorly organized. The exercises first placed in the hands of pupils when a new operation is being taught are frequently too involved and complex. In order to make them simple enough, in some cases it is necessary to present intermediate steps leading up to the new operation to be taught. What has been said before regarding the steps in learning applies here.

Second, since most textbooks do not offer enough practice material, teachers have to make up supplementary exercises or copy them from other books. In making up such exercises or in copying them, most teachers do not have the time to grade them in order of difficulty and to fill in the gaps with additional exercises of proper difficulty so that the final material represents a scale arranged in order of difficulty.

As has been indicated in a previous paragraph, the mistake here is two-fold: teachers are likely to fail to make a new operation clear in first presenting it and then to use practice materials to secure a working understanding of the operation. By the time they have

secured this understanding and before habits have been formed, they leave the operation and go to another new one.

The result is that at the end of the semester pupils have forgotten the operations taught and by the time they reach a new grade the work of the previous grade is almost unknown by them. Of course, there are many cases where the operation is set in a necessary sequence and must be used in each succeeding operation. In such cases there is more practice than when the operation is more isolated and rarely used.

V. Diagnostic Work in Arithmetic

1. Treatment of Results for Group Diagnosis

When a test that contains all significant types of examples in a process or operation has been given and the errors have been checked, the first step for group diagnosis is to discover the frequency of the various errors made. For this purpose the teacher should tabulate the number of times each example is missed or omitted by the pupils in the class. If norms of building or city scores are available, he should then find out the percent of the class missing each example. Some school systems furnish percentage tables for this purpose. This gives a measure of group disability on each type. This should be the first step in group diagnosis. The next step is the consideration of the significance of each percent missing. Table II shows the reliability, in terms of probable errors, for various scores and various sizes of class tested.

One significant phase of Table II for the treatment of results for group diagnosis is the relative unreliability of percentages missing or succeeding that lie between 75 and 25. Thus, in a class where 50 percent of the pupils fail and 50 percent succeed, the very size of these percents indicates that many of those who get the example right are insecure; on another test many of those who got the example right might get it wrong, and many of those who got it wrong in this test, on another might get it right. Also when as many as from 25 to 75 percent miss an item on a test, it is probably a case for group diagnosis of disabilities and remedial work, in order that the same error will not occur again.

If the percentage of error is less than 25, the measurement is more reliable for the group and the errors are probably individual or accidental. If they are individual disabilities, it is then a case for

individual diagnosis. A small section of a class or sections of classes require either individual diagnostic tests or longer group diagnostic tests containing more items than the sort that is used successfully for diagnosis of a larger group.

When the teacher, supervisor, or research worker has found the percentage missing each type of example in arithmetic tests, he has a measure of errors that exist in his class, but his information is incomplete until he finds out what skills and abilities are involved in each such example. A number of studies have been made of the abilities and skills, or new steps, involved in fundamentals, in integers, fractions, and decimals. Analyses of what is involved in the various types of examples in arithmetic will greatly facilitate group diagnosis. For example, if the pupils in a given class are able to add single columns of figures and double columns of figures in which carrying is not required, but fail on double columns with carrying, then the teacher may be certain that their difficulty is in carrying. A list of abilities involved in carrying makes it possible for the teacher to proceed still farther with the diagnosis. The next need is for remedial work of the type which will serve the particular purpose. Such remedial measures are discussed farther on in this chapter.

2. Individual Difficulties

After group diagnosis has been made, there will always be a small number of cases demanding a large amount of individual attention. These few cases will be, in general, the poorer pupils in the class. In addition to these, there will be another group containing some average or better-than-average pupils who will be able to do the majority of their work in arithmetic in a highly satisfactory manner, but who may have difficulties with certain processes or who may have certain cumbersome habits of work which should be corrected. For these cases, individual diagnostic procedure should be carried on. The basic principle of this procedure, it should be recalled, is that the causes of the errors should be dealt with rather than the errors themselves.

The principal questions that arise are: How do errors occur? How diverse are the causes of errors? Are the causes which account for the great majority of errors which the pupils make few or many? Furthermore, in an individual case, will one find a few or many

bad habits of work? To answer these questions, some data will be drawn from an investigation in which the individual method of diagnosis was used.

In an earlier section of this chapter, there was given a list of forty-one habits of work obtained from the diagnoses of children while multiplying (Table IV). The total frequencies given in the last column in this table make a sum total of 1713 for all the habits of work listed for these 329 pupils. Thirty-seven and eight-tenths percent of these 1713 frequencies were due to the first five of the forty-one habits of work; 61.3 percent were due to the first ten of the forty-one habits, and the first fifteen habits accounted for 75.6 percent of the total. It is apparent that while one might get an extended list of bad habits of work from a group of three hundred pupils, three-fourths of the habits could be classified under fifteen categories. The task of improving pupils' methods of work is, therefore, not a hopeless one. If 37.8 percent of the faults for this particular group of pupils could be traced to five habits, one could justify a good deal of specific teaching on these five points. For such common habits as these one might even deal with the class as a whole.

However, individual diagnosis presupposes individual treatment. It is pertinent, therefore, to inquire as to the distribution of these poor methods of work in the case of individual pupils. Does each pupil constitute a separate problem or would one find considerable similarity in a diagnosis of the pupil's procedure? Referring again to the same data on multiplication, the number of different habits of work, presumably poor ones, for each pupil may be listed as in Table V. This table is self-explanatory: in seven pupils no poor habits of work were noticed; one poor habit of work was noticed in five pupils in the third grade, eleven in the fourth, seven in the fifth, and one in the sixth, and so on. From the medians at the bottom of the table, it appears that one might expect from an individual pupil a total of from four to six specific habits of work which need correction, but in an extreme case as many as twenty-four habits might need correction—it might be better to demote such a pupil, of course.

In Tables IV and V summarized data have been given as to the occurrence of various habits of work. However, the most enlightening data are not those derived from summaries, but those which show the occurrence of difficulties with individual pupils.

TABLE V.—NUMBER OF BAD HABITS OF WORK PER PUPIL IN MULTIPLICATION

Number of Bad Habits	School Grade				Total Number of Pupils
	III	IV	V	VI	
0	1	0	3	3	7
1	5	11	7	1	24
2	6	8	10	7	31
3	9	11	14	11	45
4	10	11	11	11	43
5	5	13	11	13	42
6	3	11	8	10	32
7	3	10	5	9	27
8	4	5	9	9	27
9	0	4	9	2	15
10	1	6	5	4	16
11	0	2	3	1	6
12	0	1	0	1	2
13	0	4	0	0	4
14	0	1	4	0	5
15	0	0	1	0	1
16	0	0	1	0	1
25	0	0	1	0	1
Total	47	98	102	82	329
Median	4.3	5.6	5.5	5.6	5.3

In Table VI[7] the distribution of the specific habits of work for each pupil in a fifth-grade class is shown. The first pupil, for example, revealed eight specific difficulties in multiplication when his diagnostic test was taken. These eight specific difficulties arising in examples 1, 2, 5, 8, etc., were as follows: (1) errors in combinations, (2) errors in adding the carried number, (3) carrying a wrong number, (4) error in zero combinations using zero as multiplier, (5) omitting digits in the multiplicand, (6) using the wrong process, (7) multiplying by successive additions instead of by multiplication, and (8) omitting digits in the product. Subsequent rows in the table give the corresponding data for each pupil in this class. From such a table as this a teacher can see exactly how errors occur and how they are distributed among the various pupils in the class. It is quite apparent from a cursory observation of the table that Habits Nos. 1, 2, and 5 (Examples 1, 2, 5) are frequent, but that the other types of error are less frequent.

[7] *Ibid,* p. 129. (Data Redistributed According to Categories in Table IV.)

TABLE VI.—DISTRIBUTION OF DIFFICULTIES IN MULTIPLICATION: EIGHTEEN FIFTH-GRADE PUPILS AND FORTY-ONE TEST EXAMPLES

Pupil	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	Total
1	×	×			×			×		×				×							×				×																	8
2	×	×		×	×			×	×	×	×	×			×			×	×		×												×									14
3	×	×	×	×		×	×				×		×	×													×															10
4	×	×		×							×	×	×									×						×														8
5	×	×			×		×					×						×	×										×		×											9
6	×	×	×		×			×		×			×					×				×																				9
7						×																											×									2
8	×	×			×		×			×					×	×		×				×			×							×										11
9	×	×			×							×																														4
10	×		×	×		×	×							×							×		×																			8
11				×																																						1
12	×				×		×			×				×									×																			6
13	×	×			×		×	×	×				×																													7
14	×			×	×			×	×	×																																6
15	×				×	×	×							×			×						×				×															8
16			×								×					×															×											4
17	×				×						×						×																									4
18	×	×		×	×			×	×	×	×			×	×		×	×	×										×									×				15
Total	15	10	4	7	12	4	7	6	4	7	6	4	4	6	3	2	3	5	3	0	3	3	3	0	2	0	2	1	2	0	2	1	2	0	0	0	0	1	0	0	0	134

The problem of teaching a specific class is well pictured in Table VI. For various reasons, during the grades preceding the fifth, these pupils have acquired different habits of working multiplication examples. This variety of habits has accumulated until the distribution is as shown in the table. The teacher of the grade is responsible for seeing that children improve in multiplication. It is obvious that these difficulties which are the causes of poor work must be removed. Both common sense and experimental evidence agree that a desirable method or procedure is to attack the habits of work which need correction one by one. The point which the writer wishes to emphasize here is that these habits are widely scattered and that without attention to individual cases much teaching will be blindly ineffective and some poor habits of work may escape correction altogether.

3. The Process of Individual Diagnosis

The diagnosis of difficulties in arithmetic may be carried on either by the individual method or by the group method. Group diagnosis has for its primary purpose the discovery of types of errors. Following the discovery of these types of errors, group remedial treatment is planned which can be applied to the class as a whole. For example, if in a group diagnostic test in addition it is found that many pupils do not know how to carry, a teacher would conclude that carrying should be made the subject of special instruction. Consequently, the probable procedure would be to apply remedial drill made up of a carefully selected list of examples involving carrying of the particular type needed. The important point to note about group diagnosis is that it proceeds by discovering errors and then, by giving special instruction and drill, covering the topics in which the error occurred. It should be noted further that group diagnosis does not reveal the causes of errors except in arithmetical terms; for example, the cause of errors in addition may be lack of knowledge of how to carry, but the group diagnostic test does not show what methods of work the child used in trying to carry. Individual diagnostic testing begins at this point. Its chief function is to supplement the discovery of types of error by the discovery of the causes of error and to correct these causes by the application of the particular kind of instruction which an individual pupil needs in order to remedy the crude or erroneous method of procedure rather than by the application of more group instruction or drill. Therefore, it can readily be seen

that there is no conflict between group diagnostic testing and individual diagnostic testing. Group diagnostic testing should always come first and from it the cases which need special attention will be located. The individual diagnostic testing which follows has for its function the further analysis of the particular difficulties which the pupil experiences in order that his method of working examples may be made more effective. If individual diagnostic tests can be applied to the lowest quarter of pupils in a classroom, they will probably serve their purpose without any attempt to use them with all pupils. However, the only reason for not applying them to every member of a class is that they require more time than the average teacher is able to give to them. If classes were smaller, individual diagnostic tests could be recommended with profit for all pupils.

4. The Technique of Individual Diagnosis

In making individual diagnosis, the chief purpose is to discover how the child works when he proceeds in his ordinary manner. Consequently, the teacher needs to be very careful to avoid giving suggestions as to how work should be done; *i. e.*, the diagnostic procedure and the remedial procedure should be separate processes, and the teacher should not attempt to remedy a poor method while making a diagnosis. Since the work must be done individually, the teacher will need to assign some seat work to the class, then select a child whose work is to be diagnosed, and sit down with him at his desk or at a table in the corner of the room. He should begin by making the child feel as much at ease as possible, as the success of diagnostic testing depends upon the extent to which the teacher becomes acquainted with the details of the pupil's natural methods of work. It is important never to show disapproval even though the child's methods of procedure may be exceedingly crude and cumbersome. The time for correcting his methods is during the remedial period, never during the diagnostic period.

It is advisable to work with one process at a time, as, for example, with addition. Most normal results will be secured if the period of diagnosis is limited to from fifteen to twenty-five minutes depending upon the maturity of the child. It is important to avoid fatigue.

When ready to begin the diagnosis, the teacher should present to the child the examples to be used in the test, arranged in order of difficulty. As the child works, the teacher should make written notes

as to what he says. The child should be told that the teacher wants to know just how he gets his answers and that for this reason he is to do as much as he can orally. If the child pauses in his work and gives evidence of not expressing all his thoughts about the example, he should be asked immediately afterward to tell how he thought that out. This should be done at the end of each pause rather than at the end of the example because the child will not remember his mental processes very long. After some practice the teacher will be able to derive in this way a surprising amount of information as to the methods of work which children use.

As the child works, the teacher should check on the list of habits which accompanies the diagnostic test the particular habits of work observed and should also make as full notes as possible as to just what the child says. When the diagnosis is finished, the teacher should have a clear knowledge of the specific habits of work which are responsible for the errors in the process being tested. A concrete illustration of the results of the diagnostic test is shown in the accompanying chart. Similar illustrations can be drawn from the use of any diagnostic test.

Sample Diagnostic Chart

Name *John Doe* School *Lincoln* Grade *IV* Age *10*, I. Q. *98*. Teacher's preliminary diagnosis: *Slow and inaccurate in fundamental operations.*

Addition: (Place a check before each habit observed in the pupil's work).

x 1. Made errors in combinations
x 2. Counted
3. Added carried number last
x 4. Forgot to add carried number
5. Retraced work after partly done
6. Added carried number irregularly
7. Wrote number to be carried
x 8. Carried wrong number
9. Proceeded irregularly in column
10. Grouped two or more numbers
11. Split numbers
x 12. Used wrong fundamental operation
13. Lost place in column
14. Depended on visualization
15. Disregarded column position
16. Omitted one or more digits
17. Made errors in reading numbers

18. Dropped back one or more decades
x 19. Derived unknown combination from familiar one
20. Disregarded one column
x 21. Made error in writing answer
22. Skipped one or more decades
23. Carried when there was nothing to carry
24. Used scratch paper
25. Added in pairs, giving last sum as answer
26. Added same digit in two columns
27. Wrote carried number in answer
28. Added same number twice
29. Began with left column
30. Confused columns
31. Added carried number twice
32. Subtracted carried number
33. Added imaginary column

(Write Observation on Pupil's Work in Space Opposite Examples)

(1) 5 6 2 3 — — 7 9	*Correct*
(2) 2 8 9 4 — — 11 13	*Error in combination (Habit No. 1)*
(3) 12 13 2 5 — — 14 18	*"13 and 5 are —10 and 5 are 15, 11 and 5 are 16, 12 and 5 are 17, 13 and 5 are 18." (Habit No. 19)*
(4) 19 17 2 9 — — 11 71	*"9+2 is 11, bring down the 1." (Habit No. 4)* *"7 and 9 is 16, 6 and 1 is 7." Carried wrong number (Habit No. 8)*
(5) 6 + 2 = 12 3 + 4 = 12	*Multiplied instead of added (Habit No. 12)*
(6) 52 40 13 39 — — 65 79	*Correct*

(7) 78 46 71 92 — — 149 38	*"6 and 2 are 8, 9 and 4 are 13." Error in writing answer, omitted the "1" in 13 (Habit No. 21)*
(8) 3 8 5 7 8 9 2 7 — — 18 31	*Counted on fingers. Said "8 and 7 are 15, and 9 are —16, 17, 18, 19, 20, 21, 22, 23, 24, and 7 are —25, 26, 27, 28, 29, 30, 31." Touched one finger for each count (Habit No. 2)*

In diagnosing the case of John Doe, shown in the sample chart, the first set of examples was worked correctly, with no indication of any difficulty on the part of the pupil. Consequently, the teacher making the diagnosis simply wrote the word "correct" following these two examples. In the second set of examples, the last one, 8+4, was given the answer 13 instead of 12. Consequently, the teacher wrote "error in combination" and put a check mark before Habit No. 1. In the third set of examples, the child had difficulty with the addition of 13+5. The teacher recorded exactly what the child said, which was, "13 and 5 are —10 and 5 are 15; 11 and 5 are 16; 12 and 5 are 17; 13 and 5 are 18." In this case the child did not know the combination 13+5 but did know the combination 10+5. Consequently, he started with 10+5 and by adding one unit three times successively he was able to secure the correct answer. Habit No. 19 was checked accordingly. In this case it will be noted that the child did secure the correct answer, but the method by which he secured it was so slow and cumbersome that it should be eliminated by sufficient drill on the combinations to make them entirely automatic. In the fourth set of examples the child added the first by saying "9 and 2 is 11, bring down the 1." In this case he forgot to add the 1 that he carried to the 1 in the first number and therefore the teacher checked Habit No. 4, "Forgot to add carried number." In the second example in this same set the child said, "7 and 9 is 16, 6 and 1 is 7." Here the child carried the 6 of the 16 instead of the 1 and the teacher accordingly checked Habit No. 8, "carried wrong number." In the particular section of the test given here the teacher identified seven habits which need correction. She now has for this particular pupil specific information to the effect that he makes errors in his combinations, that he counts when he should make the combinations automatically, that he forgets to add the carried number, that he sometimes carries the wrong number, that he sometimes uses the wrong fundamental operation, that he derived unknown combinations from familiar ones, and that he made errors in writing answers. The correction of these particular habits of work then becomes a specific order for the remedial period and the child's erroneous methods of procedure are corrected before he is given further drill to do.

Some further comments should be made on the amount of time required for diagnosis. The first objection which is usually raised to individual diagnosis is that it requires more time than the teacher is able to give to it. This criticism is not entirely justified. The initial diagnosis does require considerable time. However, when this is once made, the teacher has a definite body of information upon which to base her teaching. Check diagnoses will have to be made from time to time to determine the extent to which the child's poor habits of work are being corrected. These require only a small amount of time. If one considers the amount of time that children spend in practicing things they already know how to do and the amount of time the teacher spends in instructing children in processes which some of them have already mastered sufficiently, the amount of time spent for diagnosis does not seem so large.

Furthermore, when one considers the difficulties in arithmetic which are caused by crude habits of work which are never discovered by any other method than individual diagnosis, it becomes apparent that in the long run the diagnostic method may save the teacher's time. The amount of time spent in diagnosis should be judged in terms of the time required for the entire period of teaching a given process rather than simply in term of the immediate amount of time used in making the diagnosis.

Real teaching involves the analysis of procedures for the pupil which he cannot analyze for himself. Real teaching must be individual teaching to a very considerable extent. The teacher who is unaware of the mental processes by which the pupil does his work is laboring under decided limitations. There is no reason why the same type of diagnostic exactness should not be expected as in the case of the medical specialist. Furthermore, there is no reason why the average teacher should not be able to locate difficulties in learning with the same effectiveness that the average doctor locates specific physical disabilities. There is a place for mass teaching just as there is a place for public health service, but teaching should no more be limited to mass instruction than public health service should be considered the whole of medical practice. Effective teaching is specific teaching. In the case of children who have serious difficulties, there is no way to avoid careful individual diagnosis, whatever time or effort this demands.

VI. Remedial Work in Arithmetic

1. Importance of Preventive Measures

The present emphasis on remedial work is a reflection on the lack of good teaching of arithmetic. The necessity for a large amount of remedying indicates previous faulty processes which have produced difficulties. While it is probable that no scheme of teaching will ever entirely eliminate remedial work, it is certainly to be expected that the amount of such treatment should grow notably less. The end toward which the schools should work is prevention rather than remedy. A school should pride itself on the lack of necessity for remedial work rather than on the elaborateness of this work.

It is to be expected that many difficulties in arithmetic will be eliminated by anticipating them and by making proper adjustments by means of the teacher's explanations, more favorable distributions of time devoted to various topics, possible relocation of the grade-placement of certain topics, possible changes in the sequential order of topics, and more detailed and elaborate analyses of the mental processes of pupils as they are working. If a school maintains an adequate program of tests during teaching, difficulties in arithmetic can be checked so promptly that they will be corrected during the teaching period rather than allowed to accumulate to make a problem for remedial procedure. If one could locate the breakdown in a given process as soon as it occurs, the subsequent difficulties due to that breakdown might be avoided entirely. For example, on taking a diagnostic test in addition in the fifth grade one pupil was found to have made errors on exactly half of the problems in the total test. In checking this pupil's results, it was found that she made errors in 28 of the 48 examples in multiplication. Of the 28 errors, 24 were due to mistakes in carrying. Here was a child who had not learned how to carry when it was being taught, but who was allowed to go on with this serious gap in her arithmetical knowledge. While remedial work with this child was far better than to allow her to continue in her difficulties, the most profitable treatment would have been to have located her difficulty while the topic of carrying was being taught. Some of the energy which is now going into diagnostic and remedial treatment might well be diverted into channels of preventive work. Frequent unit tests by which the child's difficulties can be discovered promptly seem to be one of the most needed elements of the testing program. One can find examples of

this attitude toward testing in some of the more recent textbooks in arithmetic and also in some of the courses of study, but in common practice faults of procedure are allowed to accumulate until eventually there is a major problem of remedial teaching which must be cared for.

One of the most pernicious causes of incorrect habits of work in arithmetic is the application of drill exercises on processes which the pupil has not yet learned to do correctly. Without some diagnostic test the teacher is not aware of the pupil's methods of work. Consequently, drill exercises tend to 'set' the bad habits of the pupil, with the result that he becomes more efficient in these bad habits. Such drilling may do more harm than good. To illustrate, if the fifth-grade girl who did not know how to carry when multiplying were given much drill in this operation, she would either become entirely confused or the erroneous habits of carrying would become so fixed that she would find real difficulty in breaking them. On the other hand, if she were given a careful diagnosis which would reveal this deficiency, specific remedial instruction could be given before drilling, with the result that the drill would automatize a correct, rather than an incorrect, habit of work. Only correct drill makes perfect; incorrect drill only makes matters worse. It is precisely at this point that prevention of errors can best be made. Regardless of how good drill materials may be or how scientifically practice is distributed, their benefits will not accrue to the pupil whose methods of work are erroneous, and only an individual diagnosis will reveal the mental processes which need to be set right before drill begins.

2. Remedial Work for the Group

a. Diagnosis through Tests and Remedial Work.—Many teachers carry on remedial work in arithmetic without having any adequate reason for doing so. It seems to be a common notion that the thing to do in teaching arithmetic is to give practice on the operation being taught. If pupils show hesitancy, then give more practice. Some teachers go farther and make the assumption that difficulty with an involved process is caused by weakness in some simpler process which is involved in the more complex one. For example, some teachers make the assumption that failure in column addition is caused by lack of familiarity with the hundred number facts in addition.

Pupils who are coming into a school for the first time find the environment new and the methods different from those they have been

taught. They often find difficulty in making adjustments to the new teacher and new textbooks. Without a thorough and systematic examination of their abilities, pupils may give the impression that they are ignorant of a process when they need only a little practice to make them familiar with the new methods and devices in use. The opposite of this is also true. Pupils sometimes give the impression that they know a process when a systematic test would show weaknesses. It is usually very wasteful to proceed with remedial work until a diagnostic test has been given which covers all essential skills in a process and reveals the weaknesses of the pupils. Such a complete diagnosis points very definitely to remedial work.

b. Class Organization and Remedial Work.—As pointed out in an earlier discussion in this chapter, many times difficulties occur for the class as a whole or for a large proportion of the class. There is nothing new about this idea. Teachers have long recognized the fact that a portion of the class has difficulties of a similar sort. The mistake has generally been made in the past to give all pupils the same drill, with the result that for some it was a waste of time because they did not need the work, while for others it was a waste of time because the drill was beyond their powers. Diagnostic test results, properly analyzed, will identify the pupils who need drill on a given process and on a given phase of a process. Advantage should be taken of such test results in organizing the class so that every pupil will get the kind of practice that he most needs.

Suppose that the group diagnostic test has been given in addition of integers. It is found that ten percent of the class are practically perfect. This group may be excused from remedial drill and receive from time to time only such practice in addition in integers as is necessary for the maintenance of skills.

Suppose that 50 percent of the class shows no ability to add long columns of uneven numbers (indicating that the empty spaces cause them difficulty). This shows clearly the type of practice that is needed.

Suppose that 25 percent of the class show lack of facility with addition in long columns of three or more figures. It is likely that careful examination will show that the same individuals who are having difficulty with this type of example are likewise having difficulty with the long column addition with missing numbers. This indicates the necessity for going through the record of each pupil to find out where

his disability first occurs. Those pupils who first have difficulty with adding uneven numbers can probably bring up their work by sufficient drill on this type of exercises. The other pupils, however, have difficulty not only with adding columns with uneven numbers, but with all examples where carrying is involved. The most economical procedure for them is to start remedial work on examples with simple carrying and not to begin drill immediately with long examples involved in both carrying and uneven numbers.

The teacher may thus have several groups within a class who need varying types of drill. The teacher may discover after he has checked up the individual records of his pupils that there are certain individuals whose work is so deficient that they require individual remedial programs. These pupils may in some cases have to start with the most elementary phases of the process involved. The teacher may also discover that the group diagnostic test has not been adequate for indicating the difficulties of certain pupils in sufficient detail so that a remedial program can be outlined. These pupils should then be given individual diagnostic tests.

c. *Furnishing Incentives for Remedial Work.*—As has been said, the normal test situation is the normal learning situation. Pupils and teachers are working toward a common end. When pupils take tests that cover the work they have studied, they are desirous of knowing the results. After tests have been scored, pupils should be allowed to see their papers; in fact, the best method is for the teacher to hand back the papers after the diagnosis has been carried on and the class deficiencies for remedial work have been pointed out. The teacher may then call together a given group, show them on their own papers the examples in which they have failed, and talk over the remedial program that they need to carry on. This is the best type of incentive for remedial work. The pupil sees the need in the same way the teacher sees it.

The keeping of a progress chart adds another incentive for remedial work, for the pupil takes pride in competing with his past record. Many types of progress charts have been developed. Some of them are individual, kept by pupils in their drill books; others are designed for the group as a whole.

d. *Furnishing Material for Remedial Work.*—The obvious thing for the teacher to do when pupils miss a given type of exercise is to give additional practice on that type. When a pupil shows from

60 to 90 percent efficiency on a given type of work, his need is chiefly practice. It would probably be uneconomical to make a careful, minute diagnosis of his errors and their causes. The chances are that intensive practice for a short time on the type of thing missed will bring his efficiency up to 100 percent; if not, then it is time for a more complete diagnosis. In many cases practice on the type of work missed, if given at the regular period, will be sufficient to bring the pupils up to the standard.

Those pupils who have missed the most difficult examples in a given process may strengthen their habits by beginning practice on the easier examples in that type of work. Some pupils who miss long examples in multiplication of integers with two-place or three-place multipliers do all other types of work in multiplication correctly. They probably need further practice on the most difficult types. But those other pupils who miss not only all of the most difficult examples but many of the easier ones as well show that they are insecure in multiplication whenever carrying is involved. They even make some errors in examples without carrying. For them the drill should begin with very simple examples in multiplication, beginning with multiplication with a one-figure multiplier and without carrying, then going on to two-figure multipliers without carrying. After they have gained facility and self-confidence in the easier forms of multiplication, they are ready for drill on the more involved forms of the process. It would be a waste of time for them to join the other individuals in the class who showed difficulty only in the most involved exercises in the test.

It may be, however, that even the easier types of examples in a process are too difficult for a pupil because when they are analyzed they are still found to be complex in nature. If the pupil who has had difficulty in division in integers is found upon drill to be unable to do any considerable number of examples where carrying is involved, then the preparatory steps in teaching carrying in multiplication should be taken up. For example, if a pupil is unable to do $51 \div 3$, he should have practice on $3 \div 3 = 1$, $4 \div 3 = 1$ and one remainder, $5 \div 3 = 1$ and 2 remainder, and so on.

The procedure for furnishing material for remedial work has been outlined in the order in which it should be carried out in remedial

programs following diagnostic testing. There are some pupils who may be excused from drill because of their high efficiency. There are other pupils who simply need additional practice on the type of work missed. There is another class that needs easy practice on the type of work missed to secure a greater facility and confidence in the process. There is still another class which requires the teaching of intermediate steps to learning the process. There is finally another group that may be found in many classes for whom complete re-teaching of the process is necessary. Many teachers make the mistake of omitting all of the steps excepting two. When they find that some members of the class are weak, they give the entire class drill on the type of work missed. When they find that certain pupils are unable to do the work, they conclude that the entire process must be taught. This is a waste of time. Only after failure on a large percentage of examples in a process should complete reteaching be done for an individual or a group. Unless pupils miss from 80 to 90 percent of all the examples in a given process, probably complete reteaching is not necessary. For others some intermediate step between the first teaching and the more involved phases of the process is probably what is needed.

3. Individual Remedial Work

A statement concerning individual remedial work should be prefaced by a full recognition of the fact that much can be accomplished by group remedial procedure. However, close analysis of the mental processes involved in working examples or problems can best be made through individual diagnosis, and the subsequent treatment will be mainly individual in character. For example, if the pupils in a class are deficient in number combinations containing sevens and nines, specific remedial drills on these combinations may be given. On the other hand, if errors are made in column addition because some pupils skip around in the column rather than adding regularly either up or down, while others make mistakes due to the failure to carry or to carrying the wrong digit, the remedies will probably have to be given individually. This section of the chapter deals only with individual remedial work.

For the sake of emphasis, the statement will be repeated that the basic principle underlying individual diagnostic and remedial work is that improvement in results will be sought by an improvement in the methods by which those results are obtained. Consequently,

the remedial procedures discussed here will aim at securing better methods of thinking while the pupil is doing his arithmetic problems. After one has made a careful diagnostic test, he will have for the pupils tested a specific list of the habits of work which need attention. The most desirable remedial procedures will be those which are aimed directly at the specific difficulties discovered. For example, if the pupil splits numbers in adding rather than adds by the complete number combinations, the question is how to make him stop splitting numbers and add the digits as they appear; or if the pupil continually forgets to add the number which is carried, how to eliminate that difficulty. To be most effective, remedial exercises should be directed at precisely those points of difficulty which are revealed in a diagnostic test. It is to be regretted that at present one cannot supply a satisfactory list of specific measures to match the list of poor habits of work revealed in individual diagnostic tests. It would be very desirable if one could give for each of the forty-one habits of multiplying noted in Table L, a specific set of directions for improving the pupil's work. Eventually this type of remedial materials will be available. At present one may find many suggestions, but very little remedial material which has really been based on experimental trial and which is known to be effective.

Recently a diagnostic and remedial experiment was conducted in which some seventy teachers made individual diagnoses of pupils and attempted to follow up the diagnoses by the application of specific remedial measures during a period of ten weeks. At the end of that period, each teacher was asked to submit a statement of the devices she found useful. Much material was submitted, most of it general statements not related to specific deficiencies and many of these clearly unverified opinions. As examples of the more specific suggestions made, the following may be given:[8]

To overcome the habit of adding carried numbers irregularly, I made use of such examples as the following:

26 I told all to add the first column, put down the 5, and remember
95 the 2. I then stopped until all got that far. Then I told them to
43 proceed with the next column, adding the 2 first. By repeating
56 this orally each day, the habit of adding the carried number first
95 seems to be firmly impressed. After adding with two columns,
— we went on to three and four columns.
315 I insisted on regular column procedure orally, the children

[8] Buswell, G. T., and John, Lenore, *Diagnostic Studies in Arithmetic.* (Department of Education, University of Chicago, 1926.)

pointing to the numbers which they read. I told them to add the column upward, writing the answer at the top. In order to prove the example, they were told to add the column downward, writing the answer below.

One pupil was troubled with zeros. The only way he could clear up his difficulty was to use simple concrete examples, such as the following: If you had no candy and divided it among four boys, how much would each get? If John had no money and you had three times as much, how much would you have?

To avoid writing rows of zeros, I told the pupils that I would show them a short cut. To do this, I took several simple problems having a zero in the multiplier and first did the work before them, writing the row of zeros. Then I did the work, omitting the row of zeros, and showed them that the results were the same. They frequently asked to see it done again, and up to this time I have not thought of a more consistent way to prove to a few doubters that it is proper to do it in the shortest way.

Some children were confused with the terms, *add, subtract, divide,* and *multiply.* I wrote these words on the board and had different children in the class tell what was wanted in each case, using examples to illustrate. (In this case the drill consisted chiefly in reading arithmetical terms; another teacher carried out a remedial plan for giving drill in reading by having the children read numbers instead of written terms.)

To avoid carrying when there was nothing to carry, we drilled for a long time on short examples, such as 45+76, 23+34, etc. Instead of giving the answers, the pupils said the word 'carry' or the words 'not carry' as was required in a given case. I gave written work and blackboard work of the same kind, and the pupils marked these with a *c* or an *o*.

To help overcome irregular procedure in column addition, I first gave the class a series of numbers to be added, pausing between each two numbers so that the class as a whole could keep up. I used such numbers as 5+2 (makes 7) + 5 (makes 12) + 4 (makes 16) + 4 (makes 20), etc. Then I called for the answer. Following this, I wrote the numbers on the board, sometimes beginning at the top of the column and sometimes at the bottom, but always keeping in order and pausing a few seconds for the children to think the answers thus far before writing the next number.

Scientific experiments could be made to determine the value of such suggestions as the foregoing, but one can readily see that this would be a tedious process. Until this is done, remedial procedure will continue to be a matter of opinion and of trial and error, involving much waste of time and much repetition of useless procedure. One could make a major contribution to teaching of arithmetic by gathering together specific remedial devices and testing their validity. Until such remedial measures are tested scientifically, the teacher must rely on her own best judgment, supplemented by trial and error. Perhaps the best procedure to recommend is that teachers try

to gather specific remedial devices for the ten or twelve most common deficiencies in each of the processes of arithmetic and then keep a record of the success which follows the application of these remedial measures. In actual practice this is apt to be the task of the supervisor rather than that of the teacher. A systematic attack upon the problem by supervisors in large city systems should contribute important results.

GUIDE TO CHAPTER VI

CHAPTER VI

THE TRAINING OF TEACHERS OF ARITHMETIC

B. R. BUCKINGHAM

If special attention is given in this chapter to training courses in arithmetic, professional or academic, this attention in no wise indicates a belief that effectiveness in teaching arithmetic depends entirely upon these courses. The arithmetic teacher must first of all be a *teacher,* and any course which helps to train a good teacher contributes to good teaching of arithmetic. It is submitted, however, that the prevalence of special courses in arithmetic argues a conviction that here is a field so important or so peculiar that specific training is desirable in addition to all the other influences to which the student is subjected. So far as the writer is aware, no institution attempts to train elementary-school teachers without one or more special courses in arithmetic. These courses are the chief concern of the present chapter.

The nature of the offerings in arithmetic at the training school depends obviously on several administrative considerations, to which attention will now be given. These are (1) the length of the training-school course, (2) the extent to which the institution offers differentiated curricula, (3) the extent to which the training-school curricula are prescribed, and (4) the location of the special course or courses in the curriculum.

I. The Length of the Training-School Course

Typically, high-school teachers take a four-year course and elementary-school teachers, a two-year course. This is a vicious distinction which outrages one's sense of fair play. There is no reason in the nature of the service why a preference should be given to the teachers of older pupils in the public schools. Some such feeling as this is leading to a lengthening of the training of elementary-school teachers. A few states—as New Jersey, New York, and Massachusetts—are developing three-year courses in their normal schools. Moreover, the single salary schedule is putting a premium upon four-year

courses for elementary teachers. The three-year course is a valuable advance. It will do much to raise the standard of teaching. It seems likely, however, that it will ultimately pave the way for a more general acceptance of the four-year course.

A progressive development of the demand for elementary-school teachers who have four years of collegiate training is to be expected. Meanwhile, more and better three- and four-year elementary-school curricula are being organized in the institutions. As these courses become more valuable, more students will take them and better teachers will teach them. Young people will prepare for teaching as for a career. A vast horde of new recruits, poured like water into a sieve, will no longer be annually required. What raised surgery from an empirical art to a profession? On the one hand, the accumulation of a body of doctrine and theory requiring for its mastery a prolonged period of exacting intellectual effort; and on the other hand, a social demand that surgeons should have the mastery of this material. The same forces and no others will raise teaching, even elementary teaching, to a profession. Teaching is an art, but when this art is exercised as the application of a body of principles, of sufficiently exacting intellectual character, it is also a profession.

The two-year course is still the prevailing one. A treatment of the training of teachers of arithmetic, if it is to be of greatest immediate use, must be made with this course in mind. It is the writer's belief, however, that the 'body of doctrine' with reference to teaching has already vastly outgrown the possibilities of the two-year curriculum. Of course, this body of doctrine is created by research, and research in education has been marvelously productive during the last fifteen or twenty years. Of what avail is it to create for the better teaching of children this essential material, if it must be disregarded in the preparation of teachers? Arithmetic is but one of many cases in point. The two-year curriculum ordinarily includes a single half-year or quarter-year course in this subject, yet so great has become the corpus of professionalized subject matter in arithmetic that no student can secure more than a bowing acquaintance with a portion of it in so restricted a course.

The long, or four-year, course in elementary education, although it already exists in a few institutions, is essentially the course of the future. It is the course which the public schools will increasingly demand as the public becomes more discriminating and the wealth of

the country tends to be applied to more worthy ends. It is the course, moreover, which the contributions of research already justify and to which they already give dignity and substance. The old question of the scholarship of the teacher is being answered in new and more relevant ways. Collegiate courses of exacting character are being worked out. They are not liberal arts courses, nor should they be. They are not, and they should not pretend to be, convertible into credits usable, hour for hour, in securing advanced standing in a liberal arts college.

The four-year course should be the next objective of every clear thinker in the field of elementary education. 'Better teachers' is a short but sufficient prescription for an improved public education. Longer and better training will secure better teachers. This is not wholly because of the direct effect of training upon a given student. The indirect effects are many and powerful. Being more exacting, the long training will be more selective. Those who meet its requirements will tend to be of better 'stuff' than those who meet a less exacting standard. For this reason and because of their finer training, elementary teachers will be more highly regarded and better paid. The service, being more attractive, will be chosen by a better grade of students, who in turn will be still more highly regarded. Under these conditions tenure will lengthen. Instead of an average service of four or five years, the average will be two or three times as long. Longer experience will itself make better teachers. Seven or eight percent of them already quit at the end of the first year, having made a ghastly crop of mistakes. The single salary schedule will no longer be a generous gesture. It will be in vigorous operation. The economic status of the teacher will be no more of a problem than the economic status of physicians.

The time is particularly appropriate for an advance to higher standards for elementary-school teachers. During the past year or two an oversupply has existed in many quarters—something which has not been true for many years. If school people do not meet this lack of quantity demand by a higher quality demand, they will miss the greatest opportunity to improve the schools which has been offered in a generation. The four-year standard should be held in view. It should be put into effect as soon as possible and as widely as possible. Nothing less than this can be regarded as a solution of the teacher-training question.

Naturally in a four-year curriculum the offerings in arithmetic would be more ample. Not fewer than twelve or fifteen semester credit hours should be devoted to this subject. Its importance in the elementary school, its difficulty for the pupil, and the rich body of helpful professional material which it possesses unite to make this a reasonable demand. If, therefore, the following scheme is somewhat in excess of what is usually permissible in a two-year curriculum, it will be understood that this is not so much a failure to recognize the practical limitations of the usual curriculum as a protest against their permanent acceptance.

II. Differentiated Curricula

Although the available relevant material in arithmetic—and the same may be said of several other elementary subjects—is more than can be presented in a single course such as the two-year curriculum can permit, the task may be reduced more nearly to manageable proportions by sharpening the objective of each student's curriculum. For example, if the student elects to prepare for primary teaching, the relevant professionalized subject matter in arithmetic, as in other subjects, will be materially different from that which will be offered to the student who intends to teach in the middle or upper grades. Moreover, it has been frequently pointed out by competent students of the training of teachers that differentiated curricula are, on every ground, more desirable than a general curriculum. For many years this has been held to be true, but as the body of relevant professional material has increased from year to year, the impracticability of the general curriculum has become still more evident.

Only for one purpose is a general curriculum admissible and that is in the training of rural-school teachers. Here a virtue must be made of necessity. Since the teacher must teach all the arithmetic, the institution must do the best it can to prepare him for his total job.

Of course the mere training of teachers in differentiated courses is not in itself an assurance that the children of a given grade will be taught by a teacher trained for that grade. Such an assurance can only be given when the state, after licensing each graduate only for the service for which he is trained, takes systematic measures to place him in the type of position which he is best qualified to fill.

The discussion so far does not touch the question of the extent to which the specialization of training should be carried. The very

forces which produced differentiated curricula now operate to carry the differentiation to further lengths. The kindergarten-primary course—now that so heavy a body of knowledge is available—is seen to be inferior to two courses, one for kindergartners and one for teachers of primary grades. Indeed, children between the ages of four and six are sufficiently different from children between six and nine to justify this subdivision of the kindergarten-primary course, even if no other reason existed.[1]

It seems reasonable that the primary curriculum may cover the first three grades. Grades four, five, and six may then constitute the middle curriculum, and grades seven and eight, the upper curriculum. The upper curriculum and the junior-high-school curriculum are sometimes combined; but this seems certainly to be a mistake. The training of teachers of junior-high-school mathematics is something more than training them to teach arithmetic. Moreover, junior-high-school teachers are not grade teachers but subject teachers. This introduces a consideration which is at variance with the point of view of this paper. Finally, the training of junior-high-school teachers is no task to be undertaken in two years. Much subject matter must be taught. Part of the reason why so much disappointment has been experienced with the new mathematics in the junior high school is that the teachers were not good enough mathematicians.

These curricula, however, should not be conducted in water-tight compartments. The argument that a teacher in a given grade should know something about the work of adjacent grades is valid. Consequently, the student who takes the kindergarten curriculum should be given a view of the work of the primary grades; the student in the primary curriculum should have a look back into the kindergarten and forward into the intermediate grades; and so on. But those views are for support to the main task; they are not the main task. They do not pretend to train for service in the adjacent grades; they merely illuminate the specialized training which the student is receiving.

III. Prescription versus Election

In this chapter attention is focused upon training for elementary teaching—for teaching in a situation where the job is clearly defined.

[1] If circumstances permit, the strength of the movement for 'unified kindergarten and first grade' justifies a kindergarten-primary course which shall go no farther than grade one.

If teachers cannot escape teaching language, arithmetic, and geography, they should not as students be permitted to escape the professional study of these subjects. Moreover, if the mature judgment of competent thinkers is that certain general courses, such as educational psychology, principles of education, and school management, have important values in relation to the work which the teacher will certainly be called upon to do; then it is far better for the student to take these courses than it is to wander through a bizzare program of his own choosing.

In considering the course in arithmetic—or any other course, for that matter—the importance of determining whether curricula are prescribed or elective is important. If each course is required, except for cause officially approved, no course is at a disadvantage. No instructor is compelled to enter into competition with other instructors in making his courses easy. Students do not desert eight-o'clock courses and afternoon courses and crowd the nine- ten- and eleven-o'clock courses. They do not take courses because a friend is taking them. Sequences can be preserved and prerequisites maintained. There is no plea here that the arithmetic course alone be prescribed. The question is much broader, involving an educational principle. It is desirable, not only from the general point of view but even for the arithmetic instruction itself, that substantially the whole curriculum should be prescribed. The lone required course in a semester is by no means in a favorable position. It becomes a tradition of the school to hate and belittle it. Students who would have no thought of being imposed upon by a prescribed curriculum which they had chosen, feel very much put upon when they are made to take one or two courses.

IV. The Location of the Arithmetic Course in the Curriculum

In the foregoing discussion the present necessity for the two-year curriculum for training elementary-school teachers has been recognized. The advantages of differentiated curricula have likewise been set forth, and the thesis has been maintained that these curricula should consist for the most part of required courses. With these premises granted, it is now possible to consider the question of the placement of any particular courses within a two-year, differentiated, required curriculum.

The course in arithmetic is only one of several special courses in elementary-school subjects. Any particular position, therefore, within

the curriculum which may be demanded for arithmetic ought likewise to be granted to these other subjects. In general, therefore, what may be said as to the placement of arithmetic applies equally well to reading, language arts (composition, grammar, and spelling), geography, and such other elementary subjects as receive special professional treatment.

Although there is some advantage for arithmetic, considered by itself, if the course is taught late in the curriculum, there are likewise advantages in placing certain more general professional courses in that position. On the other hand, it appears to be necessary that the student should bring to the special subject-matter courses (they may well be called courses in professionalized subject matter) a certain background of educational ideas. Unless the teacher of the professionalized courses in reading, arithmetic, and geography undertakes the task of developing these ideas, the course in any one of these subjects may easily lose dignity and importance. It seems clear that these subject-matter courses cannot be placed at the beginning of the curriculum any more than they can be placed at the end of it. A middle position is, therefore, indicated, a position which makes it possible to have two or three introductory professional courses before these subject-matter courses are attempted and to conclude the curriculum with two or three more advanced professional studies.

The two-year curriculum may well introduce in the first semester a survey course in education—called, perhaps, "An Introduction to Education." A preliminary course in psychology likewise belongs near the beginning of the curriculum. Similarly a course in biology —not a course for biologists, but one for teachers—may be offered during the first or second semester. Certain subject-matter courses, such as drawing and music, can likewise occupy an early position.

Most of the argument for the advantages of the history of education really rests upon the idea of a historical treatment of teaching materials. These advantages may and should be secured in the subjects to which they apply. Something about the history of arithmetic is certainly appropriate for the course in arithmetic, and the same may be said of the history of other subjects. Even the history of educational theory is likely to be more effective as part of the theory course than as part of the course in the history of education. To the writer, therefore, it appears that, within the narrow limits of

the two-year curriculum, a separate course in the history of education is inadvisable.

As to art and music, one should look at these subjects from the point of view of the elementary-school service. The two-year curriculum offers little opportunity for courses merely as cultural courses. From the studies of Payne, Holmes, and Mann it appears that each of these subjects occupies four or five percent of the time devoted to the elementary-school program. Typically 75 to 90 minutes a week are assigned to them in city school systems, and the pupils do not do outside work in these subjects. Yet at the teacher-training institution either of these subjects may be required to the extent of from two to eight credit hours.

Of course, music and art are of much larger importance in the elementary school than the time allotment in the program indicates. The teacher should be able to play and sing with the children, but a technical course in music is by no means necessary for this purpose. Several subjects in the elementary curriculum may and should be supported and enriched by means of drawing and color work, but again technical courses for the teacher are not essential.

A number of other subjects clutter up the two-year curriculum—each well enough if there were time for it, but each out of place when only the most necessary can be afforded. In the required two-year curriculum for regular teachers there is no place for such courses as gardening, house mechanics, personnel, swimming, playground, blackboard drawing, and food selection. There is scarcely room for courses in oral expression, social ethics, and library instruction. Even academic courses (required or provisionally required) in economics, sociology, European history, physics, and chemistry are of doubtful wisdom. It is indeed distressing to contemplate the omission of so many valuable fields of study. The writer, however, believes that this is a much more wholesome procedure than to weaken the entire curriculum by introducing a large number of short courses. If these subjects are desirable—and no one denies that they are desirable—then they constitute an argument for a four-year course rather than for their inclusion in a two-year course where neither they nor the other subjects may receive adequate treatment.

V. The Dominant Subjects of the Elementary-School Curriculum

At more than one place in the foregoing discussion the principles of training teachers in accordance with the service they will have to perform has been mentioned. It is easy to ring the changes on all the admirable traits of personality and character which a teacher may possess and to say, ''We desire the service of such as these; the training schools should turn out such splendid men and women to teach our children.'' It is freely admitted that nothing surpasses in importance the character and personality of the teacher—that is, of the teacher as a human being sympathetically responsive to the needs and opportunities of children. The teachers college should, as far as humanly possible, graduate such students; but it must first get students who have such possibilities. Then it must keep them and banish those who are inferior. For only to a limited degree do the deep springs of action which we cover by the term 'character' yield to training in late adolescence. In respect to these matters, the institution must be a selective agency as well as a training agency; and the selection should take place not only at the time students are matriculated, but also at various critical points throughout the course.

With respect to the activities of the teacher in the classroom, however, light may be thrown upon the nature of the teacher's service by considering the program under which he works. The latest information on this subject is contained in Doctor Mann's excellent report.[2] Mann, of necessity, takes account of the entire teaching day, including the number of minutes devoted to recesses, opening exercises, supervised study, projects and activities, unassigned and free time, and miscellaneous. For the purpose now in mind, the time allotted to these six subordinate items may be disregarded. When this is done, five subjects, namely, reading, arithmetic, language arts, geography, and history (including civics), occupy about 70 percent of the remaining time. Using Doctor Mann's figures and dropping out the subordinate items already mentioned, the figures in Table I are derived.[3]

The number of minutes in Table I are averages. They are obtained for each grade by dividing the aggregate minutes by 444, the total number of cities. Some cities, however, do not offer all these

[2] Mann, Carleton H. *How Schools Use Their Time.* New York, Teachers College, Columbia University. 1928.

[3] *Op. cit.*, p. 23.

TABLE I.—AMOUNT AND PROPORTION OF TIME PER WEEK DEVOTED TO THE FIVE MAJOR SUBJECTS BY 444 CITIES IN 1926
(Adapted from Mann)

Subject	Grade I		Grade II		Grade III		Grade IV		Grade V		Grade VI	
	Minutes	Percent	Minutes	Percent	Minutes	Percent	Minutes	Percent	Minutes	Percent	Minutes	Percent
Reading	478	44	421	36	352	28	243	18	195	14	177	13
Arithmetic	80	8	146	12	196	15	211	16	215	16	215	16
Language Arts	117	11	184	16	230	18	254	19	262	19	262	19
Geography	4	0	7	1	58	5	133	10	160	12	164	12
History	21	2	26	2	43	3	78	6	119	9	140	10
Total	700	65	786	67	879	69	919	70	951	70	958	70
Total Time for All Specified Subjects	1081	100	1172	100	1269	100	1315	100	1356	100	1367	100

subjects in all the grades. If, therefore, consideration is limited to the cities which actually allot time to these subjects in each grade, the figures are in some cases considerably higher than those shown in Table I. The typical practice, for example, of cities which offer arithmetic in the first grade is to devote about 90 minutes to it.[4]

In spite of contrary advice from high places,[5] arithmetic (if Dr. Mann's cities are typical) is definitely prescribed in the first grade in fully four-fifths of the cities of the country. The more extreme advice that the subject be deferred to the third grade is practically disregarded.[6]

VI. The Curriculum of the Training School

Any group of activities which absorbs 70 percent of the time of the school cannot be disregarded in training teachers. Even in the two-year curriculum the teacher-training institution is justified in offering and requiring professionalized subject-matter courses in each of these five subjects. It should do this no matter whether the curriculum pertains to primary, intermediate, or grammar-grade teaching—except that penmanship may take the place of history in the primary curriculum.[7] Of course, the offerings in each of these sub-

[4] *Op. cit.*, p. 79.

[5] *E.g., Fourth Yearbook of the Department of Superintendence of the National Education Association*, 1926, pp. 173-220.

[6] Mann: *Op. cit.*, p. 55.

[7] In grades one, two, and three, penmanship is shown to average 74, 80, and 84 minutes per week, respectively, or about 7 percent of the total time for specified subjects (Mann, *op. cit.*, p. 23).

jects will be different according to the curriculum which the student is pursuing. Moreover, each course must be organized with reference to the most exacting grade which the students will be eligible to teach. For example, the professional course in primary arithmetic must take particular account of the 15 percent of time devoted to the subject in the third grade rather than the 8 percent in the first grade.

1. Classification of Courses

Broadly speaking—and it is admitted that what follows is largely a matter of personal judgment—the courses in the two-year curriculum may be classified into (1) general professional courses, (2) courses in school subjects, and (3) courses in professionalized subject matter. Assuming the usual 80 semester-credit-hours as the total for the entire curriculum, something like 35 or 40 credits may be assigned to the general professional field, about 15 to school subjects, and about 20 to professionalized subject matter. More specifically, the general professional courses—each ideally somewhat different in each curriculum—may be offered as shown in Table II.

TABLE II.—SUGGESTED 'GENERAL PROFESSIONAL' COURSES

Subject	Credit hours
Introduction to Education	3
Educational Psychology I	5
Educational Psychology II	3
Educational Biology	3
Educational Sociology	3
Principles of Education	5
School Management	3
Tests and Measurements	3
Observation and Practice	10
Total	38

There are certain school subjects which the teacher-training institution must teach. This will always be true, although the particular subjects will not be the same. The elementary curriculum is always changing, and students will be called upon, therefore, to teach things they have not had an opportunity to learn prior to entering the teacher-training institution. This accounts for the present necessity for teaching in the teacher-training institutions such subjects as elementary science, sewing, physical education, drawing, and music. To

such 'school subjects,' therefore, the writer would assign about fifteen credit-hours, although in general two class-hours should be required for each credit-hour. On the other hand, it should be understood that the teacher-training institution does *not* teach arithmetic, geography, history, reading, or language as review courses. These subjects have been taught in the public schools time out of mind. Every student who comes to the teacher-training institution has had ample opportunity to study them extensively. If the student is not qualified in these subjects as taught in the elementary school, he should be either denied admission or required to make up the deficiency in non-credit review courses.

2. Courses in Professionalized Subject Matter

Finally, the professionalized subject-matter courses may be offered as shown in Table III.

TABLE III.—SUGGESTED COURSES IN PROFESSIONALIZED SUBJECT MATTER AND THE NUMBER OF SEMESTER-CREDIT-HOURS FOR EACH

Subject	Primary		Intermediate	Grammar
Reading	I	3	5	5
	II	3	–	–
Arithmetic		5	5	5
Language Arts		5	5	4
Penmanship		2	–	–
Geography		2	3	3
History		–	2	3
Total		20	20	20

The foregoing suggestions lead to a total of 73 semester-credit-hours for each of the three curricula. There is still enough leeway for two or three electives.

It is admitted that twenty semester-hours for the professional treatment of elementary studies is somewhat more than usual. Yet observation and participation in normal-school classes induces the belief that the value of such courses to the student after he takes a position justifies at least that much time for them. The importance of this type of course is attested in an interesting way by Mr. H. A. Sprague, Principal of the Montclair State Normal School.[8] The students at that institution reported in 'case' forms the difficulties which they encountered while they were engaged in practice teaching. Mr.

[8] Sprague, H. A. "The case problems of student teachers." *Educ. Admin. and Superv.*, 14: May, 1928, 314-324.

Sprague, at the time he wrote this article, had been collecting these cases for three and a half years. He had obtained 4429 of them. There were 1498 difficulties which he classified under 'techniques of teaching individual subjects.' He had six other classifications, but none had as many cases under it—not even 'discipline problems;' and discipline, as everybody knows, is the *bête noire* of young teachers.

The extent of a course in the training school depends in part on the amount of material for the course. Before the modern research movement, arithmetic in the normal school necessarily meant either a review course or some vague theorizings or both. Of late, however, the body of professional knowledge concerning the learning of arithmetic and concerning its contribution to human thought has been enormously expanded. This has been particularly true with reference to arithmetic in the primary grades. A five-hour course of dignified and exacting character containing nothing but helpful material, may now be organized in each of the three curricula—primary, intermediate, and grammar. Indeed, careful management and rigid exclusion of items of minor importance will have to be practiced or even a five-hour course will fail to cover the ground.

Where three-year and four-year elementary curricula are possible, at least two courses in arithmetic ought to be given. The first one should be of a survey character. It should endeavor to place arithmetic in its proper social setting. It would, of course, be partly historical in character, and from it the student would be expected to develop an appreciative attitude toward the subject. The second course should deal with the particular subject matter which the student will be called upon to teach.

VII. How Much Arithmetic Do Students Know?

For many years after the establishment of normal schools two opposing views of the training of teachers were held. One exalted scholarship; the other, method. The normal schools which were committed to scholarship devoted a large part of their effort not only to the teaching of academic subjects of collegiate grade but also to the reviewing of school subjects. The question to which attention is now directed is: "How much of this is necessary so far as arithmetic is concerned?"

The academic point of view is a particularly stupid one in certain respects. It holds to an exploded psychology in virtue of which

the mind is supposed to be a reservoir and to retain without diminution anything which has ever really entered it. Not only so, but anything which is not immediately reproducible is held never to have been in it. According to this theory, if you can't extract cube root, you never really knew how; if you make a mistake in adding fractions, you don't know how to add fractions and you never did know how. It is a brutal theory; it is also wrong. In virtue of it the college teaches elementary chemistry again to the boy who has already had it in the high school. It begins physics with the elements of the subject, despite the fact that half the class may have spent a year studying it.

Actuated by this sort of psychology and this lack of common sense, people have given tests and drawn all sorts of wild conclusions as to how little the examinees 'know' about this, that, or the other subject. In particular, students, upon entering the normal school, are examined in arithmetic and may be found to be below eighth-grade standard or even below that of the sixth or the seventh grade. The writer has himself given tests rather extensively to students who are about to be graduated from the normal school. He will refer to these in a moment; but whether tests are given at entrance or at graduation from the teacher-training institution, he refuses to believe that falling below a grade standard which these very students have themselves established but a few years before is serious enough to be permitted to retard and distort the work of training teachers.

Most of the arithmetic tests address themselves to one's ability to compute. Now suppose a student fails in column addition, in subtracting fractions, or in the first case of percentage. No one believes that he did not at one time know how to add a column of figures. He may have been particularly adept in subtracting fractions, and his competence in finding a given percent of a number may at one time have been quite irreproachable. The chances are that every student who enters the normal school has learned all these and many other things. The fact that, after long disuse, he or she cannot reproduce them at the moment ought to offer little disquietude to anyone who knows his psychology as well as the teacher-training people ought to know it.

The psychologist knows better than to assume that his laboratory 'subject' has no memory of a series of nonsense syllables because he cannot reproduce them. If the subject has once learned the series,

the experimenter seeks rather to find out how much less time he requires to relearn the series than he needed to learn it in the first place. Anything once learned is never thoroughly lost. It retires below the threshold, but small occasions will often serve to bring it into play. This is a part of the reason why psychologists have observed that minute stimuli often produce elaborate responses. It is because the connections are in a high state of readiness, owing to their former activity. Now the teacher, confronted with the necessity for teaching long division or denominate numbers meets with an exceedingly powerful stimulus. It is practically certain that the teacher will renew his competence, if he has seemingly lost it, in the presence of a situation so compelling. So far as the abstract computations of arithmetic are concerned, one need have little uneasiness. The student who has had a full elementary-school course and has been intellectually selected through four years of high-school and two years of collegiate work probably will not long remain unable to do examples of this sort. The reader is asked to search his memory for instances where teachers of this degree of training have been unable to do the abstract examples belonging to the grade they were teaching.

Many teacher-training institutions are now requiring 'eighth-grade standards' in arithmetic and are providing means to secure them. Professor Clyde H. Lady, at Slippery Rock, Pennsylvania, does this by means of coaching classes and a series of tests. All but about 5 percent reach the standard. "These few," he says in a letter to the writer, "are well nigh hopeless, certainly so as teachers." At any rate, no time in credit courses is devoted to a review of elementary-school arithmetic. Professor J. A. Foberg, at California, Pennsylvania, reports that the professional courses in arithmetic are four-hour courses, but that students who fail to reach a specified standard are required to meet an extra hour each week without additional credit. Professor H. C. Christofferson, at Miami University, reports the success of a plan by which below-standard students voluntarily attend two to four extra half-hour meetings per week. He says:[9] "Progress seems to be so rapid and to come as the result of so little effort as to indicate that no separate college course in mere arithmetic can possibly be justified."

[9] In a forthcoming issue of the *Journal of Educational Research.*

VIII. The Ability of Students to Solve Verbal Problems

Success in solving verbal problems depends rather more upon intelligence—as measured by the intelligence tests—than upon computing ability.[10] While we may have relatively little cause to worry about the computing ability of teachers, the situation may be quite different as regards their ability to solve problems. When a layman discusses the competence of a teacher in arithmetic, he usually refers to his ability to do hard problems such as Ray's textbook contained.

In order to find out something about the success with which students who are about to become teachers can handle verbal problems, the writer gave a test in June, 1928, to all the seniors in the normal schools of New Jersey. There were 952 of these students.

A problem was regarded as wholly wrong if the answer was wrong. This could—and did in many instances—arise from minor errors having no reference to reasoning. For example, in another run of these same tests, among 426 students 102 got a certain problem wrong, but 57 of these did so because they forgot the dollar sign. From a certain point of view this is idiotic, but the problem in question was standardized for difficulty through trial with children, and what would become of the precepts of the school if children were allowed to omit the dollar sign? Anyway, it is safe to say that the amount of failure among these seniors is partly due to causes which are relatively easily remedied.

The sixteen problems of the test varied from easy to hard. In order to institute a comparison between these seniors and pupils in the elementary schools, the test has been divided into two parts. The first part consists of the first six problems. These problems are so easy that the writer has never offered them to pupils above the sixth grade. The record in percents correct for the sixth grade and for the normal school seniors is given in Table IV.

The normal-school students did better than the sixth-grade pupils on all the problems and a great deal better on all but the first problem. That was so easy even for sixth-grade pupils that there was little room for improvement.

The story told by the remaining ten problems is more interesting. These had previously been offered to seventh- and eighth-grade pupils. The comparison is brought out in Table V.

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[10] Morton, R. L., *Teaching Arithmetic in the Intermediate Grades.* New York; Silver, Burdett and Company, 1927. P. 297.

TABLE IV.—COMPARISON BETWEEN NORMAL-SCHOOL SENIORS AND SIXTH-GRADE PUPILS ON SIX EASY VERBAL PROBLEMS

Problem Number	Percents Correct	
	Grade VI	Normal-School Seniors
1	93.6	93.9
2	86.3	94.6
3	36.9	96.6
4	39.8	79.5
5	29.8	69.5
6	47.2	87.8

The comparison between the success of the normal-school students on the one hand and the seventh- and eighth-grade students on the other is really a contrast. No doubt it is a matter of regret that as few as thirty or forty percent of students who are about to be teachers should reach the correct solution of *any* problem in the field of elementary-school arithmetic. But as for putting these students in the same class with eighth-grade pupils, it simply can't be done. The students' performance is on an entirely different level. On no problem is their percent correct less than one and a half times that of the

TABLE V.—COMPARISON BETWEEN NORMAL-SCHOOL STUDENTS AND PUPILS OF THE SEVENTH AND EIGHTH GRADES ON TEN HARDER PROBLEMS

Problem Number	Percents Correct			
	Grade VII	Grade VIII	Normal-School Seniors	Ratio of Normal-School Students to Eighth-Grade Pupils
7	42.2	47.7	73.6	1.5
8	28.8	44.3	65.2	1.5
9	26.2	45.0	86.1	1.9
10	1.8	10.1	61.7	6.1
11	4.7	16.8	53.3	3.2
12	6.6	23.3	58.6	2.5
13	6.8	7.1	36.6	5.2
14	2.8	5.0	41.2	8.2
15	1.5	4.2	30.5	7.3
16	0.0	4.9	31.1	6.3

eighth-grade pupils. Moreover the reader will observe that *the harder the problems are, the more decisively the normal-school students show their superiority.* This superiority would without question be even more manifest if the normal-school students, like the seventh- and eighth-grade pupils, had been at the peak of their form in computing ability.

This same conclusion may be reached from the investigation of H. W. Charlesworth (Master's thesis, Colorado State Teachers College) in which the Compass Diagnostic Tests were applied to students in training for teaching. On the subtest entitled "General Problem Scale, Elementary," 83 percent of the students were above the eighth-grade norm. On the more difficult subtest, "General Problems Scale, Advanced," the corresponding percent was 97; and on the subtest, "Problem Analysis," it was 99.

There is even evidence for the belief that as teachers increase their experience in service, they render better account of themselves in solving problems. In a series of problem tests given for the writer in the 1929 summer session of the George Peabody College for Teachers, the average scores for teachers of one, two, and three years' experience—that is, for the years before the selective effects of promotion and withdrawal begin to be serious—were as follows: those with one year of experience, 8.5; those with two years of experience, 9.2; and those with three years of experience, 11.3.

Thus, ability in arithmetic can be thought of in two ways: first, as ability in the abstract examples which are worked by the application of rules; and second, as ability in solving verbal problems. Students who have sustained themselves through a secondary school doubtless had brains enough in the elementary school to secure the first ability to a reasonably satisfactory degree. If this is the case, the fact that skill and accuracy may have subsequently fallen below standard is not likely to be serious. The student who has to use this ability, first in a course and later as a teacher, will 'get up' the required skill in the presence of the necessity for its use. This is not an argument for slip-shod scholarship. Indeed, the institution which gives to college students credit courses in children's arithmetic is not offering impressive evidence of its devotion to scholarship. Neither is the institution which gives a credit course, ostensibly professional, and includes in it a considerable amount of the same pabulum. There should be a 'hospital' to which students may be sent if they need treatment, or perhaps a complete non-credit course. This policy, together with an exclusion policy applicable to the worst cases, ought to relieve the professional course in arithmetic of the necessity for reviews of such topics as fractions, decimals, and percentage.

This does not, however, mean that subject matter may not properly be taught in the teacher-training institution. It does mean—at least

that is the belief of the writer—that no collegiate credit should be given for work which repeats that customarily taught in the elementary school. Ability in such work should be assumed. On the other hand, *additional* work, even in arithmetic, provided it is more extensive in character than the elementary school contemplates, may be given, and should be given, especially for students who are preparing to teach in the grammar grades. Such work is needed for mastery. Moreover, in considering the question of subject matter—again for teachers of grammar grades—considerable attention should be given to certain phases of economics and business practice. The upper-grade teacher will, as matters now stand, be required to teach the business applications of arithmetic, especially of percentage, to such matters as insurance, taxes, profit and loss, stocks and bonds, trade and bank discount, and the like. The preparation of such teachers may well include their instruction in the protection of life and property, the raising and spending of public money, trade, investment, saving, credit, and other matters relating to the work they will be called upon to do. It is not so much the computational side of these topics that should concern the teacher-training institution as the theory and practice of them.

IX. An Examination of Arithmetic Courses in Teacher-Training Institutions

In the spring of 1929, information was secured from 56 normal schools and teachers colleges concerning 126 courses.

1. Extent of Differentiation

The position has already been taken that arithmetic courses should differ according to the grade the student expects to teach. Most institutions carry out this principle to some extent. Quite typically, however, a teachers college will offer both differentiated curricula and a general curriculum. The former will carry appropriate arithmetic courses—*e.g.*, "Arithmetic in Primary Grades," "Arithmetic in Intermediate Grades." The latter—that is, the general curriculum—likewise contains a corresponding arithmetic course, such as "Arithmetic in the Grades," or "The Teaching of Arithmetic." Except in the preparation of rural-school teachers a general curriculum is not believed to be justified and the corresponding general course in the teaching of arithmetic is equally unjustified.

Truth to say, however, some of the general curricula do differentiate with respect to arithmetic. Not infrequently *one* of two or three courses will be required, these courses corresponding to different levels of elementary-school work. The fact is, therefore, that a general curriculum does not preclude some differentiation. Really, however, this practice effectually robs the advocates of a general curriculum of one of their stock arguments. They say the student is unable to choose wisely whether he will teach lower, middle, or upper grades. Yet here in the general curriculum the student is forced to make just that choice.

Optional courses afford an interesting type of differentiation. For example, at Normal, Illinois, a required course in "The Teaching of Arithmetic in the first Six Grades" is accompanied by a parallel course in "The Teaching of Arithmetic in the Seventh and Eighth Grades." Both are required as prerequisites to an optional course. The title and description of the latter as as follows:[11]

> Course 41. An Advanced Course in the Teaching of Arithmetic. This course is planned for administrators, for supervisors of arithmetic and for teachers of experience. It includes such topics as studies that have been made in the social usages of arithmetic, outstanding courses of study, methods of procedure in selected topics with investigations that have been made that pertain to them, tests, helping the backward pupil, criteria for selecting textbooks, and the equipment of the classroom.

Ypsilanti has an interesting way of differentiating its arithmetic instruction. After providing for lower- and upper-elementary courses in "Teachers Arithmetic" and an additional course in "The Teaching of Elementary Mathematics," an optional course is offered with the suggestive title "Studies in Mathematics Education." The description of this course includes the following statement:

> It is the object of this course to consider a study of the teaching of mathematics from an historical and psychological point of view.

In much the same way Bellingham, Washington, has an elective course called "Scientific Studies in Arithmetic," and the State University of Iowa, the University of Minnesota, and other similar institutions have a course in "Supervision of Arithmetic."

The State University of Iowa has carried the differentiation of courses farther than any other institution. Besides the supervision

[11] Dropping the so-called 'simplified spelling' which Normal still affects.

course just referred to, it gives three residence courses in "Modern Tendencies"—primary, intermediate, and advanced—together with *six* correspondence courses, one each for the grades from the third to the eighth. (It is assumed that no student can receive credit for more than one of each of these groups of courses.)

The number of differentiated courses among the total of 126 is 78. Thirty-nine courses, although clearly intended for graded-school teachers—most of the rest being for rural teachers—are not differentiated to any significant extent. In making these statements a course is regarded as 'differentiated' even if it does not carry its specialization very far. For example, an institution may give a course in arithmetic for the first six grades, followed by one for the junior high school. Both of these courses are here regarded as differentiated. Nine of the courses were either rural or unclassifiable. The figures indicate that, among courses designed to prepare teachers for graded-school work, about half are intended to apply to any grades from the first to the eighth, inclusive.

2. Amount of Review

In the questionnaire addressed to heads of departments of mathematics the following questions were asked and answered concerning the courses offered at 56 institutions: "Does this course include a review of arithmetic? If so, about what proportion of time is devoted to this review?"

Replies were received concerning 106 courses, but in 19 cases the first question was answered "Yes" without any answer to the second question. With respect to these 19 courses, therefore, all that can be gathered is that they did include a review of arithmetic. On the other hand, 36 courses were reported as including no review work at all. The reports on the remaining 51 courses not only indicated that time was given to reviews, but gave as well an estimate of the emphasis upon this type of work. Accordingly, the vote was 70 to 36, or practically two to one in favor of including reviews.

Only 87 reports were completely responsive under this heading. For these courses Table VI distributes the proportion of time devoted to a review of elementary arithmetic.

Although, according to Table VI the median is 20 percent, this is not particularly informing. The median of courses which devote *some* time to review is perhaps more so. It is 50 percent. The range

is from 10 percent (reported for two courses) to 100 percent (reported for one course). Six courses were reported as involving 60 percent of review and two courses as involving 66, which may be taken as representing an estimate of two-thirds of the work. Two courses were reported as including 75 percent of review work.

The significant point about Table VI is the fact that it is sharply bi-modal. Typically, either a course includes no review or it includes 50 percent of review. (All the 24 cases listed in Table VI at "50-59" were reported as "50.")

TABLE VI.—PROPORTION OF TIME DEVOTED TO ELEMENTARY ARITHMETIC

Percent of Time Devoted to Review	Number of Courses
0	36
10 – 19	6
20 – 29	5
30 – 39	3
40 – 49	2
50 – 59	24
60 – 69	8
70 – 79	2
100	1
Total	87
Median	20 percent

The reader can judge for himself the extent to which these courses, offered for the purpose of preparing students to teach arithmetic, are professional in character. They are often three-hour-a-week courses offered either for twelve or eighteen weeks. Sometimes they are only two-hour courses. Now, when it appears that a common procedure is to devote half the time to reviewing elementary subject matter, the scant treatment of the professional aspects of arithmetic is apparent. An examination of the courses reported as including no time for reviews discloses a tendency, though by no means a marked one, for these courses to be in the primary field. The extent of the tendency is perhaps indicated by the fact that among the 36 courses allowing no time to reviews, 9 are primary, 4 intermediate, and 5 upper. Most of the remainder involve other combinations of elementary grades. Four of these 36 courses may be rated as advanced courses—not advanced in the sense of referring to higher grades of the elementary school, but really advanced in the character of the professionalized treatment.

While there is a tendency for reviews to be included in courses corresponding to the upper grades of the elementary school, the practice is by no means universal. The determining cause seems to be of a quite different nature. The character of the school, and more particularly the character of the mathematics department, really decides this question. The high-grade institutions tend markedly to devote the time of the students in these courses to matters other than reviews.

3. The Textbooks Used

For a practical estimate of the nature of a course, the textbook which is placed in the hands of the students affords an important basis. One of the questions on the blank concerning the course in arithmetic called for the name of the textbook used, if any. In order to show how many textbooks per course were used, it was assumed, in tabulating, that no response meant that no textbook was used. With this assumption made, Table VII gives the facts for the 120 courses concerning which usable answers were received.

TABLE VII.—NUMBER OF TEXTBOOKS USED

Number of Texts......	0	1	2	3	4	5	Total
Number of Courses....	29	52	21	7	8	3	120

The strong tendency for teachers to use one textbook in these courses is evident. Among the 52 courses in which only one textbook is used, 17 different books are mentioned. The following are the five most popular books in these courses in the order of their frequency of mention:

	Courses
Roantree and Taylor: *An Arithmetic for Teachers*	11
Taylor: *Arithmetic for Teacher-Training Classes*	7
Morton: *Teaching Arithmetic in the Intermediate Grades*	6
Morton: *Teaching Arithmetic in the Primary Grades*	5
Smith and Reeve: *Teaching of Junior-High-School Mathematics*	4

If the 21 courses in which two books are used are considered along with the 52 courses in which but one textbook is used, the number of different books mentioned in these 73 courses is 32. Eighteen of these 32 are mentioned but once. Accordingly, in the case of 14 books, there is some degree of agreement, as will be seen in Table VIII.

Only a relatively small number of the 120 courses could be differentiated into primary, intermediate, and grammar grade (or junior

high school). Twenty were distinguishable as primary courses, 17 as intermediate, and 14 as grammar. Among the 20 primary courses, 19 different textbooks were mentioned. The only one on which there was any substantial agreement was Morton's *Teaching Arithmetic in the Primary Grades.* This book was named eight times. Lennes' *Teaching of Arithmetic* was mentioned twice, and on no other book was there any agreement whatever.

TABLE VIII.—AGREEMENT AS TO TEXTBOOKS IN 73 COURSES

Textbooks	Frequency
Morton: *Teaching Arithmetic in the Intermediate Grades*...	13
Mortin: *Teaching Arithmetic in the Primary Grades*.......	12
Roantree & Taylor: *An Arithmetic for Teachers*...........	11
Taylor: *Arithmetic for Teacher-Training Classes*...........	8
Overman: *A Course in Arithmetic for Teachers and Teacher-training Classes*	6
Smith & Reeve: *Teaching of Junior-High-School Mathematics*	4
Strayer-Upton: *Arithmetics*	4
McNair: *Methods of Teaching Modern Day Arithmetic*......	3
Stone: *How to Teach Primary Number*....................	3
Stone: *The Teaching of Arithmetic*......................	3
Buckingham and Osburn: *Searchlight Arithmetics*..........	3
Knight *et al: Standard Service Arithmetics*................	3
Brown & Eldredge: *Arithmetics*..........................	2
Denver Course of Study................................	2
Eighteen Others	18 (1 each)
Total...	94

Among the 17 intermediate courses, 8 different textbooks were mentioned. In six courses Morton's *Teaching Arithmetic in the Intermediate Grades* was named. In two courses Roantree and Taylor's *An Arithmetic for Teachers* was reported. The remaining six books were mentioned but once.

The junior-high-school list was even more straggling. In four of the 14 courses no textbook was used and for the remaining 10 courses 12 different books were reported. Ten of these were mentioned but once and the other two only twice each.

4. Collateral Readings

The heads of departments who received the questionnaire were asked to submit a "list of the five most important collateral readings, exclusive of textbooks in arithmetic." A few of those who answered this question listed more than five readings. About an equal number listed fewer than five readings. For 15 courses no collateral readings

were reported. Eighty-one books or articles were mentioned in all. Fifty-two of these, however, were not listed more than three times; in fact, 34 of them only once. The ten most popular readings were all books, as distinct from monographs, bulletins, courses of study, and articles. In other words, they belonged to the same class of material as the textbooks already mentioned.

Some of the readings less frequently mentioned are more appropriate than those which received the most votes. For example, the following were named, but in each case only once or twice: Buswell and Judd *Summary of Educational Investigations Relating to Arithmetic,* the yearbooks of the National Council of Teachers of Mathematics, the yearbooks of the Department of Superintendence, Smith's *History of Mathematics,* and the *Mathematics Teacher* (meaning the periodical). All of these should be standard sources for the professional study of arithmetic. Although a few of the important monographs were mentioned, they were mentioned very seldom. Among these were Myers, "The Prevention and Correction of Errors in Arithmetic;" Clapp, "The Number Combinations, Their Relative Difficulty and the Frequency of Their Appearance in Textbooks;" Buswell and John, "Diagnostic Studies in Arithmetic;" Knight and Behrens "Learning of the 100 Addition Combinations and the 100 Subtraction Combinations;" and Brownell, "The Development of Children's Number Ideas in the Primary Grades." The histories by Ball and by Fink were mentioned but once and Cajori's history but twice. Even Karpinski's *The History of Arithmetic* was named but four times. Considering the serious study which is now being devoted to the curriculum, one might have expected some of the new courses of study to appear among the collateral readings. Yet none of them was named, although the Denver course was twice named as a textbook.

On the other hand, there are some curious titles in the lists of collateral readings for the course in arithmetic—for example, the yearbooks of the National Society for the Study of Education. These yearbooks have contained very little on arithmetic. In fact, it is largely for this reason that the Society is bringing out the present volume. Stevens *The Question as a Measure of Efficiency in Instruction* is scarcely to be expected in a list of this sort. The same may be said of other titles such as Lord, *Getting Your Money's Worth* and Dewey, *The Child and the Curriculum.* No criticism of these books is intended. The only question is one of appropriateness in the spe-

cial course in arithmetic. Certain books of a general character would likewise seem to be out of place for collateral reading in a course of this kind—such as Charters, *Teaching the Common Branches;* Rapeer *et al, Teaching Elementary School Subjects;* Kendall and Mirick, *How to Teach the Fundamental Subjects;* and La Rue, *Science and the Art of Teaching.* These books each contain a chapter on arithmetic, but in every case the chapter is too limited to justify its inclusion among the five most important collateral readings in a course devoted to this subject.

The fact is that the same sort of materials is used for collateral readings as is used for textbooks. Time and again those who reported concerning these courses gave one of the half-dozen best books on teaching arithmetic as the textbook and then gave five similar books as collateral readings. The idea of going to the sources from which these books were in large part made up seems to have been seldom entertained. Here, as elsewhere, the failure of these courses to measure up to real collegiate standards is evident.

5. The Use of Commercial Texts

No professional course which has much regard for actual conditions in the school will neglect the instruments of instruction. Among these instruments the textbook placed in the hands of the children is easily of greatest importance—especially so in this country. In the course for training teachers of arithmetic the analysis, appraisal, comparison, and use of textbooks should be definitely provided for. Too many teachers judge a book superficially. Any textbook has scores of points which a cursory glance will fail to reveal. A bit of drill material in multiplication, for example, looks the same whether it provides for all the number facts or fails to do so. Too much regard is paid to such things as the number of colored pictures or the width of the margins. Mere prettiness scores too high and real workmanship on the part of the author in selecting and organizing material counts too low. In a course for teachers the strength of textbooks is as fundamental as the strength of materials in a course for engineers.

It is astonishing how little attention is given to textbooks in these teacher-training institutions. The heads of departments who made out the question blank were asked to list separately "the commercial textbooks in arithmetic used in the course, if any." Among 20 courses

in primary arithmetic, 12 made no report under this heading. From the wording of this direction it is a fair inference that no commercial textbooks in arithmetic are used in these courses. Among 17 courses in intermediate arithmetic, 9 failed to mention any commercial textbook; and the same lack was noted for 10 of the 14 courses in grammar grade (or junior-high-school) arithmetic. It is fair to say that more than half of the special courses in arithmetic make no important use of children's textbooks.

6. Examinations

At the conclusion of the question blank the heads of departments were asked to submit a "list of questions used at the last final examination." Such lists were received for 38 courses. As to the rest of the courses, sometimes the examination questions were simply not submitted, but more often the person who made out the report gave a reason for the omission. The commonest reason was that tests were given at various times throughout the term, the implication being that there was no final examination for the course. The next most frequent reason for not including a copy of the final examination was covered by such phrases as "not kept" or "not available."

On the average, the examinations are 30 percent children's arithmetic. This checks surprisingly well with the average of the estimates for the 38 courses in question. It will be recalled that under another heading the heads of departments estimated the amount of time devoted to review of arithmetic. For the courses now under consideration, the average of these estimates was 32 percent—that is, within two points of the proportion of elementary-arithmetic questions included in the examination papers. This is a significant commentary on the accuracy of the previous statement that, among courses which devote any time to review, the usual procedure is to give half the time to it. In judging whether a question was a review or a professional question, subject matter was regarded as professional if it was not in the elementary-school curriculum. For example, an exercise in which the student was required to change a number from the decimal to the duodecimal system was regarded as a professional question, even though it was 'straight arithmetic.'

Even if it is thought to be necessary to include in the professional course in arithmetic a certain amount of review of elementary-school arithmetic, nevertheless this, it will be generally admitted, is not the

object of the course. This subject matter is reviewed in order that the real work of the course may be done. To maintain anything else is to maintain, so far as arithmetic is concerned, an identity between the teacher-training institution and the elementary school. If the review is a necessary evil and only exists in order that something else may be attended to with success, then it is this something else and not the review which should appear in the examination. In fairness one should say that sometimes this is just what happens. A certain normal school, for example, estimates that review of children's arithmetic is 50 percent of the course, but the final examination is entirely professional.

In a great many cases, however, the tendency is in the other direction. In one school two courses were estimated to contain respectively 60 percent and 62 percent of review. The examinations in both cases were 100 percent elementary arithmetic. In another case a general course—that is, one designed to prepare for teaching in any of the grades—was estimated to contain 50 percent of review material, while the examination was just such as a county superintendent might give in connection with eighth-grade graduation.

There is a strong tendency for the examinations in the primary courses to be more strictly professional. On the average they contain but 12 percent of elementary arithmetic, whereas the examinations in the intermediate courses contain 25 percent, and those in the grammar-grade courses 64 percent. The average for the general courses was 31 percent.

In the courses for the higher grades not only is little professional work given—judging from the examinations—but that which is given is often applicable to primary grades only. For example, in one course for junior-high-school teachers, the examination consists of eight questions. Only three of these questions are classifiable as professional, and none of the three relates to junior-high-school work. These are the questions:

a. What was Clapp's study and what was its contribution?

b. What are the results of studies on

   1. How shall we subtract?

   2. Shall we add up or down?

c. Analyze the fundamental combinations in adding:

$$\begin{array}{r} 4729.7 \\ 342.02 \\ 729.39 \\ 615.26 \\ 419.09 \\ \hline \end{array}$$

Another observable fact about the examinations is that they tend to stick to the textbook. For example, it is reported concerning a course for primary teachers that a certain textbook is used. Five books are likewise given under the heading of collateral readings. What use was made of the latter one can't say, but it is a fact that not one of them was needed in answering any question on the examination paper. In other words, every question could have been fully answered by a student who had studied the textbook and nothing else.

The narrow adherence to the textbook (or possibly to the ideas of the teacher) is suggested by the wording of questions such as the following:

a. Why does the author favor teaching long division first?
b. Which method [of subtraction] does the author favor?
c. What are the three classes of problems given by—[naming the author of the textbook used]?
d. What is the number of addends and digits in each addend by the end of the 6th grade?
e. In addition and subtraction of fractions to what should denominators be limited?
f. Explain as you would to a class our procedure in the problem $\frac{1}{4} + \frac{3}{4} = ?$
g. Give the limit in the numbers of digits to an example in subtraction.
h. The danger in incidental number teaching is that it may become....
i. A flash card contains on one side the .... and on the other the.........
j. Write terse, pointed statements which will serve as guides in the teaching of arithmetic, frequently given in class.
k. Name the ten "classes" under which all addition combinations were included.

7. Outlines of Courses

The heads of departments were asked to furnish for each course a "topical outline." Some of these outlines were excellent and others were rather childish. The most convenient form of outline seemed to be one which merely set up the topics of the course in arithmetic in the elementary school. Such a classification may, for want of a better term, be called 'horizontal.' An example of such an outline for a general course follows:

1. Addition
2. Subtraction
3. Multiplication
4. Division
5. Factors and multiples
6. Fractions
7. Decimals
8. Percentage

9. Business applications
10. Lesson types
11. Tests
12. Denominate numbers
13. Commercial paper
14. Interest
15. Insurance
16. Stocks and bonds
17. Problems
18. Graphs

It is clear that not much thinking went into the making of this outline. It may, however, be distinguished from a topical arrangement which cuts across the horizontal segments of the elementary curriculum and suggests what Bagley calls "new views" of arithmetic. The following outline for the primary course as used at the State Normal School, Jacksonville, Alabama, will illustrate what may be called a 'vertical' outline:

1. Scope of the work of each grade in detail
2. The child's ability and background of experience
3. Children's present use for arithmetic
4. Methods of presenting and developing each topic or process
5. Use of charts, flash cards, games, contests, and other devices
6. Commercial materials available
7. Observation of classroom work in the demonstration school

The vertical idea was often introduced in these outlines in combination with the horizontal plan. This is capable of being illustrated in the form of a double-entry table. Table IX, for example, is adapted and abbreviated from the outline submitted from the normal school at Frostburg, Maryland.

Exactly 100 outlines of courses were submitted by the heads of departments. These were classified as horizontal or vertical accord-

TABLE IX.—AN OUTLINE FOR A COURSE IN TEACHING PRIMARY ARITHMETIC

	Counting	Meaning of Numbers	Addition and Subtrn. Facts	Processes of			
				Addn.	Subtrn.	Mult.	Dvn.
Teaching the Facts or Processes							
Organizing Drill Exercises							
Problem Solving							
Diagnostic and Remedial Work							
Judging Textbooks							

ing to the interpretation of these terms which has already been indicated. If an outline showed the vertical feature to any considerable extent, it was classified as a vertical outline, even though the horizontal features were likewise present to a marked degree. On this basis—admittedly largely subjective—62 outlines were classified as horizontal and 38 as vertical. Owing to the liberal interpretation of the term vertical, it is fair to say that the 62 outlines tabulated as horizontal were practically a series of topics set down in the order of their usual treatment in elementary arithmetic. Of course, it is entirely possible for a teacher in a normal school to give new views of subject matter without revealing in an outline the fact that he does so. It is what he does under the heading of fractions or percentage which determines whether or not his teaching is on the professional level. It is submitted, however, that an outline which contains nothing but the topics in elementary arithmetic offers no evidence of a professional treatment of these topics; while on the other hand, the so-called 'vertical' outline offers abundant evidence of such treatment. The fact is that fewer than two out of five of the courses which were outlined show any important change in the point of view from that which obtains in the elementary school.

8. Summary of an Examination of Courses

To summarize this study of arithmetic courses in the 56 institutions from which reports were received, one may observe that while in a number of instances excellent courses are offered, the survey of the entire situation is rather depressing. The brilliant accomplishments of students in the field of arithmetic appear to be too little capitalized. Thus instruction lags behind pertinent available materials of instruction.

(1) The body of professionalized subject matter justifies and requires, in order that it may receive any adequate treatment, a differentiation of courses. Half the courses examined (leaving courses for rural teachers out of account) have no differentiation and the extent of it in the others is often insufficient.

(2) A considerable proportion of the meager time available for the course is devoted to reviewing elementary-school arithmetic. Twice as many courses give time to review as get along without review; and among those courses which teach elementary-school arithmetic the commonest practice is to devote 50 percent of the time to it.

(3) The textbooks and collateral readings do not suggest—at least in many institutions—that dignified collegiate courses in arithmetic are being offered. For the most part the textbooks themselves are of an elementary character, having been written for these very courses. Moreover, the collateral readings fail to come to the rescue. They fail to include many of the important studies and sources of material. They tend to be merely more books like the textbook.

(4) Most of the courses fail to utilize, as materials for study, textbooks such as the schools are using. It has already been pointed out that owing, to the important part played in American education by the textbook, the omission of these materials from the professional courses of study is serious.

(5) A study of the final examinations and outlines used in these courses confirms the general impression as to the elementary character of the courses and the large amount of attention devoted to the arithmetic of the grades.

X. Assumptions in Regard to Professional Courses in Arithmetic

Before making more specific suggestions as to the content of courses in arithmetic, it will be well to make clear the assumptions upon which these suggestions are based. The plan here recommended contemplates three specific courses having reference respectively to the primary, intermediate, and grammar grades. One of these courses is to be required of every student. Each is a five-hour-a-week course for a semester and therefore involves 90 class meetings. It is preceded in the curriculum by a few introductory professional courses and followed by more advanced professional courses. The selection of students and the standards maintained by the institution provide for no reteaching in credit courses of the arithmetic generally taught in the elementary school.

Certain other assumptions are made in regard to institutional standards. The school which the writer has in mind is a college in fact as well as in name, although it may not be a four-year college. It is interested in quality and is not over much concerned about the size of its enrollment. It maintains collegiate standards in the personnel of its faculty, in its material equipment, and in the character of its courses. Its library facilities are adequate for the work it is attempting to do. Its practice school (or its working arrangement

with public schools) provides it with reasonable opportunity for practice teaching, observation, demonstration, and research.

The meaning of the central position of the arithmetic course (and other similar courses) in the curriculum deserves a further statement. Assuming, as has been indicated, that before students begin the required course in arithmetic they have taken a first course in psychology, together with a general survey course (now frequently called "Introduction to Education"), there are certain ideas with which they should already be somewhat familiar. Of course these ideas are not at this juncture fully understood. No matter how well they have been taught, they can hardly yet be rich in meaning. The arithmetic course, like the professional courses in reading, English, geography, and history, should contribute powerfully to the value of these concepts, and the theory courses at the end of the curriculum should make them still more productive.

Meanwhile, however, something has been gained for the courses in professionalized subject matter. It will not be necessary, for example, in the arithmetic course to present these concepts as topics for initial study and discussion. Accordingly, in the suggestions about to be offered little reference will be made to those topics of general import which may be assumed to have been brought to the student's attention in earlier courses. It is to be understood that these topics carry through all the courses in arithmetic, adding to their professional character at every point.

The following is a list of the concepts which, under the assumptions already made, may be fairly supposed to have been already brought to the student's attention in the survey and psychology courses:

- The modern social idea of the educative process
- The original nature of man and the meaning of infancy
- The general meaning of learning
  - Stimulus and response
  - Sensation
  - Perception
  - Reaction
  - Imagination
  - The concept
- Principles of learning such as recency, frequency, primacy, intensity, and mood (mind-set)
  - Laws of learning
  - Memory
  - Association

- The doctrine of transfer
- Attention
- Habit formation
- Concomitant learning—attitudes, ideals, and appreciations
- The nature of thinking
- Interest and effort
- Motivation
- Forgetting
- Fatigue
- Measurement
- Standardized tests
- The making of objective tests
- Individual differences
- Individualizing instruction
- Grouping and classifying pupils
- Types of lessons
  - Developmental
  - Drill
  - Review
  - Appreciation

XI. The Course for Teachers of Primary Grades

1. Sources of Material

Data for this section and for the next two sections have been gathered in many ways. In fact, some of the sources are so remote that they defy identification. A few specific origins, however, may be mentioned: first, the outlines of courses submitted by teacher-training institutions as described elsewhere in this chapter; second, a series of problems about the teaching of arithmetic formulated by students in training at the University of Chicago and furnished to the writer by Professor Guy T. Buswell; third, case reports (242 of which relate to arithmetic) gathered by H. A. Sprague, Principal of the State Teachers College, Upper Montclair, N. J.,[12] and contributed through Assistant Commissioner R. L. West; fourth, the catalogs of teacher-training institutions; fifth, a selected group of public-school courses of study (city and state); sixth, a selected group of textbooks in arithmetic; and seventh, books and articles on the teaching of arithmetic.

2. Assumptions as to General Concepts

Throughout the arithmetic courses it will be assumed that the general concepts just mentioned as introduced in the introductory

[12] Sprague, H. A., *loc. cit.*

professional courses will be filled out and clarified and that on the other hand the possession on the part of the students of these concepts even in rudimentary form will make the arithmetic topics themselves more significant and vital.

3. 'Transfer' as a Type

Without attempting to give in even the briefest manner the special application of all of these fundamental concepts, it will suffice to select one of them—the doctrine of transfer—and sketch a treatment.

First the teacher may review the students' knowledge of the history of the doctrine of transfer of training. The type of psychology which supported the extreme view of this doctrine, the evidence concerning transfer, the theoretical bases for it which various thinkers have proposed, and the modern view as to the amount and character of transfer may be alluded to. The point should be made that while the idea of total transfer based on the strengthening of mental faculties is believed to be a myth, nevertheless, transfer necessarily takes place within associated fields, that mental life depends upon it, and that the amount and range of transfer may be increased by a type of instruction intelligently directed to that end.

In arithmetic only a portion of the subject is taught, and transfer is relied upon to take care of the rest of it. Everybody knows that the pupil will have to use applications of arithmetic never encountered in the classroom. In teaching a child number ideas, one customarily uses relatively common units and relatively small numbers. This is done in the faith that subsequently he, the pupil, will not be helpless in the presence of large numbers and strange units. In the functional treatment of number ideas, only fragments make their appearance in the school. The child is called upon to read and write numbers, and he is expected, sometimes quite unjustifiably, to behave intelligently in performing various operations with numbers. The complex reactions and patterns of behavior of adult life which are carried out under the control of quantitative ideas must be guided largely through the operation of transfer. Only to a limited degree can the school make definite provision for contingencies so complicated and frequently so remote.

In the specific teaching of computation the school again selects the most fundamental types and relies upon transfer for the rest. By analysis it is found that in the process of division certain facts are necessary—that is, certain automatic number skills. How many there are of these is a question upon which people may differ very widely, according to the reliance they place upon transfer. The same is true in the other operations. There are systems of instruction which contemplate as few as 45 addition combinations and as many as 312. The argument *pro* and *con* will not be offered here. Fundamentally it is a question of transfer. Those who teach but a few combinations must devote considerable attention to the generalizations which each of these few facts makes possible. Each fact becomes the

captain of a host. Those who teach the greatest number of combinations say they leave less to chance. In either case, however, there is an area within which accomplishment by the pupil must rest upon transfer. A system of teaching the fundamental facts which should seek to avoid reliance upon transfer would be too extensive for the school to handle.

As will be shown later, students while doing practice teaching in arithmetic cry loudly for help in the teaching of verbal problems. Here the question is again one of transfer. The situation, however, is very much more difficult. Part of the difficulty arises from the fact that few children have adequate ideas either of numbers or of processes. Automatic response with abstract numbers is no guarantee of this adequacy. It is a common complaint that the child will say "30 divided by 6 is 5" without responding at all satisfactorily to the problem "If you buy 6 apples for $.30 what is the cost of each?" Why should he respond correctly to the latter situation upon the basis of the former? With neither interpreted, they are sufficiently unlike to baffle the keenest intellect. If the division facts are not taught with meaning, all the automatic response in the world will be like the rattling of dry bones. "Teaching with meaning" is only another way of saying: "Providing for transfer." This leads some persons to say that it is not so much what we teach as how we teach. This idea may help one to understand, but it is only a half truth. The facts must be learned. Moreover, the meanings are no less subject matter than are the facts.

It would not be difficult to maintain that the problem of transfer is the central problem of teaching—that is, a system of pedagogy can be written around transfer as the organizing principle. The normal-school student must feel that all the learning which children do in school should have a wider reference than is immediately evident. The student should be shown ways of giving carrying-power to the experiences of the classroom. It was just now suggested that the teaching of subject matter be invested with meaning. One may go farther and say that associated fields or areas should be pointed out and that the basis of their relationship to the matter in hand should be indicated. This may be no more than the pointing out of likenesses and differences.

Again, the generalizations which accompany the learning of subject matter, the ideals of precision, orderliness, neatness, and so on, which are sought in arithmetic should likewise be shown to have wider application. Not only should likenesses and differences, generalizations and ideals, be pointed out, but the pupil should be made conscious of them as such. No matter whether the carrying force is thought of as residing in the subject matter (identical elements) or is thought of as residing in the learner (ideals, abstractions, generalizations) it is submitted that this carrying force will be enormously more effective through enlisting the conscious effort of the learner. The fullest effect, therefore, of transfer will be secured when likenesses and differences between situations are brought to the active attention of the learner.

4. More Specific Items

In addition to the general concepts already listed, there are a great many topics which belong to particular subjects. These topics, so far as they apply to arithmetic, will appear in each of the three courses here considered, but in most cases with different content.

Every one of these topics is capable of extended treatment; and unless the reader is prepared to go through a book on the training of teachers of primary arithmetic, the items of the sort now under discussion will have to be little more than mentioned. In the following running comment some of the more important items will be italicized. There is a *philosophy* of arithmetic. Number is not so much a department of human knowledge as an aspect of life. According to the philosophy of arithmetic, the subject has certain *objectives*. Some of these are remote and general; others are most immediate and specific. Out of these objectives and the subject matter and methods designed to attain them arises the *curriculum* in arithmetic. At this point a great deal of pertinent professional material is available. Some of it is contained in this Yearbook. There are certain practical aspects of the curriculum which the student should obtain from this course, such as grade standards, grade placement, and time allotments.

Measurement of the product of instruction in arithmetic must be considered in this course. The history and theory of measurement in this subject may receive a glance and some of the best *standardized tests* may be studied. The various types of tests, survey, inventory, diagnostic, and so on, should have attention as well as the uses of the results of testing. The making of good local measuring instruments by objective techniques should also be studied.

In modern arithmetic *analysis* plays an important part. Thus there are analyses of subject matter, of drill materials, and of errors. The students should be brought to grips with a number of such analyses; and they should explore the contribution which each analysis makes to the teaching and learning of arithmetic. In the first place, few more hopeful changes have been made in the teaching of primary arithmetic than the analytical approach to each of the four fundamental operations. One new difficulty at a time, each previous difficulty reviewed as new ones are introduced—these are the teaching advantages made possible by such analyses. In the second place, not a few writers have made remarkably effective use of the analyses of

drill materials. The result of this form of analysis has been a better appraisal and selection of drill materials, as well as the provision of better material by textbook writers. Students should be shown how to analyze examples and other exercises after having analyzed or accepted an existing analysis of the particular purposes which the drill is supposed to serve. Students should also have some practice in making such materials. The distinction, for purposes of practice, between a teaching and a maintenance program should be made. In the third place, the doctrine that the child's work affords a clinical picture of him in action is a fruitful one. The analysis of error provides a powerful basis for adapting instruction to the pupils' needs. This analysis implies a *classification of error,* a study of the *causes of error,* and an application of *remedial measures.*

As a device in motivation, the keeping by the pupils of their *individual records* has been so generally satisfactory that the students should be shown the fact basis upon which valid record-making rests, and how to make the required records.

The *literature* of arithmetic should be made accessible to the student. He should receive, as part of his equipment resulting from the course, an annotated bibliography of at least the secondary sources of information and to some extent even the primary sources.

Mathematics has a remarkable *history*; and arithmetic, as a department of mathematics, although many of its origins are shrouded in mystery, has a particularly fascinating story. In fact, each of the major topics of arithmetic has its history. Subject matter in education has been defined as preferred modes of reaction. Perhaps the definition will not hold water, but it is a fruitful idea in connection with primary arithmetic. We teach a certain form of multiplying. This form has been the achievement of ages of evolution. Although even now more than half the world does not multiply as we do, the form we use is believed by us to be the best form. It is a "preferred mode of reaction." The history of how it came to be preferred is a proper cultural item in the course for the training of teachers of primary arithmetic.

American education leans heavily upon the *textbook.* This is particularly true in arithmetic. Since these instruments of instruction are going to play so large a part in the work of the teacher, the student should have access to a large number of textbooks for purposes of examination, analysis, comparison, judgment, and selection. Simi-

lar opportunities should be given him with reference to practice materials, workbooks, tests, and other supplementary publications.

The *interests of children* of the first three grades should be studied, not only from the literature on this subject, but also at first hand in the demonstration or practice school. A repertoire of *games* should be built up, together with an understanding of the purpose and value of various kinds of games. The role of *activities* or *projects* in primary work should receive attention, and a few activities should be worked out in essential details. Measures should be taken to permit the trial of some of these in the practice school.

A number of suggestions already mentioned may be repeated here for emphasis, such as types of lessons, including therein the question of rationalizing the processes; lesson planning; provision for individual differences; and a consideration of the changes in the arithmetic curriculum during the past twenty-five years, with a criticism of the criteria now being offered for the selection of subject matter. In this last connection a number of the best courses of study in arithmetic should be used as source material.

5. A Repertoire

From one point of view the training of teachers is intended to speed up the making of expert classroom practitioners and thus to short-circuit the wasteful trial-and-error learning of experience. Now the experienced teacher not only has greater resources in personal behavior, but also has greater resources in teaching materials. Students should be furnished with material (not only in their heads but also in their hands) which they can use in the classroom. They should, individually and in committees, devise tests, compile drill materials according to specifications and for particular purposes, write problems, organize games, make projects, and analyze processes. The results of these activities should be pooled for the benefit of each member of the class. The materials should be duplicated, classified, indexed, and arranged for filing. There is no reason why students trained for a particular teaching service should go out from the institution empty-handed. It is true that most students have a notebook to take away with them, but too often this notebook is of little use.

6. Problem-Solving

In each of the three courses here presented, problem-solving should take a prominent place, and this place should be no less prominent in

the training of primary teachers than in the training of those for higher grades. One of the questions which students ask their instructors is: "When should problem work begin?" It should begin at the beginning and proceed *pari passu* with the abstract work as long as the subject is taught. There has been a disposition to look upon abstract work as fundamental and problem work as derived. This is not true. Abstract forms and processes are by their very name and nature generalizations from concrete experience. Problems are fundamental; abstract processes are used in their solution. Outside of the school there is no other use for abstract processes. If, therefore, it is the business of the elementary school to teach fundamentals first and foremost, it is its business to teach problems first and foremost.

Each new process, each new type of computation, should be first shown in one or more described situations—that is, in one or more problems. The need for the new skill, or at least the appropriateness of it, is therefore made evident. Why should a child subtract one number from another without a glimmer of the meaning of the procedure?

After an abstract process has been shown to be useful through a problem approach, the process itself may be isolated for study and drill in order that it may run off smoothly. This drill, however, should be accompanied by more verbal problems. This position is taken for two reasons. In the first place, the problems are especially valuable as sheer drill material. In the second place, they afford a test of the thoroughness of the drill. We cannot say that a child has mastered a given process until he can use it in situations where attention must be directed to the conditions to be met. A person who learns to drive an automobile with an instructor by his side may manipulate the controls without error. His attention is upon the mechanics. The real test comes when the instructor leaves the car and the neophyte has to do his own thinking not only of the controls but also of the conditions under which he must use them. The real test of reliability of a response comes when attention is free to bestow itself upon other matters.

Children do not have enough easy problems to solve. In primary work, except where special materials may be provided for bright children, only one-step problems should be attempted. These should for the most part (and for most children) involve small numbers and

relate to familiar conditions. An attitude of confidence in attacking problems and the habit of success in solving them should be built up. Many problems in these grades should be done orally—that is, without the use of writing materials.

Pupils who can make their own problems should be encouraged to do so and to offer them for class discussion and solution. As part of the building up of richer and clearer number concepts, children should be practiced in estimating the answers to both examples and problems. As a means of grouping together problems of the same kind, the so-called 'general problem,' or problem without numbers, should be employed.

In the teaching of problem-solving an important device is classification. The first classification may be according to operation. Each operation may, in turn, be classified further. Subtraction problems, for example, may be divided into a number of types, such as, (1) how-many-are-left problems, (2) how-many-are-taken-away problems, (3) how-many-more (or less) problems, and (4) finding-the-other-number problems. It will be noticed that these depend, not upon the operation, but upon an apprehension of the meaning of the problem. Osburn has carried this idea still further in his doctrine of 'cues.' It is too early to say whether this affords a long-sought basis for "dividing and conquering" in the realm of problem-solving. At present the scheme looks promising, at least so far as one-step problems are concerned.[13]

Much depends upon the selection of problems. They should be real, vivid, and interesting. As taught, these same qualities should be heightened. The reality of a problem should be brought home to the child, not only in its expression, but also in its treatment. He should sometimes be taught to think of himself as one of the characters of the problem. Another child may take the place of a second character, if there is another. This leads naturally to the dramatization of the problem. Sometimes the pupils may draw a picture of the problem. At other times they may let counters represent the units of the problem. When an attempt is made to secure 'real' problems, one need not rule out the problem of fancy. The child's make-believe world is real to him and highly interesting. Problems about fairies or Little Boy Blue or the goblins, if not overworked, may be used with telling effect. What has just been said has quite as much to do with vividness and interest as it has with reality.

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[13] Osburn, W. J. *Corrective Arithmetic,* Volume II. Boston: Houghton Mifflin Company, 1929. Pp. 12, 16, 22-25, etc.

One of the persistent questions which teachers ask is put by a certain third-grade student-teacher as follows: "What can I do to develop the reasoning ability of my class in arithmetic? They can do the fundamentals of arithmetic very well. However, they all, with very few exceptions, are unable to tackle the most simple problems." A second-grade teacher raised a similar question, saying: "We have drills in the facts, but these do not help. I can think of nothing else to help them [the pupils]."

Poor girl! The trouble is farther back than "the facts." Indeed, it is entirely possible to drill a child so excessively upon facts that he perceives no relation whatever between them and the described situations called 'problems.' The remedy is at least three-fold; first, to teach the facts with meaning—never to let the significance of the facts be lost; second, to have many problems as part of the drill upon an application of the facts from the beginning; and third, to divide and subdivide problems into types, presenting only a few simple types in each operation at first. If these precautions are observed and if the problems are made "real, vivid, and interesting," the teacher and the school have done their part. A few children of low mentality may not succeed, but no such conditions are likely to arise as are set forth in the foregoing quotations.

7. Research

Incident to the instruction in this course and to the gathering of material for the 'repertoire,' certain shortcomings in present knowledge will be revealed. Occasionally such a hiatus will become particularly annoying. This fact should be accepted as a challenge to teacher and students to do something to meet the need. It would seem that a five-hour class conducted for a semester might be regarded as less than fully successful if it did not contribute something to the learning and teaching of primary arithmetic.

8. Special Topics

It is clear that certain topics belong in the course for primary teachers from the very nature of the public-school curriculum. At this point it is quite possible to adopt what has elsewhere been called a 'horizontal' outline and to think of such topics as the following:

number concepts	number processes
number facts	addition
addition	subtraction
subtraction	multiplication
multiplication	division
division	

These, together with a few matters such as units of measurement, telling time, vocabulary, and a preliminary glance at fractions, will represent the curriculum of the first three grades. Indeed, the processes, so far as multiplication and division are concerned, are not completed, even in their initial presentation, until the end of the fourth grade; and there is a growing tendency to postpone long division to the fifth grade.

Instead of lengthening this treatment of the primary course by a detailed statement concerning each of the topics just mentioned, an outline of one of them will be offered as a type, with the suggestion that an analysis of a somewhat similar nature be applied to the others.

a. *Number Concepts.*—Modern writers on arithmetic are stressing the fact that children are not given time enough for the development of strong meaningful concepts.[14] Part of the reason why they do no better in their problem-solving is that they have no vivid realization of the meaning of the numbers involved. Learning in respect to number concepts should proceed throughout the entire elementary school. Thus it seems a legitimate topic in each of the three courses considered in this chapter. So far as the primary course is concerned, however, these topics are appropriate:

1. Reproduction of number—the ability, for example, to give the teacher six objects or to make up a group of six objects as the result of any stimulus calling for such an act.
2. Identification of number—the ability, for example, to tell the teacher how many there are in a group of objects. This involves naming the number. Identification also calls for *any response* (any act or type of behavior) which shows that the child knows how many there are in a group presented for his consideration.
3. Matching numbers—the ability to find among a second set of objects the same number as are presented in a first set of objects.
4. Discrimination—the ability (a) to distinguish, for example, five objects from four objects or six objects, and (b) to select all the

[14] Brownell, William A. *The Development of Children's Number Ideas in the Primary Grades.* (University of Chicago Supplementary Educational Monographs, No. 35, August, 1928.)

groups, of say, five from a series of groups containing different numbers of objects.

5. The influence of patterns on all the foregoing abilities.
6. The various functional abilities described above in relation to the *units* employed, as (a) real or imagined, (b) tangible or intangible, (c) single or multiple, (d) familiar or unfamiliar, (e) large or small, (f) standard or not standard.
7. Counting as a means of building up number ideas.
8. Measuring as another means to the same end.
9. Grouping—a third basis of concepts.
10. Ratio—still another basis for number concepts.
11. Comparison—number concepts enriched by bringing them into relationship. (a) General, giving rise to a quantitative vocabulary. (b) More precise, giving rise to such statements as "three more," "twice as many," "one-third as much," "one less," and so on.
12. Size—*i. e.,* number concepts in relation to the size of the numbers.
13. Zero—the concept 'just not any;' also its use as a place-holder in notation.
14. Ordinals and cardinals.
15. Fractional numbers.

9. Case Studies

It would be possible to construct a course for the training of teachers on the basis of the problems which teachers encounter in the classroom. Of course, for such a purpose a very large number of cases would have to be assembled. Moreover, even with the cases at the disposal of the writer at the present moment such a treatment would be out of the question within the limits available. Accordingly, just a few of the cases which have been furnished from the University of Chicago and the normal school at Upper Montclair, New Jersey, will be cited.

Important questions from the University of Chicago are the following concerning the curriculum:

General questions on the curriculum as a whole:

What constitutes a curriculum in arithmetic?

What principles shall govern the construction of the curriculum?

What are the objectives of the curriculum in arithmetic?

Questions relating to child experience and life environment:

To what extent should the curriculum take into consideration the environment of the child?

Is it possible to find number situations real to the child that will give him the training that he will need in arithmetic in after life?

Do the majority of children have about the same uses for number outside of school at the same grade level?

Should the applications of the fundamental processes to life situations precede or follow the pupil's knowledge and mastery of these processes?

Specific questions on items to be included in or excluded from the curriculum:

How many combinations should be made automatic?
What fractions should we teach?
Should the multiplication tables above 9's be memorized?
What denominate numbers should be taught?
What type of interest problems should be taught in the grades?

Questions on the relation of arithmetic to other subjects:

How can arithmetic be correlated with other school subjects?
Is arithmetic to teach citizenship or is it to be largely a study of mathematical processes?
Is the teacher of arithmetic responsible for teaching the children to read arithmetic problems intelligently for the purpose of interpretation?

Psychological questions on skills, transfers, etc.:

Should accuracy or speed be emphasized in the teaching of arithmetic or should there be a certain balance between speed and accuracy?
Does increased ability in computation bring about greater reasoning ability?
How do we know that accuracy in arithmetic carries over into another subject?

The cases from Montclair are 242 in number. They relate to the problems of student teachers in the first six grades. They have been classified by Principal Sprague as follows:

Methods in Arithmetic:

1. Applying skills in problems—reasoning		7
2. How to obtain the interest and attention of pupils		7
3. Problems on subtraction		21
a. How to teach subtraction of zeros	1	
b. The addition method of subtraction	9	
c. The Austrian method	3	
d. Pupils mix addition with subtraction processes	5	
e. Other difficulties	3	
4. How to teach addition		9
5. How to teach multiplication		5
6. Difficulties in teaching division		19
a. Short division	12	
b. Long division	7	
7. Selection and application of drill		101
a. In addition	30	
b. In subtraction	5	

c. In multiplication	25	
d. In division	2	
e. Form of flash cards	2	
f. Speed and accuracy	7	
g. Other problems in drill	30	
8. Other problems on methods		68
9. Requests for demonstrations		5
Total		242

A sampling of the reports from the student teachers of the primary grades follows. The numbers correspond to those of the classification just given.

1. Applying skills in problems—reasoning

If my children are given some problems—some in which they multiply and some in which they divide—they get all confused and want to multiply all or divide all. They are given problems similar to these: "If Edward buys 6 oranges for \$.30, how much does he pay for each?" "If Edward buys 6 oranges at \$.06 apiece, how much do they cost?" If they are given problems which are all done the same way, they get them nicely.—3B Teacher.

3b. The addition method of subtraction

The class is taking subtraction by the addition method of arithmetic. They had difficulty in remembering that they should carry as in regular addition. After a great deal of intense drilling the majority of the class mastered the situation.

Now a new difficulty has arisen because of a certain example:

$$\begin{array}{r} 100 \\ 91 \\ \hline 9 \end{array}$$

They are not satisfied with leaving it with just plain 9 for an answer. They want to make it 09 and ofttimes go on working getting an answer of 109. They prove it, yet they can't find out their mistake. I have tried to bring out the fact that a zero before a number does not change its value, but it does if it comes after the number. I have also told them not to put the zero in front of the number and have explained why it is not necessary. Some of the pupils get it; others do not.

Some of the pupils are rather dubious about putting the zero in at all. They just leave it out as in this example:

$$\begin{array}{r} 632 \\ 428 \\ \hline 2\ \ 4 \end{array}$$

They prove it and get the correct answer for they have nothing to add to the 2 only 1 to carry and it comes out right.—3A Teacher.

4. How to teach addition

When a child is learning to add by endings, using an example like 36 plus 8, should he first think $6+8$ are 14, write down 4 and carry 1 to 3, then $3+1=4$ and write down 4; or should he see $6+8$ and think 1 will be carried to 3 so he immediately writes down 4 then writes down 4 under 8? Should he think immediately that the sum of 36 and 8 will be forty something? Then he sees that sum will be 44.—3A Teacher.

6a. Short division

In dividing, where the divisor is contained evenly in the dividend, my class can answer immediately; but when the divisor is not contained evenly, as $9\overline{)56}$, they cannot answer. They apparently do not think about it at all. Each time they receive such a problem, they sit staring in front of them until I ask, "What number near 56 contains 9 an even number of times?"—3rd-grade Teacher.

7a. Selection and application of drill in addition

There are some children in my class who seem utterly hopeless in arithmetic. I help them for a half hour after school drilling in addition and subtraction. We are reviewing now all the numbers whose sums are 12, 14, 16, 18, etc.

They seem to know them, but when they see them on flash cards with other numbers they don't recognize or apply them.

Will you please suggest some devices I can use?—2A Teacher.

7c. Selection and application of drill in multiplication

There are a few in my class who have difficulty in multiplication. When given a problem such as $\begin{array}{r}479,\\ \underline{3}\end{array}$ they multiply the nine by three, put down their seven, and carry the two. Then, instead of multiplying seven by three and adding the two, they first add the two to seven and then multiply, thus making their answer incorrect. What drill might I give these individuals to help them understand just how to do this work?—3A Teacher.

XII. The Course for Teachers of Intermediate Grades

The reader's attention is directed to page 352 for a statement of sources for this section and to pages 351f. for a list of prerequisite concepts. It is expected that in this course these concepts will be extended and enriched. One of them, selected more or less at random as a type, may be approached somewhat as follows.

1. The So-Called 'Laws of Learning'

The conditions under which learning takes place, the means by which it may be facilitated, and the forces which retard it are important topics in the training of teachers. Twenty-five or thirty years ago, Dr. E. L. Thorndike formulated three statements which he regarded as "laws of learning." He called them the laws of "readiness," of "exercise," and of "effect." These formulations profoundly influenced both the theory and the practice of teaching. Recently they have been attacked as insufficient and misleading. Even their author has modified his position in regard to them.

It is doubtful if the doctrines of readiness, exercise, and effect should ever have been regarded as *laws*. Moreover, the stimulus-response theory upon which they are based is looked upon with suspicion by many of the most competent psychologists. Nevertheless, the ideas for which these 'laws' stand are of undeniable importance, especially for the type of learning known as habit-formation.

The course in arithmetic offers an especially good opportunity to study the principles of learning because of the ease with which results may be measured. Whether the concept of readiness is regarded as a law or not is perhaps less important than that the meaning of the idea be apprehended. The same may be said of the concepts of exercise and of effect.

The principle of readiness must be given serious consideration as soon as a child enters school. Should he be taught number at once, and if so, what kind and in what way? To what extent do individual differences play a part? If a normal child should not begin arithmetic when he enters school, when should he begin it? The answers to these questions rest in large part upon an application of the concept of readiness. Some of the material needed for this purpose is easily available. The students should have access to it. It appears that if a child is six years of age mentally, he is quite ready to learn arithmetic—in fact, has already made some progress as a result of his experience.

The principle of readiness applies not only when a child enters school, but also throughout his entire school life. From the administrative point of view the whole question of grade placement rests fundamentally upon the fulfillment of this requirement.

There are some indications that we ought to revise our course of study in deference to the principle of readiness. Convincing direct

evidence is lacking, but there is reason to suspect that the hard multiplication and division facts should be postponed until the fourth grade, that long division should be put over into the fifth grade, and that the more difficult operations with common fractions should be postponed until the sixth grade. On the other hand, some of the material offered in the latest grades of the elementary school is easier than that which is supposed to precede it. For example, the topic of graphs which has lately been introduced into the seventh and eighth grades can, in its simpler aspects, be appreciated by children of the fourth grade and can be taught to them successfully.

One application of the idea of readiness is found in the doctrine, now receiving considerable approval, that children need more time for learning elementary arithmetic than has been allowed them. It is easy for an adult (who has largely forgotten the way he learned) to think of the subject as involving only a few topics, each of which may be summarily dealt with. Some have even advanced the theory that arithmetic can be taught the child—that is, to the normal child—in half the time now devoted to it. This notion undoubtedly arises from a distortion of adult perspective. The amazing thing is that the child penetrates as far as he does into the mystery of number.

Teachers, even those who have had considerable experience, are less successful than they might be, largely because they do not appreciate the complexity and the abstractness of arithmetic. One of the most important topics, therefore, to which students in training may give attention is the peculiar difficulty of the learning involved in this subject. Let them, for example, study the children's mental processes, using the techniques which Buswell and Brownell have successfully employed. They will then be led to see not only the modes of thinking which children utilize in dealing with numbers, but also the enormous difficulty of those more effective, but highly artificial, modes which have received the sanction of contemporary society. Modern educational theory demands sympathy for the child. Here is a way to make intelligent applications of the principle of readiness, to the end that both sympathy and guidance may follow.

The concept of exercise has to do with repetition—with reteaching, and practice or drill. It has a positive and a negative side. The former concerns learning, the latter forgetting. These may be admirably illustrated by material from arithmetic. The treatment of the sub-

ject tends to run riot unless a rigid application of the notion "other things being equal" is made.

Much of the pedagogy of arithmetic revolves about the application of the idea which Thorndike called "exercise." Some of the fundamental monographs on the subject should be made familiar to the student. Certain principles for guidance in actual teaching should likewise be brought to light.

It may not be amiss at this point to refer to the so-called "Beta Postulate" in learning. This startling proposal was first made by Professor Knight Dunlap. He proposes *practice in error* as a means of eradicating error. If this proposal proves to be sound, it will introduce a new principle of learning. In pursuance of it teachers will find out the most typical errors and train children in them *as errors* with a view to their avoidance. For example, 8 is the most common wrong answer to $1+8$. No amount of practice on $1+8—9$ seems completely to eradicate this prevalent mistake. The suggestion which is now being made is what this mistake may be eradicated by giving practice on it *as the wrong way.*

The doctrine of effect brings the emotions of the learner upon the scene. This means that learning is easy when motivation is high, and hard when it is not. At this point the whole doctrine of interest and effort and the rôle of coercion come in for consideration. There can scarcely be a more important topic within the course for the training of teachers of arithmetic than that which concerns the enlistment of the child's interest in the cause of learning. The doctrine of mind-set, the function of purposes, the rôle of knowledge of progress, and the virtues of the project method, all have their roots in the principle of effect.

These three principles of learning, as well as others not here mentioned, are all influenced by the question of individual differences. Obviously some children are ready for certain fields of subject matter before others are. It is equally clear that some children learn faster than others; that is, that they require fewer presentations of subject matter and less practice upon it before attaining a specified goal. Finally, it is equally clear that some children have stronger interests and more of them than others have.

Beginning, therefore, with the 'laws of learning' as one of the concepts which have already been discussed in an introductory professional course, it is entirely possible for the arithmetic instructor to

lead the student naturally to a consideration of several other leading concepts as applied to this subject—such as individual differences, individualizing instruction, measurement, interest and effort, motivation, forgetting, and fatigue. Indeed, an entire course in educational psychology—or in this case in the educational psychology of arithmetic—may take its rise from the principles of learning.

2. More Specific Items

As has already been pointed out in the course for the training of primary teachers, there are a number of topics which, although they may be thought of as general, nevertheless have particular reference to a specific subject. Philosophy, for example, is a general topic, but as a general topic it may have no place in a teacher-training institution. Even the philosophy of education is of wide and general scope. There is, however, a philosophy of arithmetic, and this topic has already been suggested as one which may be thought of under the heading of "more specific items."

The following is a list of pertinent topics which ought to receive considerable attention in any one of the three arithmetic courses. For students in this course the clarifying and deepening of the ideas represented by these topics should rest primarily upon the subject matter to be taught to children of the fourth, fifth, and sixth grades.[15]

- Philosophy
- Objectives
  - Grade standards
  - The course of study
- Measurement
  - History and theory of tests
    - Making
    - Using
    - Commercial
    - Types of
- Analysis of skills
- Drill
  - Materials
    - Selection
    - Appraisal
    - Making
    - Devices
    - Lessons
    - Maintenance program
- Analysis of error
- Remedial measures
- Record making
- The literature
  - Primary
  - Secondary
- History
  - General
  - Special
- Textbooks
  - Examination
  - Analysis
  - Judging
- Problem-Solving

[15] A statement in moderate detail concerning most of these topics is given on pages 355-360 of this chapter.

Lesson Planning
- Types of lessons

Collecting teaching material

The curriculum
- Changes in the past 25 years
- Criteria for selecting content

The interests of children
- Purposes
- Projects
- Activities
- Games

The special application of these topics to the subject matter of the arithmetic of the intermediate grades will generally be apparent. It would, however, be an arrant case of glossing over difficulties if the writer were to fail to point out the relative lack of materials for the intermediate grades as compared with the amount for the primary grades. It will sometimes happen that an adequate treatment of one of these topics can only be secured when the primary field is drawn upon. Moreover, a general acquaintance with the primary field is desirable—is even necessary—for the teachers of the intermediate grades. Accordingly, it is likely that in the practical working of a course for training teachers of intermediate arithmetic considerable attention will be paid to primary arithmetic. It is obvious that no extended treatment can be given here of each of these "most specific items." Accordingly, as before, one will be chosen for a discussion that may be taken as typical in scope and detail.

3. Problem-Solving

In the course for primary teachers certain ideas about problem-solving have already been presented. Since they also apply here, the reader is referred to the discussion on pages 357-360. These ideas baldly stated are:

1. Problems should be used under every topic.
2. Each new topic should begin with problems requiring the new type of computation.
3. Problems should be used to supplement drill and to form part of it.
4. Problems should be presented in miscellaneous sets.
5. Problems should be more numerous and easier than most courses of study contemplate.
6. Many of these problems should be done orally.
7. Pupils should create problems to meet certain specifications.
8. They should estimate the size of the answers.
9. They should receive practice in solving problems without numbers.
10. The teacher should classify problems and teach types.
11. Problems should possess reality, vividness, and interest.
12. Just as problems should be real, vivid, and interesting; so the teaching of them should have these qualities.

The reader is likewise referred to the treatment of transfer of training (pp. 353f.) as related to problem-solving.

Problems for intermediate grades differ from those for the primary grades in two ways. They involve harder thinking and they relate to the particular kinds of computation which first emerge in the fourth, fifth, and six grades. The difficulty of getting the right answers to the problems is affected by both of these considerations.

Teachers can use problems more intelligently if they know how hard they are. Not only in arithmetic, but also in other subjects, a vast amount of trouble is caused through ignorance of the difficulty of school tasks. Indeed, it is astonishing how hazy the notions of otherwise expert teachers are of the difficulty of instructional materials. This is particularly true, as the writer has had frequent occasion to notice, in the case of verbal problems in arithmetic. A type of activity which will prove stimulating and suggestive in a training course is an inquiry into the question: "What makes problems hard?" If such an inquiry were pursued with lists in hand of problems already evaluated, questions like the following could be considered: What effects have the size of the numbers, their integral or fractional character, the units in which they are expressed, and the processes by which they are combined? What effects have the number of steps, the vocabulary 'index' of the problems, the length of the problems, the presence of unessential detail (interesting or otherwise, verbal or numerical), the necessity for the pupil to supply information (*e.g.*, the number of months in a year), and the setting in which the problems occur?

In the intermediate grades the complex problem—that is, the problem involving more than one step—is the most important new thing in the application of number processes. It is true that problems involving long division and fractions play a vital part in the work of these grades, but the logical basis for such problems has been laid in the earlier grades. The new elements are, for the most part, the size of the numbers involved. As a matter of reasoning, therefore—in other words, purely in the problematic sense—the really distinctive feature of the intermediate grades is the complex problem.

Generally speaking, two-step problems should not be begun until the fourth grade. A broad and liberal treatment of one-step problems should be carried out during the first three grades. Only when this has been done can the two-step problem be successfully handled in subsequent grades.

When two-step problems are begun, they should be treated systematically. In fact, a great deal of the difficulty in teaching problems arises precisely from a lack of system. As Ballard says: "Problems are supposed to belong to a province of their own—a land of anarchy and confusion, where every member is a law unto himself." [16]

The virtues of *classification* have already been set forth (page 359). An obviously useful way of classifying two-step problems is by operations. If we recognize four operations, making no distinction between partition and quotition (division by measurement), there are theoretically sixteen types of problems, since each operation may be associated with all four operations. The following are the types:

Addition—Addition	Multiplication—Addition
Addition—Subtraction	Multiplication—Subtraction
Addition—Multiplication	Multiplication—Multiplication
Addition—Division	Multiplication—Division
Subtraction—Addition	Division—Addition
Subtraction—Subtraction	Division—Subtraction
Subtraction—Multiplication	Division—Multiplication
Subtraction—Division	Division—Division

If a distinction is made between partition and quotition, there will be, theoretically, twenty-five types of two-step problems.

Any of these types may occur in life, although some of them are rare. Perhaps the rarest is the addition-addition type. It is not only rare, but it may be solved by regarding multiplication as one of its steps. The following problem will illustrate: "Mr. Brown hatched 118 chicks in April and 212 in May. In June he hatched as many as he did in April and May together. How many did he hatch in the three months?"

Students should have practice in making problems of each of these types. They should also have an opportunity to examine lists of two-step problems and to classify them according to types. Such activities will tend to make them intelligent about problem materials.

Unlike some of the bases of classifying problems, there is reason to believe that this basis may well be imparted to the pupils. Nothing outside the range of their experience is involved. Moreover, the benefits are immediately evident. The pupil may be called upon to indicate without solving a given problem (or before solving it) the type to

[16] Ballard, P. B. *Teaching the Essentials of Arithmetics.* London: University of London Press, 1928. P. 205.

which it belongs. Thus he may show that the first of a series of problems is an SM problem, the second an AD, the third an MD, and so on (the letters show not only the operations, but also the order in which they are performed).

Pupils who have had frequent practice in this kind of work can easily be carried a step further. They can learn to indicate the procedure by using the numbers and the appropriate signs of operation. For this purpose they will need to introduce a letter to stand for the result of the first operation. Consider, for example, the problem, "Ellen spent $1.40 for Christmas cards at $.05 each. After she had sent some to her friends, she found she had 6 left. How many cards had she sent away?" The solution of this problem may be indicated as follows, letting F stand for the first result:

$$\begin{aligned} 140 \div 5 &= \mathrm{F} \\ \mathrm{F} - 6 &= \mathrm{Ans.} \end{aligned}$$

Or if the equation form is not favored, the operations may be shown as indicated at the right. When pupils have learned thus to indicate solutions without actually getting the answers, they can be given a great many two-step problems to be handled in this way. Problems of three steps may be similarly treated by introducing S, or any other convenient letter, to represent the result of the second operation. (The reader will no doubt note with approval the natural introduction here of the use of a letter to stand for any number which satisfies a fixed condition.)

$$\begin{array}{r} \mathrm{F} \\ 5\overline{)140} \\ \mathrm{F} \\ \underline{-\,6} \end{array}$$

When two-step problems are first introduced, all pupils should be led to see that each is in effect two problems of the kind they have already been doing. The problem about Ellen's Christmas cards, for example, breaks into two problems, one of which calls for the number of cards she bought and the other for the number she sent away.

4. Special Topics

Certain subject matter appropriate for the teachers' course in intermediate arithmetic will naturally relate directly to the course of study for grades four, five, and six. Under the heading of "Problems," reference has already been made to the complex problem, which constitutes one of the major items in the course of study for these grades. On the purely mathematical side the following topics are to be recognized:

1—Multiplication with integers completed
2—Division with integers completed
3—Common fractions

4—Decimal fractions
5—Denominate numbers
6—Measurement.

To these may be added, according to local conditions, graphs, percentage, and some of the applications of percentage.

The students should be given an opportunity to examine courses of study in arithmetic and to note the variations in subject matter and grade placement. A really fundamental treatment of the curriculum in arithmetic, however, is beyond the scope of this course. Nevertheless, an attempt should be made at least to suggest what the lines of thinking now are in regard to the content, aim, method, and outcomes of the work in elementary arithmetic.

Although the above-mentioned topics for these grades do not contain 'number concepts,' the students in this course should realize that pupils are developing *concepts* throughout their school days. This topic is emphatically *not* to be thought of as belonging exclusively to the primary grades. On pages 361-362 the topics believed to be applicable to the primary grades under this heading have been set forth. Students in training for intermediate-grade work should review these topics—the various functional meanings of numbers, their units, and the bases of apprehension (counting, measuring, grouping, ratio). Especially appropriate to the course now under discussion is the significance in connection with number concepts, of large numbers, of fractional numbers, of zero, of multiple units, and of ratio.

Carrying the topic of number concepts still further, it will be appropriate in this course to give some attention to *ordinals of special significance,* such as the first, the last, and the middle number of a series which has been arranged according to size. The middle number is called the *median,* and the difference between the first and last numbers is the *range* of the entire series. The median and the range are important values. The idea of ordinals of special importance may be carried to the point of identifying *quartiles* and *percentiles.*

The treatment of *accuracy* is in large measure a matter of number concepts. The student may be brought to see the meaning from the point of view of accuracy, of such numbers as 5 feet 4 inches, of 158 feet, and of 40 tons. Here belongs the question of accuracy as applied to decimals and the meaning of such phrases as "carrying to thousandths," or "correct to the nearest hundredth," or "correct to three

significant figures." The question of when zero is significant and when it is not likewise belongs under this heading.

The distinction between counting and measuring gives rise to a new consideration in number concepts, namely the *meaning of commensurable and incommensurable numbers.* When we count the books on the shelf, we arrive at an exact result. If we measure the shelf, we arrive at an approximate result—a result which will almost certainly be different if we use a finer scale. In measurement most magnitudes are incommensurable. All through arithmetic artificial assumptions are made without the pupil or the teacher being conscious of them. If a bushel of potatoes costs $2.00, we unhesitatingly say that five bushels will cost $10.00. We assume the possibility of measuring first a bushel of potatoes and then five bushels of potatoes. Moreover, it is assumed that the measurement of five bushels, when obtained, will be precisely five times as much as the measurement of one bushel. Sir Oliver Lodge points out: "Incommensurable quantities are.... by far the commonest, infinitely more common in fact....than the others; 'the others' being the whole numbers and terminable fractions to which attention in arithmetic is specially directed, which stand out, therefore, like islands in the midst of an incommensurable sea; or, more accurately, like lines in the midst of continuous spectrum." [17] The distinction between counting and measuring, with the resulting discrete and continuous numbers, is seldom felt except when a number which should arise from counting is treated (sometimes with humorous effect) as if it arose from measuring—as in the case of the old lady who tried to look out of the window two and a half times.

An additional topic under the heading of number concepts is concerned with the *units* by which we measure. This is a fascinating subject. The origin and gradual standardization of these units is an important chapter in human evolution; while a comparison of contemporary units used for various purposes in this country and abroad is almost a liberal education.

With more particular reference to the special topics listed on pages 373f., a single one of these topics will be selected for the fuller treatment which all of them should receive.

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[17] Quoted by Ballard, *op. cit.*, p. 241.

5. Division with Integers Completed

When pupils enter the fourth grade, they are in the midst of the mastery of the 90 simple division facts. It is true that many, perhaps most, courses of study call for the mastery of these facts in the third grade, but it is notoriously true that children do not master them in that grade. Whether, therefore, the course of study does or does not assign these facts to the third grade, considerable work will have to be done in the fourth grade by way of teaching and drilling on the harder division facts.

In earlier days it was the fashion to keep things apart much more than is now done. The child, for example, would be taught all the 90 division facts and would, perhaps, receive practice on inexact division within the tables before he would be introduced to short division. The tendency to-day, however, is to carry these matters along together. As fast as pupils learn the simple division facts, they use them in examples like $2\overline{)84}$, $3\overline{)96}$, $5\overline{)155}$, $4\overline{)120}$, and so on. Similarly, as inexact divisions are taught, it is now customary to put them to immediate use by introducing longer examples in which these inexact divisions occur.

a. Dividing by a One-Place Number—One has to decide whether to use, at this point, the long or the short form of dividing by a one-place number. There is a tendency among thinkers about arithmetic to prefer the long form—the same form as is used when the divisor consists of more than one place—postponing 'short' division to a later grade, there to be treated as a short cut. With this tendency the writer is in entire sympathy.[18]

Ballard takes the same position and does so with his customary vigor. He pleads in general for what he calls the King's Highway. By this he means that for each process one good standard mode of procedure should be adopted and adhered to. In conformity with this idea he says:[19] "The argument [*i. e.*, the argument for the short form of division] amounts to saying that the short-division road is the easier road. And this I flatly deny; at any rate I deny that it is the easier in the long run. In the long run, taking the smooth with the rough, the pupil will find it a roundabout road, and a bewildering road, and a readily forgotten road, and a road beset with pitfalls—anything, in fact, but an easy road. Moreover, it is

[18] A vote of the Committee taken by mail disclosed the fact that five of the members were in favor of the long form. The sixth member did not appear to be against it as a matter of theory, but rather to be cautious regarding it because he believed the experimental evidence insufficient.

[19] *Op. cit.*, p. 170.

not the main road—not the King's Highway, which must sooner or later be trod. And if the principles I have tried to expound are universally valid, as I think they are, to tread the King's Highway from the very beginning is always the wisest plan. . . . It is wrong to regard long division and short division as two distinct rules. They are not. There is only one rule: and that is long division."

This appears to be wise counsel. At the same time it should be pointed out that no published evidence on the question is available. Dr. Buswell, however, reports in a letter to the writer that "a carefully controlled experiment in the Laboratory School of the University of Chicago yielded differences in favor of the long form which, though not large, were nevertheless consistent." This is the only evidence on the question of which the writer is aware. It will hardly be regarded as sufficient. It does not appear to bear upon the achievements of pupils when they use divisors of two and three places. In short, it does not cover the long run. It is more than likely that the full advantage of the long form may not be apparent in the field now covered by short division, but rather in that at present covered by long division.

On theoretical grounds, however, the case appears to be clear, and we are always justified, in the absence of sufficient experimental data, in appealing to the best principles at our disposal. That is what we always do. For example, one of the members of the committee, in a letter to the writer, first advises that short division be treated as a short-cut, to be taught later as such. He then points to the superior psychology of forming early habits which will take care of all division, followed *later* by the short-circuiting of some of these habits on the easier portions of the job—*i. e.*, upon dividing by a one-place number. He believes that this is psychologically more promising than the present practice of using short-cut habits on easier portions of the job, followed by an attempt to present more basic habits when harder parts of it appear.[20]

b. Analysis of Division by a One-Place Number. It will be profitable for the students to study various analyses and to compare their advantages. These analyses are intended to make it possible for a child to encounter one new difficulty at a time. Although there are many analyses of division by a one-place number, they all have the same purpose and hence have many points in common. Almost at random, Morton's is taken.[21]

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[20] The point of view here advocated is likewise stressed by Morton. See his *Teaching Arithmetic in the Primary Grades*, pp. 167-169, and *Teaching Arithmetic in the Intermediate Grades*, p. 75. See also Johnson, T. J. "Short division or long division first?" *Chicago Schools Journal*, 12: October, 1929, 55-56.

[21] Morton, R. L. *Teaching Arithmetic in the Intermediate Grades*, p. 76.

1. The primary division facts, as $8\overline{)48}$
2. Examples having two-and three-digit quotients, no carrying
   (a) Divisor contained in first digit of dividend
      (1) Without remainders, as $3\overline{)69}$, $2\overline{)486}$
      (2) With remainders, as $3\overline{)68}$, $2\overline{)487}$
   (b) Divisor contained in first two digits of dividend
      (1) Without remainders, as $3\overline{)126}$, $4\overline{)2484}$
      (2) With remainders, as $3\overline{)127}$, $4\overline{)2487}$
3. Examples involving carrying, no remainders, as $4\overline{)172}$, $5\overline{)2325}$
4. Examples involving carrying, with remainders, as $6\overline{)232}$, $7\overline{)4538}$
5. Examples containing zeros in the quotients, with and without remainders
   (a) At end of quotient, as $4\overline{)40}$, $5\overline{)750}$, $3\overline{)32}$, $8\overline{)2643}$
   (b) In midst of quotient, as $6\overline{)1206}$, $4\overline{)1612}$, $9\overline{)5422}$, $8\overline{)3219}$

Of these five types, the first and second do not involve carrying. Since the products are always equal to the partial dividends and since the remainders are therefore always zero, it is *not* simpler to write the products and to bring down the next digit. In other words, it is simpler to regard these types as merely a succession of division facts without any complication whatever. It is obvious, too, that a portion of Morton's Type 5 should be treated in the same way. Indeed, it is doubtful whether his Type 5 is really a type at all. It is a recognition of a very persistent kind of error, but the postponing of such an example as $4\overline{)40}$ or $6\overline{)1206}$ until the last unit of work in dividing by a one-place number is probably unwise. According to the plan here advocated, Types 3 and 4, and a part of Type 5 should be handled by the full or long form.

c. Divisors of More Than One Place.—All writers testify to the difficulty of dividing by numbers of more than one place—the topic usually treated as long division. It is the writer's belief that this matter will ultimately become a fifth-grade subject with a gain not only for division but also for other parts of arithmetic.

Again the advantage of analysis into types is apparent. The following considerations play a part in making such an analysis: (1) the example does or does not offer difficulties in estimating the quotient figure; (2) the example does or does not involve carrying in the multiplying; (3) the example does or does not involve carrying in the subtraction; (4) the example does or does not involve zero difficulties in the quotient; (5) the example has or has not a remainder. All these

considerations assume a fixed number of digits in the divisor and in the quotient. If a broader view of division is taken, then two additional criteria for analysis appear, namely, (a) the number of digits in the quotient, and (b) the number of digits in the divisor.

A large or a small number of types of examples will result according as these criteria are strictly or loosely applied. Students should have an opportunity to study various lists of types and to judge their value.

In what follows, unless something is stated to the contrary, the divisor is supposed to consist of two digits only. No matter whether division by a one-place number has been taught by the long or the short form, as soon as two-place divisors are encountered one new and important difficulty presents itself, namely, estimating the quotient figure.

Accordingly, a strong effort has been made to simplify this estimating of the quotient figure. Two schools of thought prevail. One may be called the 'Two-Rule School,' the other the 'One-Rule School.' The Two-Rule School, as the name implies, uses two methods of estimating the quotient figure. It uses one if the unit figure of the divisor is small and the other if it is large, as follows—Method 1: in estimating the quotient figure, divide by the tens' figure of the divisor if the unit figure is less than 6; and Method 2: in estimating the quotient figure, divide by the tens' figure increased by 1 if the unit figure is 6, 7, 8, or 9. On the other hand, the One-Rule School uses one method only, without regard to the unit figure of the divisor, as follows—Method 3: in estimating the quotient figure, divide by the tens' figure of the divisor.

In various parts of this chapter the virtues of one general method, at least when a new process is first learned, have been extolled. The writer therefore agrees with Roantree and Taylor when, speaking of the method of adding 1 to the tens' figure when the units' figure is large, they say: "It seems to introduce needless confusion to teach two different ways of testing for finding the correct quotient figure. . . . In mechanizing a process, the more uniform our procedure, the more efficient our performance."[22]

Neither the Two-Rule School nor the One-Rule School covers all the cases. Each, in order to make its system rigorous, must supplement its first statement. The One-Rule School needs only one supplementary procedure, namely that which applies to the overestimate, for if the estimate according to Method 3 is wrong it is always too large. The child is therefore taught that if his quotient figure is wrong, he must "try the next smaller number, and then the next, and so on until his product is less than the partial dividend."

[22] Roantree, W. F., and Taylor, Mary S. *An Arithmetic for Teachers.* New York: Macmillan Company, 1925. Pp. 113.

The Two-Rule School uses two supplementary procedures, because its first rule, when it fails, leads to an overestimate, while its second rule, when it fails, leads to an underestimate. The first supplementary procedure of the Two-Rule School is the same as the One-Rule School uses; the second is "try the next larger number, and then the next, and so on until the remainder is less than the divisor." Thus the system of the Two-Rule School tends to become complicated for beginners—and it is to beginners that these rules must be offered. The peculiar difficulty of the underestimate (which the One-Rule School avoids), with the fact that the remainder must actually be obtained before the failure of the estimate can be detected, is especially troublesome.

On the other hand, if the child can use the two methods of the Two-Rule School without confusion, he will find fewer cases of its failure to work than is possible with the One-Rule School. Yet any impression which careless statements may have conveyed that the two methods of the Two-Rule School *always* work must be definitely abandoned. The following are some of the facts:

There are 49,050 possible examples with two-figure divisors and one-figure quotients, like $36\overline{)257}$. Since in long division one merely gets one figure at a time, these 49,050 examples are (except for "bringing down") *all there are* to long division with two-place divisors.

Method 3, the method of the One-Rule School, works in 29,805 of these examples and yields an *overestimate* in 19,245 of them. The supplementary procedure—"try the next smaller number, and then the next, and so on"—takes care of these overestimates.

Method 1, the first rule according to the Two-Rule School, tells the child to divide by the tens' figure if the divisor ends in 0, 1, 2, 3, 4, or 5. There are, however, 7,020 examples with such divisors in which this rule yields an *overestimate*. The supplementary procedure last quoted takes care of them. Method 2, the second rule according to the Two-Rule School, tells the child to divide by 1 more than the tens' figure if the divisor ends in 6, 7, 8, or 9. There are 3,950 examples with such divisors in which this rule yields an *underestimate*. An additional supplementary procedure is required to take care of them. The total number of cases in which the Two-Rule School yields either an overestimate or an underestimate is therefore 10,970.

Long division is the most intricate of the operations in elementary arithmetic. A book on it or a course in it is not too much to lay bare its meaning and implications. The student should be given an opportunity to study some of the interesting questions with which it bristles.

6. Common Fractions

The resort to fractions is merely an attempt to secure smaller units. When one child measures the height of another with a foot-

rule, he may find the result to be five feet and something over. He must report the result more accurately. So he uses a smaller unit, namely, the inch. He may find that the child whom he is measuring is five feet three inches tall and a little more. Now what is he to do? In reality he must do the same as before—take a smaller unit. Since there is no smaller conventional unit, he must use fractional parts of the inch.

Throughout the first four grades as well as in life outside of the school the child has already had considerable experience with fractions before the systematic treatment of them is begun. He has talked about and has used freely the idea of one-half as applied to the inch, the foot, the yard, the pound, the dozen, and the dollar. He has likewise used one-fourth (or one-quarter) as applied to the same units. Three-fourths has likewise been used with reference to common units. Probably he has also employed the unit fractions from $\frac{1}{2}$ to $\frac{1}{9}$ in finding the fractional parts of numbers—this as a part of the work in division.

The time now comes to gather these scattered bits of information and experience together for an organized treatment of the subject of fractions. The teacher at this point should give an inventory test and be sure that the desired knowledge is possessed.

In the course for the training of teachers something should be shown concerning the history of common fractions. It is an interesting subject and gives a sense of perspective.

The teacher should be sure that the various concepts of fractions are presented and used—the fraction as a part of a unit, the fraction as a part of an aggregate of units, and the fraction as a ratio.

A great deal has been said concerning the fractions which should be used in elementary arithmetic. No doubt the older tests and courses of study went too far in using unusual fractions with large denominators. Certain present-day writers on arithmetic have greatly emphasized this matter. As a result the pendulum has in some quarters swung too far the other way. The limits are drawn so tight that the child gets no richness of experience. The permitted fractions are so few that an intelligent appreciation of fundamental principles is impossible. Wilson found that ten fractions made up 96 percent of the fractions reported to him as used by adults, but neither Wilson nor any other reputable student of arithmetic has drawn the conclusion

that they were all the fractions that the child should encounter in his arithmetic course.

A reasonably generous list should be admitted for two reasons. First, a great many more fractions than the short list usually mentioned are in fact used. Some of them are not used very often, but when a person needs them, his need is just as real as if it occurred frequently. Second, practice with a larger list of fractions is needed in order that the smaller and more essential list shall be well known. Sixteenths doubtless make eighths more meaningful.

Another point should be cleared up in this connection. What does it mean to say that the fractions to be used shall be limited to those having certain denominators? The writer has seen the answers in the back of a textbook examined and the book condemned because of the presence of such fractions as fifteenths and twenty-fourths, to say nothing of fortieths and sixtieths. This is foolish. If fifths and eighths are admitted, they cannot be multiplied without being likely to produce fortieths. If fifths and twelfths are admitted, and scarcely anybody would fail to do so, they can easily lead to sixtieths. Is the child to stare at his answer without a notion of its meaning? Is he to be permitted to get it without being permitted to realize what it is? One man guesses that an auditorium is two-thirds full, another that it is three-fifths full. If they have learned no more than the minimum list of fractions which some of the 'reductionists' appear to favor, they may argue hotly about their difference because they do not understand how small it is.

Again, books have been censured, in the name of modern findings, because answers to examples in the division of whole numbers contained fractions with larger denominators than were thought to be allowable. The question of whether the complete quotient shall be required, always or occasionally, or whether its integral part together with the remainder will be sufficient is perhaps debatable. If, as is usually the case, children have been using fractions to some extent since the first grade, they may be shown the complete quotient. Certainly the average number of pages read per day by a girl who reads 131 pages in 7 hours is inadequately expressed as "18 pages with a remainder of 5." Nor is it desirable that the answer $18\frac{5}{7}$ should be debarred because sevenths are not admitted to the select society of the chosen common fractions. Indeed, if the limitation upon fractions to be used is made to include the answers in division, it is prac-

tically impossible, until decimals have been taught, to use the partition type of division unless the divisor is exactly contained in the dividend. This accounts in part for the large number of artificial division problems in which the divisor miraculously 'goes even.'

A reasonable limit should, of course, be placed upon the number of common fractions to be taught. It is not unreasonable, however, to admit all fractions whose denominators are a one-place number together with twelfths and sixteenths and to allow without cavil any result which may arise from a proper use of these fractions.

Students in training to teach intermediate grades should have an opportunity to study the typical errors which pupils make in addition, subtraction, multiplication, and division with common fractions. An outstanding analysis of this sort has been made by Brueckner.[23] In the addition of two fractions or mixed numbers he distinguishes 40 types; in subtraction, 45; in multiplication, 45; and in division, 37.

7. Decimal Fractions

The most important question under this heading is the apprehension of the meaning of these fractions as an extension of the decimal notation. Whether the approach to this desired end is through common fractions or more directly from the place value in the notation itself is a detail. If pupils have not been given genuine and liberal instruction in common fractions, the use of such fractions in showing the meaning of decimal fractions will be difficult. In the absence of experimental data we must appeal to reason, and it seems reasonable to use both methods for the support they may be expected to give each other.

If the meaning of decimal fractions is well taught, the troublesome questions about placing the decimal point will be largely taken care of. As far as may be, the explanation or at least the reasonableness of the rules for placing the point in multiplication and division may well be shown. It is certain that the mere learning of these rules on the part of the children will prove to be effective only temporarily. They must have something to fall back upon in the known nature of the numbers themselves and of the operations.

Again, for teaching purposes the various topics should be analyzed into units of work in each of which but one important new difficulty

[23] Brueckner, Leo J. "Analysis of errors in fractions." *Elem. Sch. Jour.*, 28: June, 1928, 760-770.

appears. For example, in division, the following types are readily distinguished: (1) the division of a decimal (fraction or mixed number) by a whole number, (2) the division of a whole number by a decimal, (3) the division of a decimal by a decimal. In the first type, there is relatively little difficulty; the point in the answer can be placed over the point in the dividend. In the second type, zeros equal to the number of decimal places in the divisor must be annexed to the dividend to produce an integer in the quotient. In the third type, the excess of decimal places in the dividend over those in the divisor determines the decimal places in the quotient. In all these types additional places to the right of the decimal point may be obtained in the quotient by annexing additional zeros to the right of the point in the dividend.

In decimal fractions, as in common fractions, Brueckner has done distinctive analytical work.[24] While children should be able to read decimal fractions of several places, it is idle to require them in their computation to extend their work far to the right of the decimal point. People seldom have need of decimals of more than two places, and only rarely should pupils be expected to express results to more than three places. As soon as practicable they should be taught to round off their decimals so as to report them to the nearest tenth, hundredth, or thousandth.

8. Case Studies

Students of the State Teachers College at Upper Montclair, New Jersey, during the period of their practice teaching, were encouraged to report the difficulties which they encountered. The material thus collected, so far as it related to arithmetic, has been furnished the writer. Similar material has likewise been received from the University of Chicago. So far as the Chicago material is concerned, the questions and problems are of a general character. Some of the more pertinent ones have already been listed on pages 362f. They have equal application here. The reader is referred to pages 363f. for a classification of the cases reported from the Montclair students. These students were teaching in grades one to six. At this point certain of the reports from teachers in the fourth, fifth, and sixth grades will be quoted.

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[24] Brueckner, Leo J. ''Analysis of difficulties in decimals.'' *Elem. Sch. Jour.*, 29: September, 1928, 32-41.

As was observed in connection with the primary course, the students who are doing practice teaching find the handling of reasoning problems particularly difficult. There is every reason to believe that regular teachers have the same trouble. This indicates from one more point of view the necessity for devoting serious attention in a teacher-training course to the question of problem-solving. One student teacher puts the matter this way:

> What can I do to develop the reasoning ability of my class in arithmetic? They can do the fundamentals of arithmetic very well; however, the entire class, with very few exceptions, is unable to tackle the most simple problems.—4th-grade Teacher.

Another student teacher has this to say:

> One great problem in our grade is to get pupils who do not like arithmetic to reason out the problems. They make all sorts of guesses as to what should be done next.—4th-grade Teacher.

The difficulty is, of course, not confined to the fourth grade. For example:

> Through redevelopment and drill, children seem to have mastered fairly well the mechanics of the different processes in decimals, but when given problems a good many make mistakes. Some of these say they "didn't know you did it like decimals" (fail to see application) and others ask "do you multiply?" "add?" or "divide?" etc.—6th-grade Teacher.

A number of students report their difficulties in subtraction. They are taught to use the additive method. They go out into near-by public schools to do their practice teaching. In those schools they appear to find, quite commonly, that the children have been taught take-away subtraction in the primary grades. Then they are in trouble. Their stories are interesting, but they will not be permitted to occupy space here. One variation of this rather common difficulty occurs in the case reported by a fifth-grade teacher to whom a student teacher had been assigned. According to this report the student teacher had taught subtraction of fractions by the addition method, although the children had learned subtraction of whole numbers by the take-away method. This procedure "has confused them very much" and the regular teacher now inquires: "Should I continue to drill them as they have been drilled or should I teach them according to the old method of subtraction which they know?" The writer would hazard the opinion that children should be allowed to continue in the method they were taught, whatever that method is.

Long division, as usual, proves to be difficult. Here is an interesting report of how it is made more than necessarily so by too much fussy detail:

The class is taking up long division. They are a class of average intelligence. The division is taken up in this way: this is what the children must say with the example: 21)462:

1. Try: 2 goes into 4 twice.
2. Divide: 21 goes into 46 twice because 2 went into 4 twice.
3. Multiply: 2 times 21 = 42.
4. Try: 42 is less than 46, so it is all right.
5. Subtract: 2 from 6 is 4; 4 from 4 is 0. Our remainder is 4.
6. Try: 4 is less than 21, so it is all right.
7. Bring down: bring down the 2. Our new partial dividend is 42.
8. Try: repeat same process each time a number is brought down.
9. To end up they say; the quotient is 22.

Some of them do not have any idea of what it is all about.—4th-grade Teacher.

Here is a difficulty which the reader may not have thought of:

The class has been doing long-division examples now for about three weeks. Some of the pupils almost invariably try to make the examples, which have remainders, come out even.—4th-grade Teacher.

Perhaps here is another difficulty which is not fully appreciated:

We are having long division (5th grade). It is review work to the children, but many of them don't seem to get it. I would like to know a method which they would understand. Their trouble seems to be in placing the numbers right after multiplying the divisor by the quotient.—5th-grade Teacher.

Now, a few of these student teachers are having trouble because the children do not know their primary-grade arithmetic. The following plaint is common, although the attitude is a bit peculiar:

Upon testing I find that my children need drill in the fundamentals of addition. How can I give them the needed drill without its seeming too childish for the grade?

Here is a good idea (apparently on the part of somebody higher up) carried to the point of being vicious:

The requirement of the fourth grade in the 100 addition facts is 100% with a rate of 40 per minute. So far, four of our class have attained that standard, while the rest of the class come along with a rate of between 25 and 35. Several are down as low as 17 and 19. What are we going to do with them? We have had concentrated study on this since the beginning of school—six weeks' study. I know the children are getting tired

of it, but they must have these facts before they can go on. Is there any way in which they may go on with subtraction without being up to standard in addition? Will it hinder them very materially with their other arithmetic work?

There is a considerable group of cases which seem to the writer to arise from an attempt to make progress too rapidly. For example:

> Each day they (the children) are taught a new skill and they apparently have an automatic control—but later they make mistakes which show that it has not become automatic.—4th-grade Teacher.

Does not the case just quoted illustrate admirably the novice's fault of pushing ahead with too many new things? It is so easy, unless experience or wise counsel has taught otherwise, to mistake the ready response on the occasion of initial teaching for permanent learning—and thus to fail to provide a maintenance program. The same hurry is evinced in the following case:

> Fractions are very hard for some of my children. They have only had them a week, but still they should know how to add them. A few of the children do not understand that $5/4$ is more than one whole. We have illustrated by circles, etc., and they remember for a few minutes, but when they have to add such as $1\frac{1}{2}$ and $2\frac{3}{4}$ they give us $33\frac{5}{4}$ or $4\frac{5}{4}$. They forget to change the $5/4$ to $1\frac{1}{4}$ or else forget to put down $1/4$ instead of $5/4$. Is there anything besides drill that can be done?—4a Teacher.

Fancy supposing that fourth-grade children who have had fractions only a week should exhibit the abilities here suggested. The consequence is a failure to appreciate meanings, as is shown in the following case:

> These children have no number sense at all. After spending a month in drilling them how to change fractions to common units, they insist (in a test) that $1/2 + 1/3 = 2/3$. They seem to understand the work in daily tests. When the weekly test is given, they forget what to do.—5th-grade Teacher.

Well, don't get emotional about it. Children's mistakes aren't made because they want to annoy you. They have another cause, which is not to be identified with infantile perversity. To find it is often a fascinating exercise of detective intelligence. Doubtless you are partly right. There is in all likelihood a lack of number sense. But what does a child do to get $2/3$ as an answer to $1/2 + 1/3$? Has he just been dealing with the case where one of the two denominators is the common denominator, such as $1/2 + 1/4$ or $1/6 + 1/2$? Look for some other errors and see if you can tell how they came about. Do

you not find $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$? If you don't find that type of error, you will find it later.

Another case of overambition seems to be reflected in the report of a student teacher who, after stating that the regular teacher had been working on problem solutions, goes on to say:

> The class has had all the work leading up to the problems. In class general problems without numbers are discussed and then problems with numbers. The children make up general problems and then substitute numbers. The work seems very difficult for them and when a written lesson is given, half the class fail.—4a Teacher.

Even the solving of general problems is too hard for many fourth-grade children; and to require them to originate such problems is simply to overestimate their ability—unless they are highly selected or have been exceptionally well taught.

The last case is typical of a number which indicate the general failure of the school to provide enough educative experience for the bright child. The report reads:

> In my arithmetic class there are ten children who are above average and when problems are given out, these are finished much sooner than the rest. What can I do with these children while the others are getting more attention?—5th-grade Teacher.

Such are a few of the conditions which puzzled certain normal-school students when they were doing their practice teaching in arithmetic. Some of the reports are naïve and some involve questions which are fairly easily answered. Many of the problems, however, which confront these students are as old as the school itself. They cannot be given each a general answer, either because of conflicting evidence or because the answer only becomes clear when the circumstances are intimately known. One would be interested to know what replies, if any, were given to these students and what effect both the questions and the answers had at this particular normal school upon the training of teachers of arithmetic.

XIII. The Course for Teachers of Grammar Grades

1. Defining the Objective of the Course

No detailed consideration will here be given of the curriculum of the junior high school. The preparation of junior-high-school teachers of mathematics is more than a question of arithmetic and more than a question of a two-year curriculum.

The 'new mathematics' for the seventh, eighth, and ninth grades requires exceedingly careful treatment. The teacher is, more than ever, the rock of salvation. He must be well trained professionally. Moreover, his knowledge of subject matter must be extensive. He will have to select and evaluate for his purposes material from commercial arithmetic, intuitive geometry, elementary algebra, numerical trigonometry, and demonstrative geometry. In order to do this even with modern success his knowledge of these fields and of the adjacent fields of mathematics must be considerable. A two-year teacher-training program is not to be thought of.

The prevailing American school system, however, still includes a four-year high school. Below it the elementary school may have eight grades or seven. Attention is here directed to the service in the last two years of the elementary school in such systems. Sometimes the teachers in these last two grades will be subject teachers, sometimes grade teachers. In either case the mathematics which they teach is either exclusively or dominantly arithmetic.

While, however, the training of junior-high-school teachers of mathematics is thought of as something to be provided for by another and quite different course from the one here contemplated, it is nevertheless true that even in the eight-year elementary school the influence of the junior-high-school curriculum is strong. In the last year of that school the pupils not infrequently deal with such subjects—relatively new below the high school—as formulas, simple equations, graphs, directed numbers, and the like. Indeed, even in the 8-4 organization there is nothing to prevent the curriculum of the seventh and eighth grades, so far as mathematics is concerned, from being identical with that of many junior high schools.

To a greater extent, therefore, than is the case in preparing teachers of the primary and intermediate grades, a dual objective is here indicated. There is an old curriculum and a new curriculum, and each is so prevalent that no teacher-training institution can afford to leave either of them out of account.

2. The Teaching of Academic Subject Matter

No matter whether the student is going to be called upon to teach a conservative or a progressive mathematics course in the grammar grades, two considerations make it appropriate to teach this student academic subject matter in the teacher-training institution. On the

one hand, the public-school curriculum itself, thought of as ranging from traditional to modern, is now more exacting than the curriculum which many students have been taught in the public school. On the other hand, the available professionalized subject matter exclusively applicable to the teaching of arithmetic in these grades is small in amount and poor in quality. Few real investigations, however, have been made in which the subject matter of these grades was directly involved. Each operation with whole numbers has been the subject of experiment and analysis, but no one has investigated the way to teach trade discount or analyzed it into work units or even seriously considered whether it should be taught at all. In fractions no small effort has been made to determine the commonest types of errors and to supply a preventive and remedial regimen, but no such effort has been made for mensuration or simple interest. Textbooks have been carefully analyzed for their treatment of primary arithmetic and to some extent for their treatment of intermediate arithmetic, but no one has thought it worth while to bestow the same attention upon the materials for the seventh and eighth grades.[25] ;It is in this sense that, as has been said above, "the available professionalized subject matter, exclusively applicable to the teaching of arithmetic in these grades is small in amount and poor in quality." It is for this reason that if a list could be secured of, say, the 100 topics of most worth in a course for teachers of arithmetic in these grades, a relatively large number of these topics would belong to academic subject matter.

This subject matter, however, should be concerned primarily with the activities of human beings in modern society. It should also include some arithmetic. When these students go out into public schools and attempt the teaching of seventh- and eighth-grade arithmetic, they will need sociology, some economics, and some business training; and they will also need a better grasp of arithmetic.

a. Subject Matter in Arithmetic.—So far as the arithmetic is concerned, this emphatically does *not* mean offering the same material as was taught in the elementary school. It means a more exacting type of arithmetic, a type which is not offered as a rule in the elementary school, a type which is designed to give a teacher a working margin of control.

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[25] It is true that sometimes the third book of a three-book series has been critically examined, but always, so far as the writer remembers, with references to its provision for drill in the facts and processes taught in earlier grades.

Consider, for example, the idea of *accuracy*. This is a big concept in teaching, especially in teaching mathematics. One of the desirable outcomes of this teaching is the accuracy habit or ideal. The course for teachers should stress this point. But this is insufficient. The student should learn (as he certainly has not learned in any thoroughgoing fashion in the public school) the conditions under which a desired degree of accuracy in results may be secured—likewise the degree of accuracy appropriate to a given situation. This requires a treatment of the meaning of units of measurement and of the degree of precision to which the measurement has been taken. It involves a recognition of the essential difference in precision in such expressions as 7 lb. 4 oz., 150 feet, 1,000 tons, and a 4-day boat. The difference in meaning between 27.5 miles and 27.50 miles, the fact that in the absence of evidence 16¾ inches may not be assumed to be as accurate as 12⅝, the fact that if a decimal fraction is carried, say, to hundredths, all others in the same discussion must likewise be carried to hundredths; the meaning of "correct to four decimal places," of "correct to tenths," of "correct to three significant figures"—these are some of the items to be treated.

Moreover, the effect of a smaller degree of precision in one of the measurements upon the results in different kinds of computation is illuminating. For example, if the height of one boy, measured to the nearest ¼ inch, is 58 inches and the height of another boy, measured to the nearest half inch, is 54 inches, what is the maximal possible error in taking 4 inches as the difference in their heights? A little study of this question shows that the first boy may be anywhere between 57⅞ inches and 58⅛ inches tall while the second boy will be between 53¾ and 54¼ inches. The limit of error, therefore, in taking 4 as the difference is $+$ ⅜. In other words, under the conditions, the true difference lies between 3⅝ inches and 4⅜ inches. A similar treatment of accuracy for other operations than subtraction may likewise be undertaken.

Again, with further reference to academic subject matter in the field of arithmetic, consider the matter of *number concepts*. In this course there should be given a very brief treatment of the topics already included in the primary-grade course under this heading (see pages 361f.)—the various functional considerations or ways of behaving in the presence of a number idea, the various kinds of units of which numbers may be composed, and the various ways in which

numbers may be apprehended. Then a somewhat fuller treatment may be given to the topics mentioned in the intermediate-grade course in connection with number concepts (see pages 374f.), especially to fractional numbers, zero, ratio, and multiple units; to units of measurement, and their origin and standardizations; and to the comparison between various units, domestic and foreign, ancient and modern. Some of the topics special to this course under the heading of "number concepts" might be negative numbers, general numbers, infinity, infinitesimals, limits, commensurables and incommensurables (the latter arising in measurement and the former in counting), constants and variables, functionality, and the theory of number. The matter of units may be treated under such heads as discrete and continuous, units as intervals, selection of units according to purpose, and units in fractions as determined by the denominator.

From these two instances—the consideration of accuracy and the treatment of number concepts—the reader may understand the sense in which it is said that the arithmetic to be taught in this course is not of the elementary-school type, but is designed to give the teacher "a margin of control."

b. Other Academic Subject Matter.—When it is said that such matters as sociology, economics, and business training are likewise required, it is not to be inferred that required courses in these academic fields are contemplated. There is no room for them in a two-year curriculum. The institution should offer such courses and students should be shown their advantages with a view to electing them. What is here meant, however, is that if the teacher is to handle the arithmetic of the home and of the community, a working knowledge of the home as a social unit and of the institutions within the community is required. Whether we think of this as a leaf from the book of sociology is relatively immaterial. If the teacher is to handle the arithmetic of the market, some elementary notions of economics are likewise necessary. The leaning of our upper-grade arithmetic toward business usage is well known—and sometimes deplored. In order that the teacher may cope with this situation, it is far more desirable that she should know something about our great industries and their ways of transacting their affairs than it is that she should receive renewed practice in the three cases of percentage or in finding the areas of plane figures. In short, then, the arithmetic course for grammar-grade teachers should include such headings as "Economics useful

in seventh- and eighth-grade arithmetic," "Pertinent sociology," "Business practice for the grammar-grade teacher."

The principle here invoked in regard to economics, sociology, and business practice is the same as has previously been suggested in connection with courses in drawing and music. The elementary and secondary schools in which our students have been trained cannot be relied upon to teach enough of these subjects. Hence the shortcomings of the students with reference to these subjects are quite different from their shortcomings, say, in long division or in subtracting denominate numbers. The point is that the public has provided and paid for instruction in long division and denominate numbers; that the students have unquestionably received such instruction; and that their present shortcomings are due either to temporary forgetting or to personal deficiency. If the former, they can be held to accountability; if the latter, they should be excluded.

In view of this principle, it is unhesitatingly asserted that the teacher-training institution will always have to teach non-professional subject matter. This duty will be more imperative in preparing teachers of upper grades and of the secondary school, but the duty will not disappear even in the preparation of teachers of comparatively elementary subjects. This is because the better class of public schools are now requiring certain kinds of subject matter which the people have not yet made generally available, and because it is against public policy to fail to prepare students for these schools. The nature of the subject matter which the teacher-training institution must offer will not remain the same as time passes. It will change, first with the demands of the progressive schools, and second with the amount and nature of the lag of the more ordinary schools.

This lag will be accentuated to the extent that the movement for reconstructing the curriculum gathers momentum. Those who are pushing this movement forward are doing so regardless of the teacher-training problem involved. Very likely they should disregard this problem; they have enough obstacles to contend with. The problem, however, is nevertheless very real. It will be met ultimately by one or more of these four methods; (1) refining the method and content of teacher-training courses, (2) selecting students more highly, (3) requiring preëducation courses in the secondary schools, (4) lengthening the curriculum in the teachers' colleges.

3. Professional Work

One must not infer from the foregoing discussion that strictly professional topics will fall into the background in the course here under consideration. The teacher of mathematics in the seventh and eighth grades will need professionalized subject matter regarding the reviewing of whole numbers, fractions, and denominate numbers; for such a review is definitely required in these grades, no matter whether the course of study in question be progressive or conservative. Observe, too, that this professionalized matter is not the same as that which should be offered to students in primary and intermediate courses. It is not concerned with original teaching but with review. It therefore leaves out of account much of the elaborate analysis of types of examples, unit skills, and the like, and all the body of knowledge, experimental and speculative, about methods of presentation. It does, however, concern itself with standards and objectives—with what the seventh- and eighth-grade children should know and be able to do; with the detection of error, the determination of its cause, and the application of remedies; with the characteristics of an intelligent program for maintaining and developing skills which have already been more or less acquired—with diagnostic tests, good drill materials, progress tests, the nature of the drill lesson, motivation, and so on.

The teacher of the seventh or eighth grade will likewise need professionalized subject matter of general import but with specific reference to the arithmetic of business, of the community, and of the home —in short, with reference to such arithmetic as will constitute the new work of the seventh and eighth grades. For example, the principles applicable to the exclusion and inclusion of subject matter in the curriculum must be taught to these students, but the particular incidence of these principles will perhaps be upon the "business applications of percentage." Again, the best thought about problems and problem-solving must be included in the course for teachers of grammar grades; but this material should not, except perhaps incidentally or for illustrative purposes, be made to apply to the problems of the primary and intermediate grades. It should be applied to problems which, either because of their difficulty or their subject matter, are appropriate to the grammar grades.

No matter what academic subject matter may be taught, the professional work of the teachers' colleges must be maintained at all costs. It must be maintained in the theory courses and in the pro-

fessionalized subject-matter courses such as the one in arithmetic to which attention is being directed in this chapter. The value of this professional work grows as its body of doctrine increases. It is the only basis for a true profession of teaching. To forsake it is to admit defeat. The fact that the body of professional doctrine exclusively applicable to seventh- and eighth-grade arithmetic is small affords no basis for denying this thesis.

(1) In the first place, there is an abundance of material in the application of the general ideas gathered from the introductory professional courses (see pages 351f.). A thoughtful treatment, for example, of *motivation* with sole reference to the things the student is going to have to teach—to interest, profit and loss, and mensuration—will be vastly helpful. The topic *individualizing instruction* may become a real principle of action when it is considered on the basis of the errors which pupils make in percentage or in stocks and bonds or in using the formulas for the areas of plane figures. *Concomitant learning,* after having been made more or less familiar in general courses, may now be clothed with new meaning by being applied to the making of graphs or to investment or to the protection of property.

Such general topics in education do not, to be sure, belong exclusively to grammar-grade arithmetic. They do not belong exclusively to arithmetic. They are universals. Yet precisely for this reason they will be empty and ineffective unless their application is seen in concrete instances. A part of the professionalized subject matter of every special course (reading, geography, history, arithmetic) should be devoted to this application. Moreover, the application, in the course now under discussion, should be made to the arithmetic of the grammar grades and not, as is so often the case, to that of the primary grades. This is difficult. The trail has not been blazed, and it is so easy for the instructor to slip into the broad and well-lighted paths of the number combinations and the operations with whole numbers. These are fine paths, but they do not lead these students in the right direction.

It has just been urged that, in the first place, the application of general principles constitutes an important part of the professional work in the grammar-grade course in arithmetic. The difficulty of making this application, however, brings a second type of professional activity into view.

(2) In the second place, therefore, each class in grammar-grade arithmetic should do something to *contribute* to the field. This will, it is admitted, largely depend upon the teacher, who will doubtless have to locate the problems and direct the work. But the students can help and in some individual cases or in the case of an occasional small group they may even assume some responsibility. In any event this type of work can be made tremendously valuable for the student besides adding something to the body of professional doctrine.

The fact that we have so little professional material exclusively applicable to the mathematics of the seventh and eighth grades is a challenge to the teacher-training institutions. A body of doctrine of this sort is much needed. There is only one way to create it, and that is by research methods. If the offerings of the schools in seventh- and eighth-grade mathematics are unsatisfactory—and this is generally admitted—it is mainly because the contributions to the field in question are few and unenlightening. It is of little avail to shift the seventh and eighth grades from the elementary to the seeondary school. This may be a good thing to do on general grounds. But to expect a mere device of organization to vitalize the work of these grades is childish. It reminds one of the old woman who, having observed that the second summer was the critical period for infants, proposed that we do away with the second summer by calling it the third summer.

(3) In the third place, the professionalized subject matter of this course should include the special treatment of still other general ideas—ideas which may not have been presented in the introductory professional courses, but which, being especially applicable to particular subjects, are usually taught in connection with those subjects. For example, the place of arithmetic in a plan of education should receive professional treatment. It is a general question in the sense that it belongs to arithmetic as a whole. It is not, however, so general as to belong to all subjects, and hence would not, in all likelihood, have appeared in the introductory courses. It should receive consideration in the grammar-grade course from two points of view: (a) the total subject of arithmetic, (b) the part of the subject taught in the seventh and eighth grades.

A number of other topics of a general-arithmetical character should certainly receive attention—such as the objectives of arithmetic, the arithmetic course of study, grade standards in this subject, the history

of the subject, analysis of skills, classification of errors, criticism of drill materials, tests of various kinds and the way to use them, remedial measures, the examination of textbooks, and problem-solving. In all these cases the general pattern for the professional treatment of these topics should be as indicated above, namely, first with the whole subject in mind and second with the particular part of the subject in mind.

(4) In the fourth place, as in the primary and intermediate courses, the material exclusively applicable should be taught. The fact that there is little of it has nothing to do with the principle. For example, the analysis of percentage, elsewhere presented in this Yearbook, would be an appropriate subject for study in this course. So would Morton's analysis of the errors of fifth-, sixth-, seventh-, and eighth-grade children in solving verbal problems.[26]

4. The Curriculum

Early in the course for training upper-grade arithmetic teachers the students should be made acquainted with the general features of the work they will be called upon to do. The older curriculum of the last two grades of the elementary school, as has been said, devotes much time to reviews of processes with whole numbers, common fractions, decimal fractions, and denominate numbers. This is especially true in the eighth grade, where the pupils often have to be prepared for some type of leaving examination.

This curriculum likewise includes some work in percentage and gives a great deal of attention to the so-called 'business applications of percentage.' A good bit of mensuration is likewise included. The more conservative curricula still introduce ratio and proportion, involution, and evolution; but cube root, partnership, foreign exchange, and annual interest already seem to have gone where alligation, tare and tret, and duodecimals went a generation or more ago.

On the other hand, in many places the curriculum of the last two grades of the elementary school is not much different from that of the corresponding grades in progressive junior high schools. Here the 'new mathematics' is found.

What is the 'new mathematics'? Partly it is a redistribution of subject matter; partly it is a new spirit. Those who support the

[26] Morton, R. L. *Teaching Arithmetic in the Intermediate Grades.* New York: Silver, Burdett and Co., 1927. Pp. 312-317.

new mathematics believe that we are wasting time in the seventh and eighth grades in our teaching of arithmetic, and they make out a good case for their belief. They would teach much less arithmetic in these grades and would bring in topics from intuitive geometry, elementary algebra, and numerical trigonometry. Much more significant, however, is the spirit in which this mathematics is offered. Names may not be deeply significant, but they do show tendencies; and these tendencies are evident even in the arithmetic of the new mathematics. Instead of such topics as exchange, taxes, discount, insurance, and the like, we find things called by names which are more inclusive, more intimate, and more vital. In connection with business we have sending money, keeping money busy, investing, and going into business. In connection with the home we find such topics as paying for a home, home projects, the cost of living, recipes, heating and ventilating a home, and protecting the home from fire. In connection with the community we find topics such as voting, education, the community chest, and raising and spending public money. These topics appear to be more promising educationally than the ones they supplant. At the same time they are more difficult to handle effectively. Investing is more than stocks and bonds. It includes investing in real estate mortgages, in savings accounts, in life insurance, and in annuities. It even includes investing in education. Sending money includes foreign exchange, but it also covers a great deal more. There is scarcely anything in the business applications of percentage which may not be included in a topic such as ''going into business.''

Note, too, the fact that the new curriculum is interested in science. It is likely, for example, to contain something about temperature—not merely thermometer readings, but something functionally related to expansion. Such functional relations it expresses by formulas and plots in the form of curves. It draws material from scientific agriculture and forestry, from hygiene, from mechanics, and from the chemistry of foods. It concerns itself with such physical phenomena as sound, electricity, and specific gravity. It is likewise interested, to a degree which the old curriculum is not, in the development of the science of mathematics itself. Instead of a pious discourse on accuracy, it introduces something like a systematic treatment of the theory of accuracy and of approximation. It includes as objectives the ability to understand and use directed numbers, to locate points in a system of coördinates, to graph certain equations with reference

to such a system, and to solve problems by means of graphs and by the use of equations. It lays a basis for a systematic treatment of algebra by introducing signs of aggregation, literal quantities and operations with literal quantities—to say nothing of formulas, equations, and directed numbers. It insinuates a bit of trigonometry and assures the conservative that it is *not* immoral to use facts in geometry which have not been 'demonstrated.' In line with its penchant for science the new mathematics is likely to advocate the metric system not only for scientific work but for all other purposes of measurement. Thus the new curriculum is allied to science, somewhat as the old curriculum is allied to business.

The curriculum, then, must receive considerable attention in a course which attempts to meet the needs of teachers of arithmetic in the upper grades. Evidently, there is not just one curriculum. There is a variety of curricula and many questions about which people differ. This variety should be shown by the simple expedient of examining courses of study and textbooks, and these mooted questions should be faced in free discussion. The principles on the basis of which material is placed in the course of study should be canvassed—such ideas, true or false, as tradition, transfer value, frequency of use, exigency, difficulty, contribution to liberal education, and instrumental value. A student is entitled to know the nature of the service he is likely to have to perform and, in some sort, to appraise it.

5. Problem-Solving

Summing up the opinions of writers on arithmetic, Newcomb states that skill in solving problems is agreed to be of primary importance and that speed and accuracy in abstract computation is important only so far as it contributes to success in problem-solving.[27] The writer subscribes to this view.

Usually when one speaks of 'the fundamentals in arithmetic' or 'the minimal essentials of arithmetic,' one speaks in terms of abstract examples. These fundamentals are described as the ability to add, subtract, multiply, and divide with integers and fractions. There is a very real sense in which these are neither the fundamentals nor the minimal essentials. No one ever multiplies 847 by 67 except for a purpose, and that purpose is the solution of a problem. When we figure, our figures have a concrete reference. Mere figuring takes place

[27] Newcomb, Ralph S. *Modern Methods of Teaching Arithmetic.* Boston: Houghton, Mifflin Company, 1926. P. 271.

only in school. In life the problem is the thing; and if we would make our school life-like, we should exalt the purposes for which number is used and pay close attention to problem-solving. The real fundamentals in arithmetic are problems.

This raises the question, what is a 'problem'? or more specifically, in examining a textbook which exercises should one call 'problems' and which should one call 'examples'? Customarily, we say that an exercise in which the pupil is given the operation is an example and one in which the pupil must determine the operation is a problem. This working rule may suffice for some purposes.

The real answer to this question, however, involves an unknown quantity, namely, the experience of the individual child. An exercise which is a problem for one child may be an example for another. Moreover, an exercise which is a problem for a child to-day may become an example to-morrow. Finding the discount on a bill of goods at 20 percent is a problem to a pupil who first encounters it. It presents a new situation, and it demands thought. It may remain something of a problem even when encountered repeatedly, especially if a considerable period intervenes between successive appearances. It may continue to have problematic elements for relatively dull children long after it has ceased to be anything but an example for brighter children. It is only as other exercises of the same kind—for example, finding different discounts on different amounts and on different kinds of purchases—are seen to belong to the same type, to be worked by a type procedure, that the solving of these problems is taught and learned. By that time they cease to be problems and become examples—computations with the operations indicated. In a very real sense, therefore, a taught problem is not a problem; it is an example. By the same token, teaching a child to solve problems is grouping the problems into types and reducing each type to a series of examples.

Problems may offer children training in thinking. They certainly offer them *opportunities* for thinking, and part of the method applicable to problem-solving is a method of teaching how to think. When, however, particular types of problems have been well taught, thinking is no longer required. This is a desirable outcome. If the types of problems are useful—that is, if the problem-solving is worth learning—the procedure should be reduced as nearly to a habit basis as possible. No one in his senses prefers to think when he doesn't have to.

It will be seen that classification and generalization play an important part in learning, and therefore in teaching, to solve problems. Consider this one: "Jones and Robinson lunch together and the waiter presents one check. Jones observes: 'Robinson, the check is $1.45 and you had a $.25 dessert. The rest of the bill is the same for each of us.' How much should each pay?" Confronted for the first time by these conditions, the child will find here a real problem. Judging by the awkward manipulations which Jones and Robinson probably indulge in, the situation remains a problem even for them. Certainly no small degree of intelligence will be required on the part of a child in the solution of this problem when he first encounters it.

The following situation may now be brought into relationship with one just mentioned: "Harold's motor boat will go up-stream at the rate of 8 miles an hour and down-stream at the rate of 12 miles an hour. How fast will it go in still water and how fast does the stream flow?" Here again the child will have to think pretty hard and very likely will fail. The teacher may, however, help him to see that these two problems have common essential characteristics. The essential points common to the two problems are that the sum and difference of two numbers are given with the requirement that the two numbers be found. The amount of the check in the first problem is the sum of the amounts Jones and Robinson should pay. The rate down-stream in the second problem is the sum of the rate in still water and the rate of the stream. In the first problem the difference between the amounts Jones and Robinson are to pay is the cost of the dessert which Robinson had; while in the second problem the difference between the rate in still water and the rate of the stream is the rate up-stream. This type of problem is not taken as a common one nor as a type which can be found in everyday situations. An unusual type has been intentionally selected.

Helping the child to see that these two problems embody the same principles, that they may be classified as of the same type, and that the solution therefore proceeds along the same lines—this is an important step in teaching the solution of the problems. The means by which the type solution is arrived at may vary. Indeed, the method of solution itself may vary. The writer's preference would be to introduce a few problems with small numbers. "John and his little sister buy 10 cents' worth of candy, but John is to pay 2 cents more than his sister. How much will each of them have to pay?" By juggling

the figures about (if by no other method) it can be discovered that John will pay 6 cents and his sister 4 cents. Or one may reason that for every cent *more* than half the money which John pays, his sister will pay one cent *less* than half the money. This suggests that we first get half of 10. If John pays one cent more than half (6 cents), his sister will pay one cent less than half (4 cents). The method may therefore be to find half the total amount and add half the difference to this answer to get the larger share, subtracting half the difference to get the smaller share. Of course, the neatest way of solving the problem is to add the sum and difference and divide by two to find the larger share. For example, $\frac{10 + 2}{2}$. It is essential at any rate that a method of solving this type of problem be set up and that a recognition of other problems as belonging to this type and susceptible of the same solution be secured.

The fundamental idea in problem-solving is transfer. This in turn depends upon the seeing of likenesses and the appreciation of differences. The child who can see or be brought to see that the problem about Jones and Robinson and the one about the motor boat have essential likenesses and unessential differences, has taken a long step in understanding the kind of problem. Additional problems belonging to the same type may now be introduced, and the essential likenesses and unessential differences may be inferred, or if necessary, pointed out. Thus, the cost of two articles together and the amount which one costs more than the other may be set up. Two methods of conveyance, one of which goes a given number of miles further than the other, and the total distance covered by both together may be presented. Again, perhaps more artificially, A and B are bowling partners. Together they score a given number of pins. Someone who knows this fact and doesn't know their individual scores is also told that A scored a certain number more than B.

In order to facilitate transfer through the appreciation of likenesses and differences, the classification of problems is imperative. It is possible that the greatest fault in our program of problem-solving is the fact that each problem is treated as a separate and unrelated item. Our psychology should teach us better. Things in isolation are never well learned. Even the pupil to whom problems are presented in that manner must make his own associations, from his own crude classifications, in order to get anywhere.

Almost any classifications are better than none. We have scarcely scratched the surface of the possibilities in this direction. The most obvious classification is in terms of processes. Such a classification is valuable, but insufficient. When we know, as testified by Judd, Monroe, and Osburn, that literally hundreds of expressions may be used in problems to mean the same process, we realize that something additional to classification by processes is required. Thus we must not only classify but cross-classify.

The following concepts each constitute a possible basis of classification: (a) the data given; (b) the fact or facts required; (c) the units employed; (d) the nature of the numbers involved—integers, fractions, percents, and the various subdivisions of these; (e) the fields of human activity—the home, the store, lumbering, picnicking; (f) subject-matter fields—insurance, brokerage, similar triangles, volumes of cylinders.

Unless a systematic treatment of problem-solving has been carried out in the primary and middle grades, it will be well-nigh impossible to secure high-grade problem work on the part of pupils in the grammar grades. The principles, however, are essentially the same as those in the lower grades. For a treatment of some of these the reader is referred to the topic ''problem-solving'' as it appears in the primary and intermediate courses of this chapter.

It is recommended that students who are taking the course in arithmetic for grammar grades spend an appreciable amount of time in writing problems, criticising problems, classifying problems, estimating the answers to problems, judging the difficulty of problems—in short, in becoming truly expert in this type of material. The technique of writing problems is rather exacting. Practice in it is excellent training. Sometimes the assignment of the course may be to write a set of problems in connection with some project. Sometimes projects may be related to a large topic and different students may be assigned different hints for projects. For example, in connection with food, one group of students may write on production as it relates to the farmer, another on production as it relates to the manufacturer. Other groups may write problems on the transportation of food, its storage and preservation, the work of middle men, the public market, purchasing food, preparing it for use, planning meals, cooking food, menus, classes of foods, etc. Some of these problems, after being criticised, solved, classified, and approved should

be tried in the practice school and their difficulty should be approximately determined. Finally this material should be duplicated and given to the students. Perhaps it should also be published and given to the world.

6. Problem Material

The course in arithmetic for grammar-grade teachers may well focus attention upon the whole broad question of selecting problem material—this because in these grades the cap sheaf of the elementary course is placed. In a sense all problems and all problem-solving as it pertains to arithmetic is within the province of these grades—or at least is within the province of the teacher of these grades.

It is suggested, therefore, that a good bit of study may be devoted to the kind of problems which may rightly be used in these grades. Here we are met with a host of destructive criticism—an army of don'ts which threatens to play havoc with any program which seeks to do more than devote itself to abstract examples. It seems as if most of the problems one would like to use are taboo.

"We have," says Ballard, "at various times been counselled to omit from the arithmetic program all problems which are not interesting to children; which do not serve the needs of child life; which do not serve the needs of social and business life; which have linguistic difficulties; which give no linguistic training; which do not illustrate principles; which merely illustrate principles; which work out exactly; which do not work out exactly; which can be solved by definite methods; which cannot be solved by definite methods; which contain trivialities [unnecessary data]; which refer to large transactions beyond the children's experience—and indeed many other kinds which the critic of the moment happens to dislike. Then, in the name of the great Cocker, what sort of problems *are* we to teach?"[28]

A favorite ban upon problems is "do not use problems if the answer must be known in making them." At the same time one is urged by all means to use problems that are *real.* It is certain, however, that problems whose answers are known when they are made may nevertheless conform to every reasonable canon of reality. A few days ago the writer received a card which urged him to subscribe for a certain periodical at an introductory price of $2.00 which, it

[28] Ballard, P. B. *Teaching the Essentials of Arithmetic.* London: University of London Press, Ltd., 1928. P. 214.

was stated, was 20 percent below the regular price. The first thing of interest was to find out the regular price. Yet the maker of this problem had to know the answer before he could make it. The point is that *he* knew the answer, but the receiver of the card did not. To the former it was not a problem at all; to the latter, however, it was not only a problem but a real one.

The fact is that none of the rules of exclusion is always valid. Some are merely crotchety; others are contradictory; and all are vastly inferior to a positive program. Such a program, it is suggested, may well be worked out on the basis of an analysis of thinking. If training in the use of numbers for the purposes of life is recognized as a major objective; then a liberal, constructive, systematic program of problem-solving must be set up. If practice in thinking is accepted as an important activity of the school, then an abundance of material selected with that end in view must be provided. Looked at in this manner, the little rules of exclusion are of small consequence.

In this matter, as in other matters in the work of the school, analysis of types of thinking and provision of material appropriate for these types should be a major activity. This is a curriculum problem. The writer is convinced that we shall make visible progress in furnishing stimulating problem material when we organize the ways of thinking which human beings employ, cross these with the situations in which these ways of thinking may take place, and cross both these schedules with still other criteria of problem selection, such as their supposed interest, their difficulty, the number and character of the required operations, and the units involved. Little advance will be made in furnishing valuable problem material until teachers and students of education are willing to face the question seriously. Classification and cross-classification must be made—not that these mechanical means will serve as automatic selectors, but rather that they will serve to reduce the problem of selection to one of manageable proportions. It now baffles us. It is a land of twilight across which only a few blind trails are laid down, and most of these must of necessity lead nowhere since those who made them lost their way.

7. An Instance of Analysis

Gradually the most important topics in arithmetic are being brought under control. The most potent means of doing this is

analysis of materials supported by a trial of analyzed material over against the same material in unanalyzed form.

As everybody knows, primary arithmetic was the first to receive treatment. Gradually the influence spread to higher grades. Elsewhere in this Yearbook an analysis of percentage is offered for the first time. In many courses of study, percentage is a sixth-grade subject; in a few it is a seventh-grade; in some it is spread over both these grades.

The following analysis is adapted from the treatment given in Chapter IV, Part I, of this Yearbook. Aside from the fundamental operations with fractions, common and decimal, and the identification of the three cases, there are only four new difficulties in percentage. These involve the equivalence of common fractions, decimal fractions (hereafter called merely "decimals"), and percents. The four cases are given in Table X.

TABLE X.—TYPICAL DIFFICULTIES IN PERCENTAGE

Type	Given	To Be Changed to
I	Fraction	Percent
II	Decimal	Percent
III	Percent	Fraction
IV	Percent	Decimal

Each of these four cases arises frequently enough to justify separate teaching and practice. Each is so dependent upon the other three that to leave any out or to give it insufficient treatment is to weaken the whole subject.

Type I is likely to occur whenever fractions and percents are given in the same connection. One may be told, for example, that 3/4 of the children got one example right while only 68 percent got another example right. One must know that 3/4 is the same as 75 percent in order to understand the situation. In the same breath one hears of "1/5 off" and "20 percent off." Frequently these matters are confined to aliquot parts, but by no means always.

Type II occurs most frequently in working Case II of percentage. For example, when we find what percent 38 is of 45, we get a decimal, namely 0.844. This must be converted into a percent.

Type III is not unlike that of Type I in its occurrence. When a fraction and a percent are thought of together, it is often more il-

luminating (and sometimes easier) to bring them into comparison by changing the percent to a fraction than by changing the fraction to a percent. This is especially true if the percent is exactly equivalent to one of the frequently used fractions.

Type IV is probably used more frequently than any other. It is always used in Case I of percentage. If for any reason one has to find $4\frac{1}{2}$ percent of a number, one must begin by expressing $4\frac{1}{2}$ percent as a decimal. Similarly, percents must be changed to decimals (for the purpose of dividing) whenever Case III of percentage comes up.

Having set up these four types, one may take an additional step in the analysis. In considering the difficulty of any of these types the size of the percent is especially important. Table XI gives the significant distinctions.

TABLE XI.—SIGNIFICANT DISTINCTIONS IN SIZE OF PERCENTS

Size of Percent	Example
a. Less than 1	½%
b. 1-9 (whole numbers)	6 %
c. 1-9 (fractions)	5½%
d. 10-99 (whole numbers)	40 %
e. 10-99 (fractions)	37½%
f. Over 99	125 %

Since each of the six distinctions in Table XI may be associated with each of the four types in Table X, it is obvious that we have 24 items to consider. Of course, it is to be understood that in Type I where a fraction rather than a percent is given, distinctions a, b, c, etc., apply to the answer. For example, the fraction may be $\frac{3}{400}$, with a resulting percent which is "less than 1." Similarly, if the fraction is $\frac{1}{20}$, the percent is in the group "1-9 (whole numbers.)"

From this point the analysis may take a number of directions or it may proceed no further. It is entirely possible to recognize the distinction between examples and problems or between a type separately set up (*e.g.,* 0.844 equals what percent?) and the same type in use to serve a larger purpose (*e. g.,* interpreting the answer in Case II of percentage). One may analyze from the point of view of

the common fractions involved—the occurrence of the percent equivalents of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, etc. Finally one may follow the trail of these types and distinctions into the various applications of percentage where even the percent often takes on another name.

Without, however, carrying the matter further than the 24 items mentioned above, one may use them to make one's teaching more intelligent. Textbook and drill materials may be examined for the extent to which they provide for these necessary skills. Moreover, diagnosis of pupils' needs is greatly facilitated by raising these 24 items to consciousness. Causes may be determined and remedies provided. Finally, such an analysis provides a basis for fruitful investigation into method. Are the lessons derived from more elementary arithmetic applicable here? Will a method prove to be superior which introduces each of these skills one at a time, reviewing the old ones and combining them with the new? Will drill material prove more effective which neglects none of these skills but provides for an abundance of distributed drill on all of them?

The foregoing analysis is not offered entirely for its intrinsic value. It is offered as a type of what may be done and should be done with a number of other topics in the seventh- and eighth-grade curriculum, such as interest, simple equations, and graphs. It likewise suggests an analysis of the essential knowledge, skills, ideals, attitudes, and appreciations involved in a number of topics of a quite different order, such as earning, spending, giving, and investing money; the cost of living, the standard of living, making a living; group action for protection (insurance); raising and using public money; buying and selling; and banking. In such analyses the computational aspects of the topics will find a place, though by no means a dominant one. If it is decreed that boys and girls thirteen to fifteen years of age shall be taught business practices, there is more sense in teaching them to understand these practices than there is in confining their attention to the figuring involved in them.

PART II

RESEARCH IN ARITHMETIC

CHAPTER I

THE PURPOSE AND PLAN OF PART TWO

Part One of this volume consists in a systematic treatment of the general field of arithmetic. Part Two is devoted to research relating to this subject. In this part of the Yearbook the Committee is presenting two types of material. The first of these, contained in Chapters II and III, is a classification of the entire body of published research in arithmetic arranged under the nineteen techniques which have been employed in such investigations. The second type of material consists in a series of new research studies relating to several aspects of arithmetic.

The materials in Chapters II and III are introductory to the studies which follow. The general survey of research in arithmetic should enable the reader to evaluate specific studies in a more intelligent manner. The Committee believes that one of the most productive uses of Chapter II will be in connection with advanced classes dealing with problems relating to arithmetic. At present there is no other source in the literature in which the instructors of such classes will find as comprehensive a guide to the existing studies in this field. Furthermore, the research worker will find in the classification of research problems according to various techniques some stimulating suggestions for the application of certain techniques to problems with which they have not previously been used.

In Chapter III no attempt has been made to review, one by one, the previous research studies in arithmetic. Such summaries are already available in previous publications. The purpose of Chapter III is to give a general overview of research in arithmetic and to point out the trends of research in this field.

The studies reported in Chapters IV to XV were selected after a careful canvass of a large number of projects which were submitted in various forms by members of this Society in response to a letter which was sent by the chairman of the Committee to all the members inviting contributions to this part of the Yearbook. Approximately seventy replies were received, some of which were accompanied by finished reports, but many of which were simply statements of possible research studies which the author was willing to undertake.

The Committee canvassed all of these replies and selected a number of projects which seemed most promising for use in the Yearbook. From this group of promising leads a second selection was made after the complete reports were in hand. More studies were submitted than could be used in the space allotted to the Yearbook. Consequently, some reports which the Committee would have been glad to include had to be excluded. These will probably appear as articles in some of the educational journals. In making the final selection of studies, the Committee tried to include a fairly representative body of material, covering various types of problems and using various kinds of techniques.

Chapter IV is presented as an illustration of what the Committee considers a desirable form of a rather elaborate report of research. The Committee allowed more space to this study than to others in order to illustrate the value of reviewing the previous literature in a field and of making a full interpretation of the data which were secured.

Chapters V, VI, and VII report studies dealing with various aspects of the content of arithmetic. Chapters VIII and IX deal with the transfer of training and with the improvement of arithmetical reasoning. Chapter X is the only study which deals with the problem of measurement. It contains an account of the construction of a test in arithmetic for measuring general ability. Chapter XI deals with the problem of motivation and is an abstract of a very elaborate study of the effect of awareness of success or failure. Chapter XII is an illustration of one type of diagnostic procedure in analyzing pupils' misunderstandings in percentage. Chapter XIII is an attack upon one of the most difficult problems in arithmetic, namely, the proper grade-placement of instructional materials. This chapter had to be very much condensed and is based upon a rather comprehensive investigation of this problem. Chapter XIV consists in a rather detailed review of one specific problem. It is a type of treatment which might have been desirable to give to many different topics, but which could not be done on account of limitations of space.

The general impression which the Committee has of the present status of research in arithmetic is that, while a very substantial number of studies are now in published form, the majority of fundamental problems in arithmetic cannot be given final answers until more elaborate research studies have been carried on. Much of the research in arithmetic which has been published is brief and inconclusive, and a

considerable proportion of it deals with trivial problems. There are numerous fundamental problems which should receive the attention of men who are mature enough to devote more time to investigation than can ordinarily be given by a graduate student in preparing a thesis. The Committee hopes that the summaries of research afforded in this part of the Yearbook will stimulate fundamental studies of this type.

For the Committee,
G. T. Buswell

CHAPTER II

THE TECHNIQUES OF RESEARCH EMPLOYED IN ARITHMETIC

W. A. Brownell

I. Purpose of the Chapter

Exact and valid knowledge concerning the many problems of teaching and learning in arithmetic is to be had only from fundamental research studies which are carefully planned, painstakingly prosecuted, and wisely interpreted. In the last analysis the central issue in such studies consists in the development of a technique adapted to the type of problem which is to be investigated.

If the truth of the two preceding statements is conceded, it should prove valuable to collect and to examine the various techniques which have been devised for the study of problems in the field of arithmetic. This the present chapter purposes to do. In the pages which follow there will be found, first, an account of the procedure by which the materials here reported were brought together; second, some note of the limitations of the study; third, a descriptive list of the research techniques which have been employed in arithmetic investigations; fourth, a classification of research studies in arithmetic on the basis of the problems studied, the techniques employed, and bibliographical citations to the original reports of the investigations; and fifth, certain comments on some of the more significant features of the classification. It should be apparent from this outline that the chapter represents merely a convenient summary of the present situation with regard to arithmetic research from the standpoint of technique; it does not offer much by way of critical evaluation. Other portions of the Yearbook are designed to serve that purpose.

II. Procedure Employed in the Present Study

The basis for the present study was provided by Buswell and Judd's *Summary of Educational Investigations Relating to Arithmetic*,[1] published in 1925, and by the four annual supplements

[1] Buswell, G. T., and Judd, C. H. *Summary of Educational Investigations Relating to Arithmetic.* (Department of Education, University of Chicago, 1925, pp. viii + 212).

published since that date by Buswell. The uses made of these materials and the steps in the procedure employed in the present study are outlined briefly in the following paragraphs.

Step 1. The Buswell and Judd summaries were read very carefully page for page by a group of five persons.[2] A card was prepared by each person for every investigation as soon as specific mention was made of that investigation in the body of the summary report. Data were entered on these cards to cover the following items: (1) the problem or problems investigated, (2) the technique or techniques employed, (3) the name of the investigator, (4) the reference number of the original report of the investigation according to the Buswell and Judd bibliographies, and (5) the number of the page in the body of the summary in which the investigation was cited.

Step 2. When, at later points in the summary, additional references were made to investigations for which cards had already been prepared, any new information was recorded on the corresponding cards.

Step 3. When the résumé of any investigation as reported by Buswell and Judd seemed to be incomplete from the point of view of the present study, the original report of the investigation was examined. This was done in the case of fully 70 percent of the references. In this way there were uncovered in several instances additional problems and occasionally additional techniques.

Step 4. Cards similar to those described in Step 1 were also prepared for those investigations which were not treated in the summaries.

Step 5. When all the summaries had been read, the five members of the group independently prepared complete lists of problems which had been investigated in arithmetic, using as the basis for these lists the entries on the cards. A total of 127 such problems was thus secured.

Step 6. Then in joint session the members of the group, each supported by his own list and his own cards, classified and reduced the number of problems from 127 to the present number, 67.

Step 7. New cards, called "problem cards," were then prepared, so that each card represented one of the 67 problems. On these cards there were gathered together from the original cards complete data regarding the techniques which had been employed in connection with the given problems and regarding the reference numbers of the investigations.

Step 8. The problem cards were then utilized to make a master chart, on which the 67 problems were listed along the left-hand margin and the techniques along the top. Opposite each problem entries of all relevant investigations were made under the various technique headings. The result was a complete classification of the research investigations in arith-

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[2] The writer desires to acknowledge the assistance of the following students in his course in "Investigations in Arithmetic" who took part in this project: Paddy Blair, Lillian Burdette, Mary L. Gabbert, and Lorena Stretch.

metic on the basis of (1) the problems investigated, (2) the techniques employed, and (3) specific bibliographical references.

Step 9. From this master chart it was then possible to draw off the separate charts which are to be presented later.

III. Limitations of the Study

1. Articles Omitted

Buswell and Judd have summarized to date the contents of 532 separate investigations. Of this number, 142 are omitted in the present classifications. Table I lists the reasons for omitting investigations, together with the number of investigations omitted for each reason. Table II lists by number the specific investigations omitted in this study.[3]

Table I.—Disposition of Investigations as Summarized by Buswell and Judd

Source: Summary List of Year	Number Omitted for Certain Reasons							Total Number of Investigations		
	Reports Unavailable	Arithmetic Instructional Materials	Texts on Methods of Teaching	Routine Testing	Reviews and Summaries	Texts in Educational Psychology and Measurements	Discussions Not Quantitative	Reported	Omitted	Classified
Up to 1925	1	6	13	3	6	7	19	314	55	259
1925		5	1		6		12	58	24	34
1926	4*	1	1	1	5		13	52	25	27
1927	3†	1	3		2	7	2	52	18	34
1928					3	1	16	56	20	36
Totals	8	13	18	4	22	15	62	532	142	390

*Three of these reports were not regarded by Buswell as of sufficient importance to demand special mention in the body of the summary report.

†Two of these reports were unimportant according to the same criterion.

[3] There is, of course, no intention to reflect discredit on any investigation by failure to include it; that is, the absence of any study in the classifications should not be interpreted as evidence that the study is of little value, for in some instances quite the contrary is true. Some of the omitted articles which are critical in nature are of very much more worth than some of the less careful experimental studies which are included. The general criterion for admitting or omitting investigations was whether or not they were quantitative in nature. Only one type of investigation which was not quantitative was included, namely, the type based on historical and anthropological techniques.

TABLE II.—REPORTS OMITTED, BY REFERENCE NUMBER* ACCORDING TO THE BUSWELL AND JUDD BIBLIOGRAPHIES

Source: Summary List of Year	Reports Unavailable	Discussions Not Quantitative
Up to 1925	175	1, 4, 5, 11, 23, 25, 32, 58, 65, 68, 78, 83, 84, 99, 109, 125, 130, 143, 146, 150, 153, 155, 160, 161, 182, 185, 200, 205, 210, 219, 226, 228, 234, 249, 251, 252, 261, 262, 267, 268, 270, 271, 274, 284, 292, 298, 301, 305, 306, 308, 314, 315, 317, 319
1925		505,* 506, 508, 509, 513, 514, 516, 518, 520, 521, 522, 524, 539, 540, 542, 544, 545, 547, 549, 551, 552, 553, 554, 555
1926	1,† 13,† 14,† 43 †	606,* 608, 609, 616, 618, 619, 620, 622, 624, 625, 626, 627, 629, 632, 633, 636, 645, 646, 650, 651, 652
1927	3,† 14,† 29	701,* 706, 710, 711, 712, 720, 721, 728, 732, 733, 734, 738, 743, 745, 746
1928		801,* 803, 810, 811, 817, 822, 823, 825, 826, 830, 831, 836, 837, 841, 842, 846, 847, 848, 849, 852

*For purposes of economy of space, all references to the 1925 summary are numbered beginning with 501. Thus, the first reference in the 1925 summary is throughout this chapter indicated by the number 501 instead of by the number 1. Similarly, the references in the 1926 summary are made to begin with 601, and those in the 1927 and 1928 summaries, to begin with 701 and 801, respectively.

†Not regarded by Buswell as of sufficient importance to demand special mention in the body of the summary report.

2. The Problems

The 67 problems finally selected for the purposes of classification represent a reasonably detailed analysis. Certain obvious limitations may be noted in passing. In the first place, any one of the problems could be further broken down into subproblems. Thus, for example, the first problem, "Adult uses of arithmetic," might easily be divided into minor problems in terms of the types of adults investigated—business men, teachers, superintendents, etc. If, however, analysis were carried to such a point of refinement, the number of problems that could be distinguished would be too large to admit of useful treatment. In the second place, the problems could, by further telescoping, be still further reduced in number. Thus, for example, all the problems relating to the curriculum could be bulked together.

Such a procedure would, however, probably tend to lose important subdivisions of the whole topic. This matter has been handled by classifying together in the tables all related topics and investigations. In the third place, the problems overlap considerably. Thus Problem 27, "Methods of teaching," overlaps with Problem 17, "Effects of varying lengths and distributions of drill periods," with Problem 26, "Instructional difficulties reported by teachers," and so on. In the case of such overlapping a given investigation was cited under as many different headings as it contained problems which were actually investigated.

3. The Techniques

In this study nineteen research techniques are recognized. As in the case of the problems, it was necessary to stop somewhere in recognizing differences in techniques. Each of the nineteen techniques here described represents a type of technique or a group of techniques rather than a specific research technique. Thus, for example, a number of techniques might be substituted for the "Laboratory technique" in terms of the specific forms of apparatus utilized, etc. Likewise, the technique "Analysis of texts" is found in a variety of forms—in measurements in terms of square inches of space, of number of pages, of frequency of occurrence of a given item, of frequency of occurrence of types of items, etc.

IV. Classified List of Research Techniques Employed in Arithmetic Investigations

The nineteen research techniques, in the meaning of that term as set forth in the preceding paragraph, are next classified and described under six categories. The techniques are numbered consecutively from I to XIX without regard to the special classifications into which they are divided. Enough of the characteristics of the different techniques have been given to enable the reader to identify their use in any given investigation.

A. Techniques Requiring Immediate Contacts with Individual Subjects

Under this heading there are four techniques; the difference between them lies in the degree to which the observation of behavior is intimate, complete, and controlled.

(I) The Personal Interview Technique is essentially a form of oral testing. It differs from oral testing chiefly in its flexibility and in its

thoroughness. Example: G. Stanley Hall's study (98)[3] in which he secured information regarding the knowledge of certain of the smaller numbers possessed by children upon entering school.

(II) The Case Technique surpasses the technique of the personal interview in the searching nature of the questions, in the extent of the questioning (concerning matters related to the one specifically under study), and especially in the opportunity afforded to observe accompanying behavior which may be of significance. The aim here is to determine, not merely the *what* (with the answer to this question the personal interview stops), but also the *how* and the *why*. Example: Buswell's investigation (607) of children's processes in dealing with the fundamental operations.

(III) The Laboratory Technique is the most refined of the four techniques here classified together, since it permits the most complete control of the experimental situation and since it yields the most objective of measurements. Example: Heilman and Shultis's effort (110) to discover the relative difficulty of the number combinations by a study of reaction-time.

(IV) The Biographical Technique is essentially the case method prolonged over a considerable period of time. Example: Court's reports (51 and 52) of the developing number consciousness of one child up to age eight.

B. Techniques Equally as Well, or Better, Adapted to Contacts with Groups or Subjects

As indicated by the heading, the five techniques which are grouped together here differ from the first group of four techniques in that they do not require individual contacts.

(V) The Observation Technique. Here the experimenter simply watches the subjects who are at work and without questioning them secures data concerning their processes by noting significant behavior. Example: Steinway's study (245) of children's interest in number games.

(VI) The Consensus Technique. According to this technique subjects are requested to express their opinions regarding some phase or other of the problem under investigation. Example: Lewis's study (148) of children's attitudes toward arithmetic.

(VII) The Testing Technique. This technique is sometimes called "experimental," with the result that the latter term has almost ceased to have any definite meaning. The term "testing technique" is here used to describe the procedure of the investigator who, without setting up any special scheme of training, merely measures status as he finds it. "Testing," therefore, includes even those studies in which groups or tests or both are rotated. Example: Ashbaugh's survey (4) of the arithmetic ability of Iowa children.

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[3] The numbers throughout this section are those assigned in the Buswell and Judd bibliographies to the investigations cited.

(VIII) The Special Training Technique. Here the investigator sets up some special form or course of training in order to study its effects. Thus, this technique includes testing as a subtechnique, but goes beyond it to include the establishment and prosecution of some special training. "Special training technique" is not used, however, to cover investigations in which paired groups are used to provide comparative data. Example: Holloway's attempt (112) to discover the relative difficulty of the number combinations by teaching the combinations to children who did not know them.

(IX) Controlled-Group Experimentation. This technique includes both testing and special training, but adds something more. Here the research worker sets up an experimental situation for the purpose of evaluating one type of procedure by comparing its results with the results of other procedures. He is, therefore, especially careful to have paired groups, parallel sections, check classes, etc., and to control or at least measure all variables in addition to the one which is the chief object of study. Example: Kelly's evaluation (136) of the Courtis, Studebaker, and ordinary methods of administering drill.

C. Techniques Based on the Results of Subjects' Work

(X) Correlation and Refined Statistical Technique. Statistics must, of course, be utilized in treating the data secured from all, or nearly all, the techniques. The technique here referred to implies that data secured through testing have received more elaborate statistical treatment than is to be found in the usual study. Example: Cobb's investigation (42) of the inheritance of arithmetical ability.

(XI) Analysis of Pupils' Written Work. The typical instance of the application of this technique is the study of pupils' arithmetic errors by an examination of their papers. Example: Counts (50).

D. Techniques Based on a Study of the Material to be Learned

(XII) Logical Analysis. According to this technique the researcher makes an analysis of the material which is to be learned by scrutinizing and classifying the mathematical requirements of the material. Thus, Kallom (132) early reported the various types of examples to be found in the addition of fractions.

(XIII) Analysis of Texts, Tests, etc. Criteria are set up and instructional materials are examined to determine how well they satisfy these criteria; or a list of items to be taught is prepared, and texts are analyzed for the frequency of appearance of these items, etc. Example: Knights' comparison (526) of five sets of drill material.

E. Techniques Based on Contacts Other Than Those with Subjects Engaged in Arithmetic Work

(XIV) The School Pupil Survey. This is the technique which has been developed especially by Wilson (283), in his case for the purpose of securing

information regarding adult uses of arithmetic from parents through the coöperation of their children.

The *Consensus Technique* might well be mentioned here again because of its frequent use. It is not, however, given a separate number, for the reason that its re-appearance in this connection is due, not to a difference in the technique, but rather to a difference in the type of person addressed.

F. Techniques Based on Administrative Materials

(XV) Analysis of Courses of Study and School Programs. This technique is essentially self-explanatory. Example: Glass's study (86) of curriculum practices through the analysis of courses of study.

(XVI) Analysis of School Records and Marks. Again no explanation is necessary. Example: Judd's study (128) of promotion in arithmetic.

G. Techniques Resisting Classification

Three techniques are relatively independent of each other and of the sixteen which have been heretofore presented.

(XVII) The Comparative Technique. Only one use has been reported of this method—that by Brown (22), who compared mathematics in the United States with mathematics in Europe.

(XVIII) The Analysis of Non-Instructional Material. This technique is the same as Number XIII, analysis of texts, etc., save for the materials used, and frequently the purpose of the study. Example: Scarf's investigation (638) of the mathematics necessary for reading periodicals.

(XIX) The Historical and Anthropological Technique. This technique is self-explanatory. It is the only non-quantitative research procedure which is included in the list of 19 techniques in this study. Example: Stone's historical account (642) of changes in methods of subtraction.

V. Classification of Investigations in Arithmetic

In this section of the chapter there are presented in several tables the results of our effort to classify arithmetic investigations which has been described in the foregoing paragraphs. In Table III the bibliographical references to 390 research studies, historical or quantitative in nature, are listed by reference number[4] under headings which set forth the problems attacked and the research techniques employed. The references to the problems (1-67) are listed along the left margin and the techniques (I-XIX) along the top. Two keys are provided that explain this numbering. Thus, reading from left to right, one discovers in Table III that Problem 1, "Adult uses

[4] Reference numbers in the 500's refer to Buswell's 1925 supplement; in the 600's, to his 1926 supplement; in the 700's, to his 1927 supplement; and in the 800's, to his 1928 supplement.

of arithmetic,'' has not been investigated by Techniques I, II, III, IV, or V; but has been investigated by Technique VI, ''Consensus Technique,'' in References 39, 286, and 287.

In the remaining tables the specific bibliographical citations have not been preserved. There is shown merely the number of investigations made on a given problem by a given technique.

Table IV contains a classification of all the investigations of problems relating to the arithmetic curriculum, grouped according to specific problems studied and techniques employed. Thus, the problem, ''Adult uses of arithmetic,'' information concerning which should supply valuable data regarding the curriculum, has been studied three times by Technique VI, three times by Technique XIV, and five times by Technique XVIII.

Table V presents the corresponding data regarding methods of teaching arithmetic. Certain problems which appeared in Table IV as affecting the arithmetic curriculum are repeated here again (for example, Problem 9, ''Correlated mathematics'') because of the obvious importance of such studies in determining methods of teaching.

Table VI is devoted to studies relating to pupils' reactions; Table VII, to studies relating to the measurement of the results of teaching; Table VIII, to studies relating to the organization and grade placement of arithmetic materials; Table IX, to studies relating to the construction of instructional materials; and Table X, to four miscellaneous studies which can not be classified under these headings.

Key to Problems

1. Adult uses of arithmetic
2. Arithmetic in foreign countries
3. Arithmetic objectives
4. Attitude of non-school people toward arithmetic
5. Children's attitudes toward arithmetic (and toward particular numbers)
6. Children's uses and needs of arithmetic
7. Comparative difficulty of the number combinations
8. Comparative values of types of drill
9. Correlated mathematics
10. Counting
11. Critical evaluation of instructional materials
12. Development of arithmetic as a school subject
13. Disability in arithmetic
14. Effects of drill on individual differences (initial ability, etc.)
15. Effect of size of class and of school

16. Effects of vacations on arithmetic ability
17. Effects of varying lengths and distributions of drill periods
18. Efficiency of non-elementary pupils and adults in arithmetic
19. Eliminations from the arithmetic curriculum
20. Effects of fatigue
21. Grade placement in arithmetic
22. Home vs. school work in arithmetic
23. Hygiene of arithmetic
24. Individual differences in arithmetic ability
25. Inheritance of arithmetic ability
26. Instructional difficulties reported by teachers
27. Methods of teaching arithmetic
28. Motivation (including utilization of personal experiences, testing, etc.)
29. Nature and development of number consciousness
30. Number games
31. Number prodigies
32. Oral vs. written work in arithmetic
33. Origin and development of number anthropologically
34. Overlapping as between grades
35. Periodicity in arithmetic
36. Permanence of the effects of drill
37. Persistence of errors
38. Place of imagination in arithmetic
39. Preparation of arithmetic teachers
40. Promotion and retardation in arithmetic
41. Pupils' errors and difficulties in arithmetic
42. Pupils' processes in arithmetic
43. Reaction time for various arithmetical operations
44. Relation of amount of drill to effects produced
45. Relation of arithmetic ability to age
46. Relation of arithmetic ability to amount of schooling
47. Relation of arithmetic ability to general ability (intelligence)
48. Relation of arithmetic ability to language ability (reading, vocabulary, etc.)
49. Relation of arithmetic ability to other school abilities (reading excluded)
50. Relation of arithmetic ability to social status
51. Relation between neatness and accuracy in arithmetic
52. Relation between speed and accuracy in arithmetic
53. Relation between various arithmetic abilities
54. Reliability and validity of arithmetic test scores
55. Remedial instruction in arithmetic

56. Rural vs. urban schools
57. Schedule hour for arithmetic
58. Sex differences in arithmetic
59. Space allotment in arithmetic
60. Specific items to be taught in arithmetic
61. Standards in arithmetic
62. Steps in difficulty in arithmetic processes
63. Time allotment in arithmetic
64. Time for beginning systematic instruction in arithmetic
65. Transfer in arithmetic
66. Value of incidental instruction in arithmetic
67. Value of systematic instruction in arithmetic

KEY TO TECHNIQUES

I Personal Interview Technique
II Case Technique
III Laboratory Technique
IV Biographical Technique
V Observation Technique
VI Consensus Technique
VII Testing Technique
VIII Special Training Technique
IX Controlled-Group Experimentation
X Correlation and Refined Statistical Technique
XI Analysis of Pupils' Written Work
XII Logical Analysis
XIII Analysis of Texts, Tests, etc.
XIV School Pupil Survey
XV Analysis of Courses of Study and School Programs
XVI Analysis of School Records and Marks
XVII Comparative Technique
XVIII Analysis of Non-Instructional Material
XIX Historical and Anthropological Technique

TABLE III.—THREE HUNDRED NINETY REFERENCES TO INVESTIGATIONS IN ARITHMETIC, CLASSIFIED IN TERMS OF THE SIXTY-SEVEN PROBLEMS INVESTIGATED AND THE NINETEEN TECHNIQUES EMPLOYED

Prob.	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX
1						39 286 287								283 285 296				159 173 303 309 638	
2																	22		
3						546							546		617 819				510
4						35 36 186 246 247 283													
5	221				245	148													
6	98 193 238					33 39									819			635	
7		26 237 827	110 237 501 827			501	112	112 716 827		827 845 855	31 40 41 112 206 501 845 855								
8									14 136 139 162 163 213 236 241 264 519 812 834 835 838										
9						191 546							546						12 503
10	75 100 209 805		3 28 77 127 154 169 273 724	51 52 89	805	149 209 704	116 515 805												

TABLE III.—(Continued)

Prob.	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX
11						269	829		136 162 163 644	41 723	41 112		141 142 179 180 198 199 239 269 318 320 526 556 631 713 715 726 731 735 806 816 856						
12													82 91 118 133 134 174 178 227 504 531 831						12 20 66 82 91 118 133 134 135 174 178 227 229 230 231 232 233 503 504 531 548 717 737 744 831
13		19 220																	
14								67 96 213 259 260 263 266 276 807											
15							72 108 307 634												
16							21 81 104 112 138 215 223 277 294 541 705 809 839	839											
17									18 96 138 216 289 719 727										
18							6 15 39 55 70 101 145 165 171 188 224 237 263 275 523 647 749 812												

TABLE III.—(Continued)

Prob.	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX
19						36 39 120 121 122 124 186 247 285 286	39							283 285 296	120 124				
20							113	202	18 105 106 293										
21						124 191									86 92 124 819 853				
22						634			107										
23		30				30													
24		550					18 53 55 72 181 206 236 515 532 550 705												
25		19								42									
26						225 250													
27	833	8 115		69		184 225 507 527 634	8 13 43 70 151 165 255 625 740 814 850	115 215 264 282 320 529 543 602 615 648 716 718 820 833 843	14 49 96 136 138 139 144 164 193 216 236 241 245 289 294 295 511 519 610 630 648 707 708 709 719 727 730 750 812 834 835 854						617				642
28						196 288 649	10 81 223 278 840	215 537 718 739	73 114 245						617				
29	75 98 100 209 805	82 805	3 28 77 127 151 169 273 525 724	51 52 69 203	805	209	87 116 515 805												46 47 48 64 66 88 126 157

TABLE III.—(Continued)

Prob.	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX
30					245	196 245 649			114 245						617				229
31		172		172 203															
32							235			281									
33																			46 47 48 64 66 88 157
34							2 53 54 55 95 244 634												
35										71									
36							21 138 705 727 741 752			277									
37	833	190 827									190								
38		115 147 279 280				34 35 79 115	97 535 736		240	204 208 209									
39						225									851				
40							55 265									55 128			
41		19 168 177 190 207 235 236 237 256 272 302 320 550 607	237 256 607				177 550 821 854	112 716			2 31 40 41 45 50 85 111 119 131 140 152 168 171 189 190 194 195 206 221 222 235 236 243 258 303 517 528 533 534 543 550 604 607 612 621 623 702 725 742 749 751 802 807 808 812								

TABLE III.—(Continued)

Prob.	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX
42	75 100 209 805	3 8 19 26 80 110 168 172 177 207 235 236 237 256 273 295 302 320 550 607 623 724 805	3 28 77 110 127 154 169 213 237 256 293 515 607 623 722 724		805	149 209 507	8 13 43 53 116 151 176 177 255 502 511 550 740 821	716	49 164 708 709	59	2 31 40 41 45 50 85 111 112 119 131 140 168 171 189 190 194 195 206 221 232 235 236 242 258 302 517 528 533 534 543 550 604 612 621 623 702 725 742 749 751 802 805 807 808 812 828	526 556 604 611 628 715 726 815	631 724						
43			110 129 237 273 501 607 623 724 827				112												
44							61 217 218		7 254	41			41						
45							8 9 17 50 53	266 705	.	805									
46							15 44 50 62 145 265 307 805			6 62 102 220 805						102 220			
47		19 87					74 87 166 167 240 320 511 512 558 752 809	320		27 44 93 530 533 640 805									
48		256 257 320	256 257			39	39 63 550 736 821	282 529 615	192 730	93 102 533 550			39 320 603 615 724 816					635 638	
49							166 167			108 641 804									
50							294												
51							297												

TABLE III.—(Continued)

Prob.	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX
52			129				43 50 60 94 95 206 235 263 289 610 805		630	16 44 57 59 530	206 235								
53						35	53 60 76 116 188 235			15 17 24 44 62 90 187 221 248 290 533 805 827	235								
54			129				37 38 60 90 94 95 181 212 222 294 512 557			56 57 59 117 201 818									
55		207 236 843						31 207 235 543 558 621 716 741 812 840 843	168 236 838		235 704								
56						225	108 253 299 307 634												
57									105 106 293						86 94				
58						79	6 8 17 72 76 89 113 187 211 212 234 244 265 276 294 515 705												
59													29 239						
60						33 36 103 120 121 122 124 186 246 286 287					31 112 159 173 303 309 635 638	197 526 611 639 806 807	29 141 142 158 179 239 526	285 286 296	29 86 120 121 122 124 156 246 617				

TABLE III.—(Continued)

Prob.	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX
61							8 9 50 53 54 55 60 89 90 94 95 183 244 248 300 302 517 637 802 824 844												
62							511					116 132 526 556 604 611 628 639 715 726							
63						122 634		644							29 86 94 218 536 853				
64	98 238	30		51 52 69		30 103 123	94 95 853	833	7 254						156				
65								168	144 214 243 290 291										
66				51 52 69			165												
67							10 290 291	31 61 67 90 95 137 259 260 264 266 276 277 278 529 537 558 605 702 739 747 748 752 812 813 820	21 49 73 96 114 138 162 163 192 211 213 236 241 245 289 511 519 538 610 737 838										

TABLE IV.—CURRICULUM STUDIES CLASSIFIED ACCORDING TO TECHNIQUE OF RESEARCH

Specific Problems Studied (See key)	Techniques of Research (See key) I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX
1						3								3				5	
2																	1		
3						1							1		2				1
4						6													
5	1				1	1													
6	3					2									1			1	
9						2							1						2
19						10	1							3	2				
30					1	3			2						1				1
60						11					8	6	7	3	9				
61							21												
62							1					10							

TABLE V.—STUDIES OF METHODS OF TEACHING ARITHMETIC, CLASSIFIED ACCORDING TO TECHNIQUE OF RESEARCH

Specific Problems Studied (See key)	Techniques of Research (See key)																		
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX
8									14										
9*						2							1						2
15							4												
17									7										
22						1			1										
26						2													
27	1	2		1		5	11	15	32						1				1
28						3	5	4	3						1				
30*					1	3			2						1				1
32							1			1									
38		4				4	3		1	3									
44							3		2	1			1						
55		3						11	3		2								
65								1	5										
66				3			1												
67							3	25	21										

*See Table IV.

TABLE VI.—STUDIES OF PUPILS' REACTIONS TO ARITHMETIC MATERIALS, CLASSIFIED ACCORDING TO TECHNIQUE OF RESEARCH

Specific Problems Studied (See key)	Techniques of Research (See key)																		
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX
5*	1				1	1													
7		3	4			1	1	3		3	8								
10	4		8	3	1	3	3												
13		2																	
14								9											
20							1	1	4										
23		1			1														
24		1					11												
28†						3	5	4	3						1				
29	5	2	9	4	1	1	4												8
31		1		2															
35										1									
37	1	2									1								
38†		4				4	3		1	3									
41		14	3				4	2			46								

*See Table IV. †See Table V.

TABLE VI.—(Continued)

Specific Problems Studied (See key)	Techniques of Research (See key)																		
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX
42	4	23	16		1	3	14	1	4	1	47	8	2						
43			9				1												
45							5	2		1									
46							8			5						2			
47		2					11	1		7									
48		3	2			1	5	3	2	4			6					2	
49							2			3									
50							1												
51							1												
52			1				11		1	5	2								
53						1	6			13	1								
58	1					1	17												
62*							1					10							
65†								1	5										

*See Table IV. †See Table V.

TABLE VII.—MEASUREMENT STUDIES OF THE RESULTS OF TEACHING, CLASSIFIED ACCORDING TO TECHNIQUE OF RESEARCH

Specific Problems Studied (See key)	Techniques of Research (See key)																		
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX
8†									14										
13‡		2																	
14‡								9											
15†							4												
16							13	1											
17†									7										
18							18												
24‡		1					11												
34							7												
36							6			1									
40							2									2			
46‡							8			5						2			
47‡		2					11	1		7									
48‡		3	2			1	5	3	2	4			6					2	

†See Table V. ‡See Table VI.

TABLE VII.—(Continued)

Specific Problems Studied (See key)	Techniques of Research (See key)																		
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX
49‡							2			3									
50‡							1												
51‡							1												
52‡			1				11		1	5	2								
53‡						1	6			13	1								
54			1				12			6									
61*							21												

*See Table IV. ‡See Table VI.

TABLE VIII.—STUDIES OF THE ORGANIZATION AND GRADE PLACEMENT OF ARITHMETIC MATERIALS, CLASSIFIED ACCORDING TO TECHNIQUE OF RESEARCH

Specific Problems Studied (See key)	Techniques of Research (See key)																		
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX
3*						1							1		2				1
6*	3					2									1			1	
7‡		3	4			1	1	3		3	8								
9*						2							1						2
10‡	4		8	3	1	3	3												
11						1	1		4	2	2		21						
21						2									5				
29‡	5	2	9	4	1	1	4												8
30*					1	3			2						1				1
34§							7												
45‡							5	2		1									
46‡							8			5						2			
48‡		3	2			1	5	3	2	4			6					2	
57									3						2				
59													2						
61*							21												
62*							1					10							
63						2		1							6				
64	2	1		3		3	3	1	2						1				

*See Table IV. †See Table V. ‡See Table VI. §See Table VII.

TABLE IX.—STUDIES OF THE CONSTRUCTION OF INSTRUCTIONAL MATERIALS, CLASSIFIED ACCORDING TO TECHNIQUE OF RESEARCH

Specific Problems Studied (See key)	Techniques of Research (See key)																		
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX
8†									14										
11‖						1	1		4	2	2		21						
14‡								9											
22†						1			1										
23‡		1				1													
26†						2													
27†	1	2		1		5	11	15	32						1				1
28†						3	5	4	3						1				
30*					1	3			2						1				1
32†							1			1									
36§							6			1									
38†		4				4	3		1	3									
48‡		3	2			1	5	3	2	4			6					2	
52‡			1				11		1	5	2								
55†		3						11	3		2								
61*							21												
62*							1					10							
63‖						2		1							6				
65†								1	5										
67†							3	25	21										

*See Table IV. †See Table V. ‡See Table VI. §See Table VII. ‖See Table VIII.

TABLE X.—MISCELLANEOUS ARITHMETIC STUDIES CLASSIFIED ACCORDING TO TECHNIQUE OF RESEARCH

Specific Problems Studied (See key)	Techniques of Research (See key)																		
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX
12													11						25
25		1								1									
33																			7
39						1									1				
56						1	5												

VI. Comments on the Classifications

The classifications presented in Tables III-X will repay careful study. Limitations of space prevent any degree of elaborate comment in this place; it is possible here to do no more than to list some of the items of greatest significance.

1. The tables provide classified bibliographies with respect to 67 problems of large importance in arithmetic. It is interesting to compare the number of different investigations which have been made of the various problems. If this is done, it becomes at once apparent that the relative frequencies are not a good criterion of the comparative importance of the problems in the field of arithmetic as a whole. Rather, certain ones of the problems seem to have become favorite ones for research, while others have been avoided.

2. The tables also show the techniques which have been used in investigating the various ones of the 67 problems. As in the case of the problems, an examination of the comparative frequencies of use of the techniques shows that certain ones have been perhaps over-used proportionately, while others have been under-used. Thus, according to Table III the Testing Technique has been employed 125 times. This technique is one of the easiest to use. On the other hand, the Laboratory Technique has been used but 22 times. It is perhaps the most difficult to employ. While it is true that the types of problems which can be investigated by the two techniques are not identical, there is still some reason to believe that research workers in arithmetic are loathe to utilize the more difficult technique where it might be used to advantage.

3. The blank spaces in these tables are fully as significant as are the spaces in which numbers have been entered. These blanks indicate that for some reason a specific problem has not been investigated by means of a given technique. Those interested in furthering research in arithmetic may well scrutinize these blank spaces, for each one raises the question: Can this problem be investigated by this method? If so, will the results be apt to add needed information to our understanding of learning and teaching in the field of arithmetic?

4. On the other hand, the mere fact that a specific problem has been studied a number of times by a given technique is not to be interpreted that all that should be done has been done. One of the

greatest needs in arithmetic is the checking of investigations by repetition.

In Chapter III the reader will find a discussion of some of the more fundamental problems relating to research in arithmetic. His appreciation of the points which are there made will be increased by the general survey of research in arithmetic which has been provided in this chapter. He will find it of value to return occasionally to the classifications of investigations in Tables IV-X, the contents of which have been arranged with due regard to the organization of the next chapter.

GUIDE TO CHAPTER III

CHAPTER III

A CRITICAL SURVEY OF PREVIOUS RESEARCH IN ARITHMETIC

G. T. Buswell

No attempt will be made in this chapter to present a detailed summary of the research studies in arithmetic which are now in print. Such detailed summaries have already been published and may be procured easily. The chapter will deal with the nature and findings of investigations in arithmetic without attempting to mention all the persons who have made contributions.

The studies with which this chapter is concerned may be defined as those which are either critical or quantitative in character. No attempt will be made to deal with reports which are based upon opinion only or with reports which are merely descriptive in character. The purpose is to summarize the facts concerning arithmetic which are based upon adequate objective data, to note the trends which characterize research in the various phases of arithmetic, and to present what might be called a list of accepted generalizations relating to arithmetic.

I. The Scientific Literature of Arithmetic

If a somewhat liberal criterion is applied and all the studies relating to arithmetic which are non-critical or non-quantitative in character are eliminated, there will remain, up to January 1st, 1929, some 518 references. The trend of interest in this subject may be revealed by a tabulation of the number of critical and quantitative studies which appeared during successive quadrenniums beginning with 1890. From Table I, which indicates the numbers of studies published during each successive four years, it appears that interest in arithmetic showed a very slow development up to about 1913, that from 1913 to 1916 there was a marked increase in the number of investigations reported, that contributions lagged somewhat from 1917 to 1925, but that during the last quadrennium there has been a con-

spicuous development of scientific interest in this field.[1] From the trend exhibited in Table I, it is evident that arithmetic is entering into a stage of critical investigation such as has characterized the subject of reading during the past twenty years. It is highly important that this large flow of energy be directed toward crucial problems rather than be dissipated among trivial studies.

TABLE I.—NUMBER AND SOURCE OF PUBLICATION OF QUANTITATIVE AND CRITICAL STUDIES RELATING TO ARITHMETIC

Date	Books	Periodicals	Bulletins	Total
Prior to 1893	2	4	0	6
1893-96	1	8	0	9
1897-1900	2	13	0	15
1901-4	2	7	1	10
1905-8	3	13	0	16
1909-12	5	23	1	29
1913-16	9	54	20	83
1917-20	9	49	15	73
1921-24	12	42	12	66
1925-28	23	129	59	211
Total	68	342	108	518

The material to be presented in this chapter will be organized on the basis of five major subdivisions of the field of arithmetic, which are as follows: (a) The Curriculum, (b) Methods of Teaching, (c) Pupils' Reactions: Analysis of Errors and Methods of Work, (d) Measurement of Results, and (e) Construction of Teaching Materials.

II. RESEARCH RELATING TO THE CURRICULUM

1. Historical Background

Recent and present interest in arithmetic can be best understood in the light of the historical development of the subject. Without such an historical background much of the research relating to the curriculum would appear to be meaningless.

In the early Colonial period arithmetic received little attention in the schools; frequently it was omitted altogether. Its appearance in the curriculum seemed to be related to the practical demands for

[1] An interesting comparison may be made with the number of investigations relating to the subject of reading. Prior to 1890 there were two published scientific studies in the field of reading. From 1891 to 1900 there were 12 studies, from 1901 to 1910 there were 20, from 1911 to 1920 there were 200 studies, while during the period from 1921 to 1928 there were 545 studies reported.

the subject in different communities. However, by 1800 arithmetic appeared to be a common subject in the schools. It consisted chiefly of instruction by rules, which were memorized and applied without any attempt at rationalization. The work was individual and was written. The formality of this instruction by rules was broken by the introduction of Pestalozzian influence through the textbook prepared by Warren Colburn in 1821. Colburn instituted a radical reform in the subject of arithmetic, the main elements of which consisted of (a) the abandoning of the teaching of rules, (b) the introduction of drill methods to secure better mastery of the number combinations, (c) the introduction of practical problems, (d) oral instruction, and (e) the use of some objective materials. Colburn emphasized both the practical and disciplinary values of arithmetic, but he considered the practical values as the primary ones. This emphasis upon practical values diminished gradually until by 1860, under the influence of Ray, the disciplinary values were made the chief objectives of arithmetic. During the period from 1821 to 1900, arithmetic expanded too greatly; in some schools as much as one-half of the total school time was devoted to this subject. Toward the end of the nineteenth century, Colburn's influence diminished and textbooks again tended to be formal in character.

About 1880 the first public criticism of the enormous amount of time devoted to arithmetic appeared. In 1893 and 1895, the Committee of Ten and the Committee of Fifteen urged reform in this subject and pointed out particularly the desirability of covering the entire field of arithmetic in a period of five years, beginning with the second grade and ending with the sixth grade. The tenor of the report of these committees pointed to a practical emphasis upon the subject rather than a disciplinary emphasis.

The developments in arithmetic up to 1900 were based chiefly upon empirical observations and upon *a priori* considerations. Few objective data relating to the problem were available prior to 1900. In 1902 Rice published his articles based upon the objective testing of the outcomes of arithmetic, while in 1908 and 1909 Stone and Courtis began their testing. These tests revealed wide differences in the achievement of pupils, and these differences were related to the course of study. The earlier criticisms of overexpansion of arithmetic began to bear fruit; numerous studies appeared which dealt with the problem of elimination of subject matter. These attempts at elimination

constituted the first direct result of scientific research applied to arithmetic.

It soon became apparent that simply eliminating subject matter might correct somewhat the overexpansion of arithmetic, but that it would not supply a fundamental principle on which a course of study could be based. In the search for such a fundamental principle which has dominated the research in this field during the past fifteen years, two main types of investigation have been carried on. One of these has consisted of an analysis of social demands, on the theory that only such arithmetic should be taught as met some actual need. The second type has consisted of an investigation of the psychological considerations relating to the mastery of the number system and its application.

The present situation may be summarized somewhat as follows: Arithmetic experienced a great period of overexpansion extending from the time of Warren Colburn's influence down to 1900. The first problem to receive the attention of the scientific student of education was the determination of some means of reducing this overexpansion and the elimination of useless subject matter. This led at once to studies of a more positive character to discover some fundamental principle on which the construction of the curriculum might be based. This search for a fundamental principle has stimulated two main types of research: first, investigations based upon the principle of social utility or social demand, and second, investigations based upon a psychological consideration of the nature of the number system and the difficulties which children encounter in mastering it.

2. Studies of Elimination

Among the earliest research studies relating to the curriculum in arithmetic were comparisons of courses of study and of practices in different cities. The objective facts revealed by these studies furnished the first tangible basis for a reduction of amounts of time devoted to arithmetic and of the number of topics treated. Early studies by Stitt and by Jessup, and later ones by Glass, showed a marked variation in practices among different cities. Jessup also compiled the judgments of superintendents as to subjects which should be eliminated from the schools. Further support of this move toward elimination was given in books on the teaching of arithmetic by Brown and Coffman, Lennes, and D. E. Smith. One of the earliest examples

of an objective application of this policy of elimination to a school's course of study is furnished by G. M. Wilson in the course of study for the Connersville, Indiana, school system.[2]

The principal basis upon which Wilson recommended elimination of some topics and emphasis upon others was business usage. It can readily be seen that it is only a short step from deciding to eliminate certain topics because business needs did not employ them to the inclusion of other topics because business practices did employ them. By a simple expansion of the concept 'business usage' to cover social usages of a wider variety, one arrives at a theory of social utility as a positive basis for curriculum construction.

3. Studies Based upon the Theory of Social Utility

In 1913 Courtis published the results of an investigation in New York City which showed the differences in standards of achievement found in commercial activities in New York and in the public schools of New York. Mitchell made analyses of advertisements, pay rolls, and cookbooks to determine the kind of arithmetic needed. Wise, from a study of the problems which adults are called upon to solve, reported that in 85 percent of these one or more of the four fundamental operations were involved. Woody made an extensive analysis of sales slips to determine the kind of arithmetic needed in that connection. Camerer made a questionnaire study of the arithmetic involved in banking operations. Other investigators analyzed books, magazines, and newspapers to determine both the problems which needed to be solved and the arithmetical information needed by persons reading those materials. Among the most extensive investigations following the theory of social utility are those reported by Wilson. Wilson supplements his findings by detailed proposals as to their use in applying the theory of social utility to curriculum construction. Recently a number of critical articles have appeared which take exception to the entire theory of social utility as a basis for curriculum-making. Knight, Myers, Harrop, Judd, and others have expressed decided opposition to the unlimited use of this theory.

The application of this theory of social utility to a school program is illustrated concretely in a monograph written by Wilson in 1926. Wilson describes the application of the theory of social utility

[2] See G. M. Wilson, *Connersville Course of Study in Mathematics*. Baltimore: Warwick & York, Inc., 1922.

to the construction of a course of study in arithmetic in Marshalltown, Iowa. He quotes, with apparent approval, the conclusions of the committee in charge of this investigation. These conclusions are as follows:

1. The problems of arithmetic used by the public are simple. The necessary tool material could be taught in four years. We suggest grades 3, 4, 5, and 6. During these grades and after, motivated problems should lead the children out into actual business situations.

2. Much useless material is now in the curriculum in arithmetic. It should be eliminated. Only such material as will be useful in everyday life should be included.

3. The problems in arithmetic should center around business situations because life's problems in arithmetic do.

4. The community's arithmetic is sufficient to form the basis of the general arithmetic work in the elementary schools.[3]

The objections to this theory of social utility seem to be due chiefly to the proposal that the theory should constitute the sole basis of curriculum construction. There is no objection to gathering objective facts concerning social usage in the field of arithmetic. Furthermore, there is little objection to the use of these facts as a partial guide to the selection and distribution of problems. The objection is that this theory does not constitute a sufficiently broad basis to guarantee a suitable course of study. Specifically, it is urged that a tabulation of social usage will throw no light whatever on the kinds of arithmetic which *might* be useful were people sufficiently trained to use them with ease. Again, it is pointed out that the solving of problems such as are encountered in one's daily experience constitutes only one aspect of the total contribution of arithmetic and that it neglects entirely the development of abilities which may be highly valuable in quantitative thinking. Still again, when the course of study in arithmetic is based upon the arithmetic needed in a particular community, as was suggested in the last paragraph of the quotation given, the course of study will likely be extremely meager; and if this theory is followed for successive generations of pupils, the teaching of arithmetic may be reduced to a vanishing point. There is marked evidence of a general dissatisfaction with this method of curriculum construction, which is apparent in other subjects as well as in that of arithmetic. The limitations of the applications of

[3] Guy Mitchell Wilson, *What Arithmetic Shall We Teach?* pp. 58-59. Boston: Houghton Mifflin Co., 1926.

this theory are so marked that its uncritical acceptance is no longer justified.

4. Psychological Studies

Among the early contributors to the psychological studies of arithmetic were G. Stanley Hall and his students at Clark University. They made extensive studies of children's number experiences and made a number of proposals which have direct bearing upon the curriculum. Among these proposals there is expression of the view that the number series is the primary fact in arithmetic and should be particularly emphasized in the earlier grades; that the number combinations should be taught before practical applications are given; that drills should begin with the small numbers and proceed gradually to the higher numbers; and that motivating devices, such as games and puzzles, should be introduced into the course of study. Some of these conclusions obviously lacked adequate objective support. Several psychological studies have emphasized the difficulty of the arithmetical operations, showing that arithmetic causes more failure than any other subject in the elementary school, that it is particularly difficult for children of low intelligence, and that the child's interests need to be supplemented by various forms of motivation.

Various psychological treatments relating to the content of the curriculum in arithmetic have been influential. In one of them, by McLellan and Dewey, emphasis is given to the statement that early number experiences should begin with measurement. In another, by D. E. Phillips, the position is taken that measurement should not be considered as the essence in the development of numbers, but that the series idea and counting should receive greater emphasis. Phillips presents a very considerable body of new data to support his view. In still another treatment, by Thorndike, arithmetic is presented as a subject in which there must be set up in the minds of the pupils a large number of associative bonds to be made permanent by the proper administration of drill. Thorndike has supplemented his psychological treatment by a treatment of method based upon psychological study in which emphasis is given to inductive methods and rationalization of number ideas.

Another type of psychological study, which has appeared more recently, consists of an attempt to analyze the subject of arithmetic

into certain unit skills which should be presented to children in a predetermined order. Such detailed analyses were exhibited in the revision of the curriculum in arithmetic in the Denver Public Schools and have also been exhibited in the work of Knight, Osburn, and other writers in this field. The aim of these persons is to apply the accepted principles of psychology to the detailed organization of materials to be included in the course of study in arithmetic.

Following a survey of research dealing with the course of study in arithmetic, certain observations seemed warranted: first, a suitable course of study in arithmetic cannot be built through the process of elimination alone; second, some positive principle of curriculum construction is needed, although at the present time no single principle has met with universal acceptance; third, the application of the theory of social utility has apparently encountered certain insurmountable obstacles which prevent one from following this theory to its logical conclusion; and fourth, increasing emphasis is again being laid on certain psychological considerations which underlie curriculum construction.

The schools have just witnessed a period of intense activity in the field of curriculum research. The most outstanding characteristic of this research has been an emphasis upon activity analyses. These analyses may be made objectively and their results may be expressed in quantitative terms. Frequency tables can be constructed showing in detail the number of specific activities related to each subject in the curriculum. However, at this point the objectivity of this technique of curriculum construction ceases. In the attempts to interpret the meaning of frequency of the various activities and in the attempts to translate the objective data into actual courses of study, one finds either an unintelligent attempt to apply the data in purely mechanical fashion or a reliance upon personal or group opinion which counteracts all the objectivity of the method of collecting activities. For example, the analysis of the activities involving fractions shows that the fraction ½ is used far more commonly than is any other particular fraction. The first inference from this finding is that the fraction ½ is a very important one to teach. In building a curriculum, does this mean that much more time must be given to the fraction ½ than to ⅓ or ¼ or ⅕? An investigation of the mental concepts of children indicates that the notion of one-half is very well developed even before instruction in fractions begins. Consequently,

instead of requiring more time in a course of study, the teaching of the fraction $1/2$ requires less time than do other fractions. How then shall one interpret the meaning of frequency? In another activity analysis data are reported showing that "transactions in stores" occurred in 30 percent of all the activities noted, whereas "using calendars" occurred in only 2 percent of the activities. Furthermore, "reading numbers on tickets" occurred in only one-tenth of one percent of the activities. Just how is the curriculum-builder to make use of these numerical facts in constructing a course of study? It is in the interpretation of the results of activity analysis that the proponents of this theory of curriculum construction have failed to make out a case.

Curriculum construction in arithmetic has apparently come to a cross-road. A more adequate technique needs to be discovered. Among the various suggestions which are encountered two seem to be promising. One of these is that a good deal of light might be shed on the problem by more detailed analyses of pupils' reactions in dealing with quantitative situations, letting the difficulties which these exhibit point the way to the necessary instruction. A second suggestion is that the curriculum-builder might give more attention to the number system itself and the many possible uses that may be made of it. This latter suggestion is illustrated in Osburn's tabulations of the possible number combinations, in Knight's analyses of the possible combinations in long division, in analyses of types of examples in fractions, from that of Kallom in 1920 to that of Brueckner in 1928, and in the unit skills worked out by the Denver Curriculum Committee.

III. Research Relating to Methods of Teaching

As has been pointed out earlier in this chapter, arithmetic was not an important school subject prior to 1800. During the early period of teaching in American schools, the chief method of instruction was the dictation of practical problems by the teacher and the copying of these into 'ciphering books' by the pupils. The use of textbooks by pupils was rare at this time. The necessary explanations of the methods of working examples were given by the teacher and copied directly by the pupils. Arithmetical training was consequently of a most routine sort. The most common method used in the solu-

tion of problems at this time was the 'rule of three,' which in our present arithmetics is called proportion. The outstanding characteristic of the methods of teaching during this early period was the use of a collection of rules which were memorized and applied mechanically to the solution of problems. No reasons were given for the various methods of solution; hence, to the majority of pupils arithmetic was a body of arbitrary procedure of which they had no rational understanding. It was in response to this situation that Warren Colburn published his first book, which initiated a radical reform in methods of teaching. Colburn recognized the utilitarian value of teaching as the principal value, although he considered mental discipline to be a desirable outcome of the subject. As the result of Colburn's reform, the ciphering books were discontinued and textbooks substituted for them. Colburn not only revitalized the teaching of arithmetic, but also systematized it to a considerable degree. Still further attempts at more rigid systematization followed, as illustrated in the Grube method, which emphasized repeated drill in the primary grades on those combinations of numbers below ten. By this insistence on the complete automatization of combinations of numbers lower than ten, the Grube method resulted in a very tardy initiation into larger number experiences. Following the Civil War, Colburn's inductive treatment of arithmetic was gradually supplanted by formal methods and more or less obsolete materials which continued to be used until the reaction of the present period set in. During the past two decades the treatment of methods of teaching has been characterized chiefly by an elaborate development of drill for the purpose of automatizing the combinations, together with an attempt, none too successful, to motivate arithmetic by relating its problems to the familiar life experiences of the pupils.

The general literature dealing with methods of teaching arithmetic is voluminous. Only a minor part of this literature can be classified as scientific studies of method, but even this minor part now includes a considerable number of significant quantitative studies. The materials dealing with method will be reviewed here under the following subheadings: (1) incidental teaching, (2) drill, (3) some specific problems of method, (4) motivation, (5) remedial methods, and (6) transfer of training.

1. Incidental Teaching of Arithmetic

It is sometimes said that the schools are overdoing the teaching of arithmetic and that children would learn much of it incidentally without any formal training. Court has given a detailed account of the number experiences which one child developed in the first eight years of life without any systematic treatment. Drummond presents further material of the same type. Meriam has reported the results of incidental teaching of arithmetic during the elementary-school period and has shown that children so taught are able to do efficient work in the high school. In contrast with the few quantitative statements of the results of incidental teaching, there is a large body of quantitative data which shows the effect of specific drill under controlled conditions. These results make such a clear case for systematic teaching that the proponents of incidental teaching have comparatively little on which to base their claims. Incidental teaching may be valuable, but it is in no sense a substitute for systematic teaching. One should recognize, however, that some teaching which may appear to be incidental from the standpoint of the pupil may actually represent careful planning on the part of the teacher. This type of incidental teaching may possess merit, but objective proof of it is yet to be supplied.

2. Studies Relating to Drill

In 1914 Jessup investigated the amount of drill being given in numerous public schools. Wood has transposed the data from Jessup's investigation into a table which shows the amount of time

TABLE II.—*PERCENT OF RECITATION TIME SPENT IN DRILL AS REPORTED BY JESSUP IN 1914

Grade	One-Fourth of Schools Spent Less Than Indicated Percent	Median Percent	One-Fourth of Schools Spent More Than Indicated Percent
I..............	25	43	80
II..............	27	50	81
III..............	34	52	74
IV..............	31	46	64
V..............	26	39	49
VI..............	16	31	43
VII..............	10	22	36
VIII..............	6	17	26

*Wood, E. R., "Investigations in arithmetic," *Kentucky High School Quarterly,* Oct. 1919, p. 68. (Table constructed from data reported by Jessup.)

devoted to drill in the various school grades. The most striking fact shown is the excessive variation in the time devoted to drill. Since only a few sets of commercialized drill materials were on the market in 1914 in comparison with the enormous quantity of them now sold annually, a repetition of Jessup's questionnaire at the present time would certainly reveal interesting data.

Recent literature indicates that the purposes of using drill are becoming more specific. Drill as a device of learning, for instance, is distinguished from drill for the purpose of maintaining learning. These differences in purpose are reflected partly in the distribution of practice periods for drill and partly in the selection of materials to be made the subjects of drill. While drills for the purpose of learning should be given at regular intervals of the same duration, drill for purposes of maintenance should be so distributed that the amount of time between drills on the same fact increases until no further drill is required at all. This latter type of drilling for maintaining skills once formed has been given little attention until recently.

One of the important conditions affecting drill is the length and distribution of the drill periods. Kirby has made an extended study of the effect of drill in periods of 22.5, 15, 6, and 2 minutes. He found that the same total amount of practice, when distributed according to these various periods, produced gains in the relation of 100, 121, 101, and 146.5, respectively; that is, the shortest practice period was the most effective. Hahn and Thorndike made a similar experiment with pupils in Grades IV to VII in which they distributed 90 minutes of practice in intervals of 22.5, 20, 15, 11.25, 10, 7.5, and 5 minutes. The evidence from their experiment favored long rather than short periods of practice. In both of these experiments one must note that the total time allowed for practice was brief as compared with the total amount of time that would be given to drill during a given semester of school; also that in the case of the longer periods the number of practices was quite small.

There are some reports which show the results from continuous drills running over several months of time where a single length of drill period was used. Comparisons in these cases are difficult to make because of the variations in conditions other than length and distribution of drill period. Certain general principles would seem to apply in any case. (1) Practice periods should be sufficiently frequent that habits begun in one period would not fade out be-

fore the next period of practice begins. (2) The duration of a period of practice should be long enough to give repeated contact with the materials used, but not long enough to induce marked fatigue. Scientific evidence does not justify a statement at the present time as to the precise length of the most effective drill period, but from the total body of evidence available, regular drill periods should probably not be longer than ten minutes. The optimal length of time is a function of motivation, amount of trouble encountered, monotonous vs. varied content, etc.

Several studies show that marked improvement may result from a relatively small number of drill periods. For example, studies covering a period of two weeks or four weeks show that drill produces an enormous improvement over initial ability. Much caution must be exercised in interpreting these data. The improvements which are shown depend very much upon the nature of the initial ability of the pupils when the drill experiment was begun. For example, if the experiment includes drill upon combinations which were once thoroughly learned but have been partially forgotten, the results of a short period of drills will differ from the results of similar drills given to pupils who have never learned the combinations included in the drills. Furthermore, drills given to pupils whose initial state of practice is high, owing to previous exercise, will show different results from drills given to pupils covering processes which may have been once learned but which have not been used for some time. The school is chiefly interested in the permanent effects of drill. In general, these permanent effects cannot be shown by a brief experiment. What the brief experiment shows is that one may increase his present ability markedly by special drills, but that this increase in ability is likely to be temporary rather than permanent. It is for this reason that many of the experimental studies are of no particular significance to the school. What the school generally wishes to know is the permanent value of a systematic set of drill materials as compared with incidental drills given for a brief period and without any reference to their place in a general drill program.

The most marked contribution to drill during the past decade has been the studies which have resulted in the improvement of drill materials. In 1921 Thorndike published an analysis of the practice provided in certain arithmetic textbooks. A part of his data are reproduced here as Table III. As shown in this table, the combina-

tion 2×2 is presented in the textbook 668 times, while the combination 9×8 is presented only 82 times. Evidently the combination 2×2 will be greatly overlearned or the combination 9×8 will be greatly underlearned by the pupils using this textbook. Other gross inequalities in drill are indicated in the table. As a result of this analysis by Thorndike and of other similar analyses, much attention has been given recently to the construction of drill materials. While the outlet of much of this energy has been in the form of practice exercises and workbooks, there is evidence that textbook writers are now giving the same kind of attention to drill materials provided in their textbooks. By employing well-known techniques of analysis, drill exercises may be constructed to provide any desired amount of practice for any particular combination. The problem which remains is to determine how much drill shall be given to the various combinations and how these combinations shall be distributed in a given set of materials. At this point authors have generally turned to the studies of relative difficulty of number combinations and have attempted to give a greater amount of drill to the presumably difficult combinations and less to the presumably easy combinations. This scientific construction of drill materials is probably one of the most important contributions of the past decade to the teaching of arithmetic.

3. Studies of Specific Questions of Method

It is to be hoped that at some time the majority of the questions relating to methods of teaching may be answered on the basis of specific experimental evidence. At the present time few issues can be settled on that basis.

The fact that for so many questions of method no experimental data are available should stimulate the scientific student of education to an increased enthusiasm for research. Whenever the notion thoroughly percolates into the teaching profession that the way to answer a question of method is to set up a crucial experiment which will furnish objective and verifiable data bearing upon the issue, one may expect an age of real progress in education. As long as teachers are willing to accept the dictates of *a priori* theorizing in the place of clear-cut quantitative evidence, there is little reason to hope for any marked improvement in the teaching of arithmetic.

TABLE III.—*NUMBER OF PRACTICES PROVIDED ON VARIOUS NUMBER COMBINATIONS IN CERTAIN ARITHMETIC TEXTBOOKS

Multipliers	Multiplicands										Total
	0	1	2	3	4	5	6	7	8	9	
1	299	534	472	271	310	293	261	178	195	99	2,912
2	350	644	668	480	458	377	332	238	239	155	3,941
3	280	487	509	388	318	302	247	199	227	152	3,109
4	186	375	398	242	203	265	197	163	159	93	2,281
5	268	359	393	234	263	243	217	192	197	114	2,480
6	180	284	265	199	196	191	168	169	165	106	1,923
7	135	283	277	176	187	158	155	121	145	118	1,755
8	137	272	292	175	192	164	158	157	126	126	1,799
9	71	173	140	122	97	102	101	100	82	110	1,098
Total...........	1,906	3,411	3,414	2,287	2,224	2,095	1,836	1,517	1,535	1,073	21,298

*Edward L. Thorndike, "The psychology of drill in arithmetic: the amount of practice," *Jour. of Educ. Psych.*, 12:1921, 183-194

a. Subtracting.—However, such a question as the best method to be used in subtracting may be referred to as a concrete illustration of the experimental method of settling questions in education.

The problem of methods of subtracting reduces itself in school practice to two specific questions: (1) Shall one use a subtractive method or an additive method? (2) After having decided on either the subtractive or the additive method, shall one subtract by decomposition (involving borrowing) or by equal additions (involving carrying)? There are, therefore, four possibilities in subtraction which may be illustrated as follows:

First, the subtractive method using decomposition: Taking as a stock example 71 — 39 = 32, those who would subtract by this first method would say: "11 minus 9 leaves 2; 6 minus 3 leaves 3."

Second, the subtractive equal-additions method: Using the same example, those employing this method would say: "11 minus 9 equals 2; 7 minus 4 equals 3."

Third, the additive method using decomposition: In this case pupils would say: "9 and 2 are 11; 3 and 3 are 6."

Fourth, the additive equal-additions method: Following this plan pupils would say, "9 and 2 are 11; 4 and 3 are 7."

There have been published ten quantitative studies which deal with these various possibilities of subtraction. There are in addition hundreds of pages devoted to *a priori* theorizing as to which might be the best method to use. The ten studies are not directly comparable, since some make comparisons between one pair of the four possibilities while others use different combinations for purposes of comparison. Furthermore, the studies are limited to abstract combinations of whole numbers and do not cover verbal problems, denominate numbers, or fractions. Before one can consider the case closed, further studies and check experiments must be made. However, of those studies which related to the additive method, only one indicated any superiority attaching to it and in that case the superiority was very slight. Of the two possibilities using the subtractive method the experimental evidence favors the equal-additions procedure as compared with the decomposition procedure. While the experimental evidence is not satisfying, it is certainly more convincing than *a priori* theorizing presented with no supporting data. Until further evidence is at hand, one must interpret the facts as indicating that the subtractive equal-additions method is superior and should be used in the schools. This tentative conclusion should be considered subject to change only

on the basis of quantitative evidence. Pronouncements which are made without supporting quantitative evidence are becoming less and less convincing to that increasing group of teachers in whose training scientific methods of study have been emphasized.

b. Number Combinations.—Most questions of method are susceptible to specific experimental study like that pertaining to subtraction. In a few cases one finds such objective data relating to the specific problems involved. For example, the question of the relative amount of time which should be devoted to the teaching of the different number combinations can now be dealt with in terms of a number of objective studies of the relative difficulty of these combinations. True, the methods of determining relative difficulty have been so varied that it is difficult to draw final conclusions, but it should be noted that the studies which have been made are subject to repetition and verification. Certainly no one would suppose that *a priori* reasoning would give a better indication of the relative difficulty of number combinations than the experimental studies have done.

c. Adding.—A third example of a specific attempt to answer a question of method by objective study rather than by rationalization relates to adding upward or downward. Obviously this issue is on a different level of importance than is the issue of the best method of subtracting, but, even so, there are some objective data available for deciding which procedure to accept.

4. Motivation

It seems to be rather generally assumed that the subject of arithmetic will not be interesting to primary children unless motivated by something outside of the subject itself. Consequently, the literature is full of descriptions of devices and games which may be used to stimulate interest. As would be expected, one finds few scientific data relating to motivation, though, aside from the tedium of making such studies, there is no reason why the scientific method may not be applied to problems of motivation just as well as to any other problems in education. Obviously the place to begin is with the more significant problems of motivation rather than with the trivial devices which are so frequently used with so little regard for the reason why.

Why has the number system itself been so little used as a motivating device? The psychological inventories of children's interests have

always revealed a fundamental interest in number and its possibilities. In place of the compilation of endless games and devices which now characterize the literature relating to motivation in arithmetic, the students of education should build up a body of scientific studies dealing with crucial issues in this field. At the present time the main body of objective literature relating to this topic deals chiefly with the use of tests or drill as motivating devices.

5. Remedial Methods

Remedial methods are simply the result of an attempt to deal with difficulties in arithmetic in specific terms. Emphasis upon them has followed naturally the construction and use of diagnostic tests. Diagnostic tests furnish an adequate list of specific difficulties for which remedial methods should be devised, but there are few specific remedial devices the value of which has been determined by objective data. There are published lists of specific remedial devices which certain teachers have found valuable, but the supporting evidence is empirical rather than scientific. The most common remedial method now used is to supply drill on combinations needing further practice. Some excellent material is available for this, consisting of good diagnostic tests with remedial practice exercises keyed to the errors which may be made in the test. This use of drill is good to the extent that the pupils' methods of working are satisfactory. If the pupils' individual methods of work are not good, remedial measures to correct these bad habits should be applied before drill is continued.

6. Transfer of Training

The problem of transfer of training may be reduced to more specific questions, such as: (1) Does arithmetic provide general mental discipline for the pupil? (2) To what extent will specific training on certain topics carry over to other topics without specific teaching of them? (3) To what extent will the teaching of arithmetic become generalized into quantitative methods of thinking?

Since the first question is so general and relates to all subjects in the curriculum equally, it will not be dealt with here. It constitutes a general psychological problem rather than one which is particular to the subject of arithmetic.

The second question touches the subject of arithmetic at many specific points. The study by Knight and Setzafrandt is a good illustration of this problem. In that study an experiment was made to determine the extent to which the teaching of fractions with even denominators would carry over to fractions with odd denominators. The results showed that there was considerable transfer, without any specific teaching, to the fractions with odd denominators.

Another illustration is Osburn's contention that number combinations up to 40 should be taught specifically. This is chiefly a question of how much specific instruction must be given before transfer will take place. If one learns the combinations $7 + 8$, $17 + 8$, and $27 + 8$, will he be able any better to meet the combination $57 + 8$ than if his original teaching dealt with only $7 + 8$, and $17 + 8$; or with only $7 + 8$? The question here is simply how much specific teaching of number combinations is required before the pupil can generalize his learning to meet combinations of higher order not met before. Questions of this sort are entirely susceptible to experimental treatment. Until experimental data are available, Osburn's contention must be considered as an hypothesis which may or may not be supported by fact.

The third question—to what extent instruction in arithmetic results in generalized modes of quantitative thinking—will require a much longer period of study and a much more complicated technique than do questions of the second type. The discovery of effective methods of developing quantitative modes of thinking by arithmetical or other instruction ranks among research problems of the highest order. Further information upon this problem would constitute a contribution of the greatest importance.

IV.—Pupils' Reactions: Analyses of Errors and Methods of Work

As might be expected, more studies have been made of pupils' reactions than any other single aspect of arithmetic. Under this rather general heading are included the following types of studies: analyses of the processes by which pupils carry on the various operations in arithmetic, such as counting, adding, working with fractions, solving problems, etc.; analyses of pupils' errors and the causes of these errors; studies of individual differences, including on the one hand the study of number prodigies and on the other hand

the study of pupils with seeming disabilities in arithmetic. These studies have thrown much light upon the intimate mental processes of children in doing arithmetic work and have included some of the most important and substantial contributions to methods of teaching. Some of these studies will be briefly cited here.

1. Analyses of Mental Processes

The studies of pupils' reactions in carrying on the various operations in arithmetic are too numerous and detailed to be fully reviewed here; a few samples will be helpful, however.

At the first level of arithmetic one finds detailed analyses of pupils' reactions, such as Judd's elaborate study of counting, while at the other extreme one finds analyses of the complex reactions of problem-solving as illustrated in a recent study of Hydle and Clapp. Among the most valuable reports of pupils' reactions are those studies which give in detail the methods of thinking which the child employs in solving problems or working examples. The following quotation taken from a study by Winch[4] shows very clearly the complex mental processes which occur before a pupil is ready to write his answer on paper.

Cissy F———, aged 10 years, 0 months, dealt with a sum containing several noughts in the minuend in a similar manner.

400,000
59

She said:

(1) 9 from 0 I can't; go next door I can't; go next door I can't; go next door I can't; go next door I can't; go next door, take 1, leaves 3, and that makes that (pointing to the nought immediately to the right of the 4 in the minuend) 10. 9 from 0 I can't; go next door I can't; go next door I can't; go next door I can't; go next door, take 1 from the 10 leaves 9, and makes that one (pointing to the nought in the second place from the 4) 10. 9 from 0 I can't; go next door I can't; go next door I can't; go next door, take 1 from the 10 leaves 9 and makes that (pointing to the third nought) a 10. 9 from 0 I can't; go next door, take 1, leaves that a 9 and makes this a 10; 9 from 10 leaves 1.
(2) 5 from 9 leaves 4.
(3) 0 from 9 leaves 9.
(4) 0 from 9 leaves 9.
(5) 0 from 9 leaves 9.
(6) 0 from 3 leaves 3.

The problem of teaching a child to subtract is obviously a complex one. Such detailed analyses of pupils' mental reactions as that just

[4] Winch, W. H., "'Equal additions' *versus* 'decomposition' in teaching subtraction: an experimental research," *Jour. of Exp. Ped.*, 5: 1920, 213.

quoted from Winch give a quite different view of the task of teaching arithmetic from that which one would get in reading some of the theoretical discussions of methods of teaching subtraction in which one is given the impression that the whole process is simply the forming of associations which may later be reproduced in automatic fashion. Ultimately this automatic type of subtraction will probably be attained, but the task of teaching is always that of leading a pupil to such automatic responses and the initial stages of teaching are certainly far more complex than many teachers realize. Teachers who are not aware of the complex mental processes which go on in the pupil's mind before he is able to write his answer are ignorant of the main body of facts which should direct their teaching. The many reports of pupils' reactions are well worth careful study.

It should be noted that the early studies of pupils' reactions make up the background for much of the present diagnostic work which is being carried on in schools; that is, many of the present diagnostic tests are nothing more than systematic devices for objectifying pupils' reactions which would otherwise escape the notice of the teacher. It is through these diagnostic tests that the preliminary work in analyzing pupils' mental processes is being translated into school practice. This is a capital illustration of the fact that research studies which in their initial stages may seem to have little actual relationship to the teaching process may eventually prove to be of the greatest value in the classroom. If this fact could be realized more fully, it would doubtless stimulate a greater amount of fundamental research in the place of some of the trivial studies which make a rather general appeal because of their seeming immediate practical value.

2. Analyses of Errors

Many of the studies of pupils' reactions may be classified as analyses of pupils' errors. These analyses have sometimes been made by studying the written tests of pupils and inferring from the errors in the answers the possible causes for the errors. Other studies have dealt with the oral work of children. By asking questions and interrupting the pupils as they work their examples, analyses of errors may be made which are more detailed than those that may be obtained from a study of written tests. From these analyses of errors teachers may obtain useful facts for directing their classroom procedure.

The usefulness of these analyses of errors depends upon the detail to which the study has been carried. For example, in one report of the errors occurring in problem-solving the statement was made that 30 percent of all the errors were due to total failure to comprehend the problem, 20 percent to procedure which was partially correct but in which one or two essential elements were omitted, 10 percent to failure to respond to fundamental quantitative relations, 20 percent to mistakes in the fundamental processes, and 20 percent to miscellaneous causes or to causes which could not be discovered. While this study was a commendable contribution to the literature at the time that it was made, it is obvious now that listing 30 percent of the errors as due to "total failure to comprehend the problem" needs further analysis. This study was made by the group method of analyzing results from test papers. If individual tests could have been made of the children who failed to comprehend the problem, a more detailed analysis of the causes of failure doubtless could have been secured. It is of some value for a teacher to know that a child cannot solve a problem because he cannot comprehend it, but it is of far more value to tell the teacher the probable reasons why children fail to comprehend problems. The more recent studies of errors reveal greater detail in the analyses. Some very good analyses of errors exist at the present time as unpublished Masters' and Doctors' dissertations. These studies should be made available by publishing comprehensive abstracts of them in some of the educational journals.

3. Studies of Individual Differences

Many of the data concerning individual differences in arithmetic have been derived from standardized tests, particularly from diagnostic tests, one of the chief purposes of which is to detect individual difficulties. Tests have also been used to show the individual differences in arithmetical ability among persons in different occupations and for students majoring in different academic departments. Several studies have been made of sex differences in arithmetic. While it appears that sex differences in arithmetical ability are not large, the differences which do exist are in favor of the boys.

Considerable attention has been given in the literature to the description of number prodigies; *i.e.*, those individuals who are able to perform mathematical operations which seem beyond the ability of the ordinary person. An excellent summary of the descriptive ac-

counts of these prodigies has been given by Mitchell, who also made an attempt to explain their extraordinary performances. Mitchell's conclusion is that the number prodigy does not possess a special type of ability but merely exemplifies a special development of one type of association. The typical number prodigy seems to be a person who has a specialized interest in counting and, in addition, an unlimited opportunity to practice under conditions where his whole attention is absorbed in the relationships between numbers and the possibilities of their recombination. There is little evidence that the schools have profited to any extent from the study of methods employed by number prodigies.

At the other extreme are those who may be described as possessing some special disability in arithmetic. However, psychological theory leads one to be suspicious of the idea that children who are otherwise normal may often suffer from a special disability in arithmetic. Of course, children who are mentally deficient find arithmetical processes among the most difficult operations which the school presents. Several studies have been reported of the difficulties encountered by children whose I.Q.'s range from 67 to 70. Following a study made of mentally deficient individuals, Goddard writes as follows:[5]

> If we have hit upon a right explanation of our data, we have a most important conclusion. Most of our feeble-minded children have been arrested in their development before the concept of number had come to them. It is a waste of time and energy to attempt to train them in abstract number. Simple combinations of concrete things are all that they can master.
>
> In the public schools the number concept does not develop for many children until some time, perhaps years, after they usually begin number work. This time is wasted; and worse than that, the child is confused and discouraged and effectually driven away from ever having that love for number which leads to its mastery.
>
> Thus the data from the feeble-minded children have given us a starting point which may enable us to interpret the facts of normal life and possibly discover a principle of great importance to the public-school teacher.
>
> If anyone is disturbed by the thought that perhaps it is not safe to argue from feeble-minded children to normal children because they are radically different, it is sufficient to remind such a one that no one who has studied them at all any longer regards them as radically different. They form a continuum, and what is true of one is true of the other, but in a different degree.

[5] Henry H. Goddard, "A side light on the development of the number concept." *Supplement to the Training School,* No. 1 (December, 1907), 23-24.

While individual differences in general mental ability are important matters for consideration by the school, it is probably true that the teacher will be more directly interested in those individual differences among normal children which express themselves in a great variety of habits of work in carrying on ordinary arithmetical operations. It is in the understanding of these differences that the teacher will find the greatest help in instruction.

V. Tests and Measurements

It is doubtful whether any single event has stimulated as much research in arithmetic as has the construction and standardization of objective tests of achievement. Particular credit for the early studies in testing must be given to Rice, who in 1902 reported the results of testing some six thousand children in arithmetic, and to Stone, who in 1908 reported the results of a series of tests given to sixth-grade children in order to determine the achievement resulting from six years in arithmetic. However, the major stimulus to standardized testing in arithmetic came from Courtis, who in 1909 published the first of a series of articles dealing with testing in arithmetic. To the energy and initiative of Courtis, who carried on the first elaborate piece of research for the purpose of standardizing a test in arithmetic, teachers owe much. Courtis's work developed so rapidly that by 1915 an adaptation of his materials was used in the Cleveland School Survey. Further improvement was contributed by Woody through the introduction of more types of examples and by Woody and McCall in providing mixed tests. Other standardized tests began to make their appearance, and the use of such instruments in carrying on school surveys became a standardized procedure. The important research contributions relating to testing consisted of those studies which were made for the purpose of standardizing materials and also studies made to determine the validity and the reliability of the tests. During the later period of testing, valuable contributions were made by applying statistical procedures to determine the relative difficulty of the various items in a test and to check the distribution of the items which occurred in the test to see that they represented adequately the general field being tested.

While during the past decade there has been an enormous amount of literature relating to testing, the great majority of it has consisted of reports of the routine application of tests to school systems in

which the findings are chiefly of local interest and the contribution to fundamental research is negligible. The earlier studies were given over to the construction and standardization of survey tests. During the more recent period much attention has been given to constructing diagnostic tests whose chief purpose is to give direction to the teacher rather than to provide general facts for the school administrator. A final stage in the development of testing appears in the organization of bureaus of measurement in which a systematic application of tests has become an organic part of the school's procedure. This acceptance of tests as a part of normal teaching procedure is a logical outgrowth of the earlier stages of the testing movement.

VI. The Construction of Teaching Materials

The most common kind of research relating to teaching materials has consisted of analyses of textbooks and practice materials. As already noted the study by Thorndike in 1921 showed wide variations in the distribution of practice in textbooks and suggested that many of the authors were entirely unaware of the distribution of practice in their own tests. Similar analyses of both textbooks and practice exercises have been made by Knight and his students. Investigations such as these are now bearing fruit in the increased amount of care which authors and publishers are devoting to the construction of teaching materials. It should be possible soon to secure from an author or publisher a certified analysis of the content of a book.

A second type of analysis relating to teaching materials is illustrated in a monograph by Monroe in which all of the problems in a group of books are classified according to certain categories or types.

A third type of analysis is illustrated by Wilson's investigations of the kinds of problems which are common in social life. While it is considerably more difficult to apply the results of this type of analysis to the construction of teaching materials than is the case with the results of the preceding two types, they do throw light upon the concrete social situations in which arithmetical problems arise.

Numerous other problems of textbook construction have received minor attention in the research studies; much more attention is necessary before full application of the findings is warranted. One notes, for example, that the use of pictures in color is becoming common in arithmetic textbooks. This produces a large increase in publishing costs. The value of colored pictures needs to be studied carefully.

There is little doubt but that they add to the attractiveness of a book, but the relationship between attractiveness and mastery of arithmetic has not yet been made clear.

VII. Conclusion

In 1900 there were but thirty published studies of arithmetic which could claim to be quantitative and scientific; in 1929 there are approximately five hundred such studies in print—evidence of the amount of energy which has been devoted to arithmetic. The amount of actual contribution to knowledge which has resulted from all this effort may be disappointing to many readers. However, a long step has been taken from an attitude of mind which was satisfied with authoritative conclusions based upon *a priori* reasoning to a spirit of inquiry which demands objective data as a basis for conclusions.

New techniques of investigation will undoubtedly be developed which will give further stimulation to research. There is obvious need for more major studies which are carried to the point where more valid conclusions may be drawn. There is also a definite need for check experiments and for studies which will bring together the many partial investigations which have been reported.

GUIDE TO CHAPTER IV

CHAPTER IV

THE NUMBER ABILITIES OF CHILDREN WHEN THEY ENTER GRADE ONE

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I. Introduction: The Importance of the Problem

The conventional arithmetic textbook, a book intended to be placed in the hands of third-grade children, begins with the presentation of a gaudy picture in which certain priggish children, impossibly clean in appearance, are occupied with hoops, marbles, dolls, or whatnot of the apparatus of play. The author tells the pupil to look at the picture. "How many boys are there?" "How many girls are there?" "How many girls are sitting down?" "How many boys are sitting down?" "How many boys are standing up?" "How many children are there?" "How many children are sitting down?" "How many children are standing up?" "How many dolls are there?" "How many boys are playing marbles?" and so on.

Very likely the author now calls upon the child to count from one to ten, and a few lines later from one to twenty, or possibly even to one hundred. He probably shows the pupil the figures from zero to nine and asks him to name them.

Within half a dozen pages, however, the scene abruptly changes. The child is now asked to solve problems involving the adding of one-digit numbers and to complete abstract number facts, such as four and eight, six and three, five and nine, and so on. A few more pages and the child is very likely in the throes of subtraction, solving problems and making abstract statements, such as seven less two are five, eight from eleven are three, fifteen minus nine are six, and so on.

One gets the impression from all this that progress is exceedingly rapid. Within the space of twenty pages the child has apparently progressed from telling how many girls there are in a picture to the solving of a problem such as, "Harry was 4 years old when his sister Julia went to school. Julia has been in school 8 years. How old is Harry?"

The conventional third-grade arithmetic can be right at only one end or the other of the material in the first score of pages. If there is anything educative in counting the three girls who have dolls in their hands, or the four boys who are playing marbles; if there is any learning taking place when the pupils are counting to twenty or writing and reading the numbers to nine, then the problems and abstractions involving the one hundred addition combinations are quite beyond the powers of these pupils.

This, of course, raises the question of the assumptions underlying the course of study in arithmetic at the point where a textbook is first commonly employed. What do children know upon entering the third grade? Obviously an answer to this question, even in the most general terms, depends upon the existence of a more or less standard environment. In particular, it means the identification of the school environment as it affects children before they enter the third grade. Here the problem arises of the teaching of number in the kindergarten and in the first and second grades. The American school is wabbly on this point. Some say teach number in the first grade as a regular prescription; others say wait until the second grade; and a few would wait until the third grade. A distinction is made between formal and informal teaching of number. It is impossible to be sure what this distinction really means. It is clear, however, that if we can decide where teachers should begin to subject children to number experiences, to the end that they may learn about quantitative matters, then the work at that point should be definitely provided for. Teachers will want to know, and they are entitled to know, what is expected in the way of a learning product. Whether they secure the desired end by formal or informal treatment of the subject may be immaterial, except to the degree that it is material in any other subject.

The question of number teaching and number learning in the early grades depends primarily upon the so-called 'law of readiness.' Of course there are those who argue that reading in the first grades is so important that number work should not be begun lest it interfere with the reading. It is true that reading is of first importance, but no teacher need avoid even a systematic treatment of number because of any supposed damage to the reading. It is entirely possible to make the reading of the first grade so continuous and unrelieved as to pass into the area of diminishing returns.

Accordingly, as has already been said, the 'law of readiness' offers the best standard by which one may determine when to introduce the teaching of number. The main question at issue is whether the child's maturity, his outlook upon life, and his interest in his environment are such as to make him need and desire the learning of number in order to make his life fuller and more meaningful.

A very strong suggestion as to the answer to this question—in fact, practically speaking, a determination of the answer—is to be found in the extent to which children, prior to entering school, have already acquired an interest in number and a knowledge of it. If, at the age of six, when he enters school, the pupil has passed beyond the period of counting the four boys who are standing or of telling the number of pink rabbits; if he has a functional grasp of the small numbers and can enter into communication with other children and with adults on the basis of a common knowledge of small quantities, then the indications are that this child is ready for our ministrations, and that to delay the teaching of number is to retard his growth.

What number, then, does the six-year-old child know when he enters the first grade? In seeking an answer to this question, the authors note that they have been preceded by a goodly array of inquiring spirits who have likewise addressed themselves to this question or to questions of similar import.

II. What Has Thus Far Been Acomplished

Many reports—often presented in foreign publications—have been made concerning the number knowledge of young children. Some of these reports are based on observations of individual children, while others are the result of testing large groups.

1. Records of Individual Children

Binet tells of a little girl who at four years, three months, could indicate with reasonable assurance (one error in nine trials) the larger group within the following pairs when composed of similar objects: 15 and 18, 16 and 18, 17 and 18. When counters of different colors and different sizes were used, the results were so confused that Binet remarks: "These experiments only prove the very great difficulty which the child (any child) has in perceiving a mass in any other than continuous form, even when he should perceive it as a large

number of units."[1] He continued his experimentation with small numbers and concluded that the perception of number for the four-year-old was four, "after some practice, five." The corresponding limit for a two-year-old whom he likewise studied was lower. "She could not recognize with assurance numbers beyond three."[2]

Stern, who carefully followed the progress of his three children, quotes remarks made by them at early ages which indicate a feeling of serial order and later the understanding of two and then of three.[3] Drummond recounts in some detail the progress of one child in the mastery of simple number relations.[4] The most detailed and consecutive report is that of Decroly, who tells of the different evidences of understanding number relations which his daughter showed from her fourteenth to her fifty-eighth month.[5]

A more complete understanding of the number ideas of young children has been gained from these detailed descriptions of the progress of individual children. Deucher analyzed number concepts into five processes, and upon this analysis he arranged a series of tests for children from three to seven years of age.[6] No record of the results of these tests was available, but Deucher's fundamental assumptions were quoted later by Filbig and Beckmann. Beckmann in his turn differentiated the following possible components of number-consciousness: act of hearing numbers, act of speaking numbers, act of producing numbers, act of differentiating numbers, act of finding numbers, and act of naming numbers.

The number activities of a child are particular reactions to a definite situation. They are not comparable to the highly abstract number concepts of an adult. The child may be able to reproduce a group of three objects without yet having mastered the ability to count three or having acquired the ability to name the group without counting—an ability of still later appearance.

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[1] Binet, Alfred. "La perception des longeurs et des nombres," *Revue Philosophique*, 30: Juillet, 1890, 68-81.

[2] *Ibid.*, p. 80.

[3] Stern, William. *Psychologie der Frühen Kindheit.* Leipzig, Verlag von Quelle & Meyer, 1927. pp. 342ff.

[4] Drummond, Margaret. *Five Years or Thereabouts.* London, Edward Arnold and Company, 1921. pp. 119-135.

[5] Decroly, M. le Dr., and Degand, Mlle. Julia. "Observations relatives a l'évolution des notions de quantités, continués et discontinués chez l'enfant," *Archives de Psychologie*, 12: May, 1912, 81-121.

[6] Deucher, G. "Psychologische Vorfragen des ersten Rechen Unterrichts," *Zeitschrift f. Pädagogische Psychologie*, 13: 1912, 36-52.

Beckmann notes that the child's particular "manner of reaction may be a thoroughly correct expression of his capacity for performance and may indicate its limits." [7] Continuing, he says: "Only the sum of all number acts appropriate to the child, and correctly carried out by the child, guarantees a full number-consciousness in relation to a certain number. Hence, if we speak of a qualitative analysis of the number performances of the child, the meaning can only be that we investigate the number-reaction of the child in relation to many varied combinations and determine their nature definitely." [8]

2. Experimental Studies

a. Studies with Preschool Children—Counting is an early, but not necessarily the earliest, number process noticed by the biographers of children. Rusk has said, "The ability to count or repeat numbers, say up to ten, only indicates that a series of closely associated terms have been learned." [9] To Binet belongs the credit of having detected the relation, which later investigations have verified, between the ability to count and normal mental development.

One of the earliest studies of young children's number ideas was made by Phillips. He examined several groups of children himself, interviewed many teachers, and sent out a questionnaire dealing with several phases of the subject. He summarizes thus his findings regarding counting:

> "Children first name the series without reference to objects of any kind. Every attempt to instruct children in numbers shows that the series-idea is highly abstract. On making a test of thirty-nine children in the kindergarten, I found that thirty-three of them counted without the slightest reference to the objects to be counted, always running the series far ahead of the objects. Nearly every primary teacher I have consulted says that for some time children count in this manner." [10]

Decroly devised a series of tests which were based upon his observations of the way in which number ideas are manifested by young children. The most extensive use of Decroly's tests has been reported by Descoeudres, who tested more than three hundred children from

[7] Beckmann, Hermann. "Die Entwicklung der Zahlleistung bei 2-6 jährigen Kindern," *Zeitschrift f Angewandte Psychologie*, 22: 1923, 1-72.

[8] *Op. cit.*, p. 9.

[9] Rusk, Robert. *Experimental Education.* 1913, p. 295.

[10] Phillips, D. E. "Number and its application psychologically considered." *The Pedagogical Seminary*, 5: October, 1897, 221-281.

two and a half to six years of age. Limitations of space confine discussion here to only two sorts of tests made by Descoeudres—those of counting and of number reproduction.

The directions given by Descoeudres for the test, To Count Numbers in Order, are: "Listen carefully. I am going to count; you repeat after me what I am going to say." She says: "The fact of freely repeating or saying the series of numbers need not influence the results greatly, since children's memories hardly allow them to repeat more than two or three numbers in succession if they do not know them already."[11] She finds that, at two and a half, children can repeat the numbers in order from 1 to 3 and that the proficiency in this regard increases regularly to the ability to count to 6 at four, to 8 at four and a half, and to 9 at five.

The second test of counting is more difficult. Its directions are: "You see these pebbles. Count them, touching them with your finger." The child not only must know the number names, but he must also be able to count the objects in serial order. The results of this test differ from those of the other tests of the series in one respect. The author says that, "while in all the others the children pass gradually from one number to another, here we see the children of four years succeed in counting from 1 to 6, and those of five years to 10." The younger children cannot accomplish this feat.

Similar tests given at the Child Welfare Station of the University of Iowa gave quite different results. In counting by rote, the average of the two-year-olds was 1, of the three-year-olds was 3, of the four-year-olds was 10.6, of the five-year-olds was 23.7, and of the six-year-olds was 25.3. The children in a second test were directed to count marbles. The averages for the same successive age-groups were: 1, 2.1, 9.2, 21.3, and 27.6.[12] These Iowa children were pupils in the nursery school or kindergarten, while the Geneva children whose scores were reported in the earlier paragraphs were younger brothers and sisters of the pupils at the Institut J. J. Rousseau, and children who were playing in the public parks, many of whom may have had only casual opportunities to learn about numbers.

Various sorts of tests have been used by the different experimenters to determine children's ability to recognize numbers. For example,

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[11] Descoeudres, Alice. *Le Développement de L'Enfant.* Paris, Delachaux & Niestle, S. A. (No date) pp. 271-294.

[12] Baldwin, Bird T., and Stecher, Lorle. *The Psychology of the Preschool Child.* New York, D. Appleton & Company, 1924, pp. 152-169.

matching the number of objects shown by the same number of fingers and matching by a group of objects the number of fingers raised were tested by Descoeudres.

Beckmann, a German investigator, intensively studied the familiarity with numbers from one to five which was possessed by 465 children whose ages ranged from two to six years. All the testing was done with objects—small cubes, dice, marbles, and the like. He designated four sorts of number familiarity. The directions for the first test—reproducing—were: "Give me 3 dice." Number differentiating depended upon the child's choice, which was directed by the question: "Is this 2 or 3?" In the finding test the child was asked to find 3 dots, and in the naming test he gave the number name of a designated group of objects. Beckmann found that reproducing or imitating numbers developed first and that number naming developed most slowly. The children whose ages were two and a half had an average record for reproducing of 1.4 and for naming of 0.1, while the averages of the six-year-olds were 4.66 and 4.07.[13] The variety of tests and the thoroughness with which Beckmann analyzed the number activities of these children merit longer description than can be given here.

Filbig, another German investigator, tested the number ability of 102 children enrolled in the kindergarten. His findings and those of Beckmann and Descoeudres have been arranged by Stern. As summarized by him, the numbers 2, 3, and 4 were mastered by the children at the following ages:

	2	*3*	*4*
Beckmann	3:6 to 4:0	4:0 to 4:6	5:0 to 5:6
Filbig	3:9	4:2	5:6
Descoeudres	3:0	4:0	5:0

Stern says further: "Arithmetical development makes its greatest progress about the age of four, namely at the time when the child masters the distinction between 1 and 2. As soon as the 3 idea is conquered, the way is apparently clear for the higher numbers."[14]

The combining of numbers in addition was studied in detail by Beckmann. For this purpose he used dice on which the numbers were represented by dots. From his study he decides that the order

[13] Beckmann. *Op. cit.*, p. 20.

[14] Stern, *Op. cit.*, p. 416. This is taken from the English translation by Anna Barwell (London, George Allen and Unwin).

of difficulty for the combinations used was: $1+1$, $2+1$, $3+1$, $2+2$, $4+1$, $5+1$, $3+2$, $4+2$, $3+3$.[15]

Some of the children 'saw' the sum; others counted from one die to the other. Among the older pupils a third type appeared. Children of this type knew the sum by memory and said: "I know that 2 and 2 are 4." Both Beckmann and Filbig found that the earliest addition results were obtained at four years.

b. Studies with First-Grade Children.—One of the frequently quoted contributions to education made by G. Stanley Hall is his article, "Contents of Children's Minds on Entering School." In it he not only refers to his own study with first-grade children at Boston, but he also tells of similar studies made at Berlin, Annaberg (Germany), and Varde (Denmark). In 1869, the 2,238 children who entered the schools of Berlin were asked a variety of questions, and their knowledge of the numbers 2, 3, and 4 was tested. It was found that 74 percent had an idea of 2 and 3 and that 73 percent knew 4.[16] The similar test at Boston was given in 1880; 92 percent of two hundred first-grade children were acquainted with 3; 83 percent with 4; and 71 percent with 5.[17] Hartmann gave number tests to the 1,312 children who entered the school of Annaberg during a five-year period (1881 to 1884), and he found that 66 percent of them could count from 1 to 10.[18] Of the 5,600 boys and girls who entered the school in Varde in 1898, 90 percent of the boys and 85 percent of the girls could count to 10.[19] Differences in age may be responsible for the discrepancies between the percentages recorded for these last two studies, because the children at Annaberg were between five and three-quarters and six and three-quarters years of age, while those at Varde were between six and seven. The results obtained at Varde rather closely agree with the findings of the present study.

Fifty boys and fifty girls whose average ages were less than six years were tested by Yocum on the first day of school. Twenty percent of each group could count by rote beyond twenty, and 32 per-

[15] Beckmann, *Op. cit.*, p. 54.

[16] Hall, G. Stanley. *Aspects of Child Life and Education.* Boston, Ginn and Company, 1907, p. 4 (percentages computed by the writers). In somewhat similar form this article appeared in the *Princeton Review* (II, pp. 249-272, 1893) and in pamphlet form under the same title.

[17] *Ibid.*, pp. 16-17. Percentages originally given in terms of "children ignorant of it."

[18] *Ibid.*, p. 45.

[19] *Op. cit.*, p. 51.

cent of the boys and 38 percent of the girls counted with objects beyond 20. Two was recognized by at least 70 percent of the children and 6 was known to more than 12 percent.[20]

Eckhardt gave a series of arithmetic tests to 45 boys who were entering the beginning class of a *Mittelschule* at Frankfort during the first week of a school year. Eleven percent of the boys could count to 10, 53 percent of them to 100, and 7 percent to 1,000. A test which was expressed by the question, "Which is more, 9 or 6?" was carried to 100 by 7 percent of these boys. Addition to 100 by 2 and by 5, 6, or 7 was successfully achieved by 36 percent and by 18 percent respectively.[21]

Eckhardt's results, therefore, imply a criticism of the earlier studies. These investigators, influenced by the prevalent belief that the children knew very little, asked less. Instead of searching for the limits of the children's number knowledge, they merely assured themselves that they knew what they were already expected to know.

c. *Probable Sources of Children's Knowledge.*—Meumann, commenting on a "recent study"—evidently that of Eckhardt—said: "It is found that the number concepts of children just entering school show a wide variation in development. . . . This variation may be traced to preschool influences; and the more active these influences have been, the more number is the child likely to know. . . . It cannot be urged that the child's life does not compel him to acquire this essential concept; on the contrary, there are many occasions in his daily experiences which lead to its development. Games, toys, various exchanges of gifts, articles of food, objects about the house—all these should lead to a knowledge of number relations and stimulate the development of number concepts."[22]

Eckhardt commented on the dissimilarity, as to quality and quantity, of home instruction, which he thought usually came by association with older brothers and sisters.

Filbig prefaced the account of his study of 102 kindergarten children with a reference to their social background, the sorts of

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[20] Yocum, Albert Duncan. *An Inquiry into the Teaching of Addition and Subtraction.* Philadelphia, Avil Printing Company, 1901, pp. 13-14.

[21] Eckhardt, Von K. "Beobachtungen über das Zahlenverständnis der Schulrekruten." *Zeitschrift f. Experimentelle Pädagogik,* 8: 1909, 232-235.

[22] Meumann, Ernst. *Vorlesungen zur Einführung in die Experimentelle Pädagogik und Ihre Psychologischen Grundlagen.* Dritter Band. 1922, pp. 642ff.

activities in which they engaged in the kindergarten, and the games and rhymes involving number which they used in their play.[23]

Many first-grade children, of course, have received some informal instruction in number from their parents. They have also learned much from their older brothers and sisters while playing school or games involving numbers. Smith some years ago by personal interviews with five hundred first-grade children at Detroit found some of the uses which these children made of arithmetic in their out-of-school lives. The relative frequencies with which some of the situations involving arithmetic were mentioned by these children were:

	Percentage of Frequency
Transactions carried on in stores	30
Games involving counting	18
Dividing part with playmates or pets	6
Playing store	3
Measuring	2
Setting the table	1

Addition was needed in 35 percent of the situations; counting, in 23 percent; subtraction, in 12 percent; and fractions, reading Arabic and Roman numerals, measuring, comparison, multiplication, and division each in less than 8 percent.[24] Since the home experiences of five-year-olds are not markedly different from those of six-year-olds, we may infer by analogy that some of the familiarity with number which first-grade children possess when they come to school has been gained in similar ways.

d. Studies from the Psychological Laboratory.—Lay studied in great detail the comparative value of different forms of grouping. His preferred form was quadratic, each number being represented by a series of squares of four dots each with part of a square added in case the number was not a multiple of four. Cards bearing these "number pictures" were exposed to view, but the time of exposure was too short to permit counting. Lay carried on a series of experiments with kindergarten and first-year children in which he compared his quadratic pictures with several other sorts of number pictures.

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[23] Filbig, Joseph. "*Untersuchungen über die Entwicklung der Zahlvorstellungen im Kinde.*" *Zeitschrift f. Pädagogische Psychologie*, 24: 1923, 105-113, 156-168.

[24] Smith, Nila B. "An investigation of the uses of arithmetic in the out-of-school life of first-grade children." *Elementary School Journal*, 24: April, 1924, 621-626.

He concluded that the quadratic pictures were superior to other known pictures and particularly to rows.[25] Our interest in Lay's experiment is not controversial, however, but informational, for he found that numbers up to 10 when represented by dots in various arrangements and briefly exposed were easily and accurately apprehended by children in the kindergarten and first year of school.

Howell repeated and extended Lay's experiment. He concluded from preliminary testing that the numbers 1 to 4 were recognized by all the children. From the results of his testing of first-grade children with the numbers from 5 to 12, he found that the order of difficulty in recognizing the numbers (beginning with the most difficult) was 11, 7, 9, 10, 12, 5, 6, and 8.[26] It was therefore his belief that the odd numbers were especially hard to apprehend.

Howell's results have recently been severely criticized by Brownell, who in his turn experimentally studied the ability of children to apprehend numbers of dots after short exposures. He did not limit himself to one form of grouping, but used six different forms. His returns from first-grade children led him to conclude that Howell's decision that odd numbers were more difficult than even numbers, was not due to the 'oddness' of the numbers, but to the irregular arrangements of the dots in the pictures of the odd numbers. He then derived a composite ranking of the numbers for difficulty, using all six arrangements. The order of numbers then became 12, 11, 10, 9, 7, 8, 6, 5, 4, 3.[27] His general conclusion was that "when the factor of objective grouping is eliminated, the difficulty of apprehending visual concrete numbers is in direct proportion to the number of objects exposed. . . . The order of difficulty of apprehension is the order of the notation system." Again he says "there is reason to believe that children normally acquire their abstract ideas of number in the order of the notation system."[28]

Three tests were given to a group of kindergarten children by Douglass, and, since one of them was a short-exposure test, they will arbitrarily be classified with the psychological experiments. The first was a dot-recognition test and the second a dot-selection test. In the

[25] Howell, Henry Budd. *A Foundation Study in the Pedagogy of Arithmetic.* Macmillan Company, 1914, p. 162.

[26] Howell, *Op. cit.*, p. 219.

[27] Brownell, William A. *The Development of Children's Number Ideas in the Primary Grades.* Chicago, University of Chicago Press, 1928, p. 25.

[28] Brownell, *Op. cit.*, p. 31.

third test the child was asked to estimate the number of marbles which the experimenter briefly uncovered. This experimenter concludes that children of these ages ($4\frac{1}{2}$ to 6 years) have "completely accurate concepts of 1 and 2, very serviceable and accurate concepts of 3, and a very serviceable concept of 4, and of 5, 6, 7, 8, 9, and 10 rather vague concepts, though serviceable to a slight degree." [29]

These experimental studies of the number concepts of young children under restricted conditions resembling those of a psychological laboratory have not only conclusively proved children's ability to apprehend groups of objects in systematic arrangements when briefly displayed, but have also determined rather convincingly the order of difficulty involved in the apprehension of the numbers from three to twelve.

The foregoing account of "what has thus far been accomplished" shows a wide range of activity. The reader, however, has no doubt observed that the investigations are fragmentary and that the reports are sometimes conflicting. The investigation which is reported in the following pages attempted to cover more ground and to reconcile conflict by the use of a large number of cases.

III. Description of the Primary Investigation

During the summer of 1928 the writers brought to completion an interview test, the purpose of which was to measure certain aspects of the number knowledge of young children.[30] This test was divided into six subtests, of which the first two concerned counting; the second two, number concepts; and the last two, number combinations. It is painfully evident that this is not all that might be desired in testing the number abilities of young children. There is nothing here about comparison, nothing about a quantitative vocabulary, nothing about fractions, nothing about measurement. The job which the teachers had to do in giving the test was already exacting. More would have been an imposition. Some time when occasion serves, richer data may be secured. Meanwhile, however, the writers are grateful for what they have, and they recommend the same state of mind to the reader.

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[29] Douglas, Harl R. "The development of number concept in children of preschool and kindergarten ages." *Jour. of Exp. Psyc.* 8: December, 1925, 443-470.

[30] A copy of the test is given at the end of this chapter.

1. The Circumstances of the Testing

The returns upon which this report is based were obtained from the following cities and villages in Ohio: Akron, Canal, Winchester, Cincinnati, Cleveland Heights, Groveport, Hamilton, Hilliards, Kent, Lakewood, Lancaster, Lima, Marietta, New Albany, Portsmouth, Toledo, Upper Arlington; and from Port Arthur, Texas. In addition to the returns from these cities and villages a few replies were obtained from rural districts.

All the testing was done by first-grade teachers before they had taught the children any number work. In nearly all cases the children were tested during the first two weeks of school. Only first-grade children who had entered school at the opening of the semester in September, 1928, were given these tests. Moreover, except in the case of Cincinnati, the teachers excluded from consideration all children except those whose ages were between six and six-and-a-half. At Cincinnati, owing to the desire of the local officials for complete returns for their own use, each first-grade teacher interviewed all her pupils. Moreover, these teachers likewise secured certain additional information, such as the intelligence of the children and whether or not they had attended kindergarten. For these reasons the returns from Cincinnati were not received until the returns from the other sources had been tabulated. Accordingly, the investigation as conducted in districts other than Cincinnati will be regarded as the 'Primary Investigation.' The results from Cincinnati, so far as they relate to children six to six-and-a-half years old, will be reported in a later section as supplementary data.

In the school districts other than Cincinnati where the first-grade teachers selected their pupils on the basis of age, the method of selection was as follows: The teacher first arranged the names of all her pupils alphabetically, according to the last names. Then, beginning at the top of the list, she selected the first six pupils whose birthdays were between March 1, 1922, and September 1, 1922, and who were entering grade one for the first time. These six pupils were the ones to be tested. If, however, the teacher was willing to test more than six pupils, she simply moved on farther down the alphabetical list, taking each pupil, without skipping any names, who qualified as to date of birth and first entrance to grade one, as already described. It is to be understood, therefore, that the pupils who participated in this experiment were between six and six-and-a-half years of age;

had first entered grade one in September, 1928; had received no instruction in number; and were selected at random from all the children in the various classes represented to whom this description applied.

In all, 1,356 children were interviewed (exclusive of Cincinnati), but owing to defective reports the totals of some of the tests are less than the entire number of children. Table I shows the number of children from each district from whom usable returns were received.[31]

2. The Nature of the Test

As has already been indicated, the interview blanks consisted of six tests. Test I called for counting by rote. It consisted of two parts, (a) counting by ones and (b) counting by tens. Provision was made for two trials of both types of counting, but the second trial was not administered until additional testing had intervened. In fact, throughout the entire interview, where more than one trial of a test was required, no two trials of the same test were consecutive.

TABLE I.—DISTRIBUTION OF PUPILS ACCORDING TO DISTRICTS AND TESTS

Districts	Test I		Test II	Test III	Test IV to Test VI
	A	B			
Akron	569	522	533	591	591
Canal Winchester	21	15	15	22	22
Cleveland Heights	74	72	69	79	79
Counties	43	41	39	47	47
Groveport	20	13	16	20	20
Hamilton	67	62	66	66	67
Hilliards	14	5	11	14	14
Kent	6	5	6	6	6
Lakewood	100	80	94	113	113
Lancaster	15	17	16	20	20
Lima	104	84	105	110	110
Marietta	34	29	33	34	34
New Albany	5	5	3	7	7
Portsmouth	115	78	115	119	119
Toledo	59	54	59	60	60
Upper Arlington	20	19	20	20	20
Port Arthur (Texas)	24	20	22	27	27
Total	1290	1121	1222	1355	1356

[31] In this as in other connections, the returns from rural districts are grouped under the heading "Counties."

Test II was likewise concerned with counting. This time, however, concrete objects were counted and enumerated by the child as he pointed to them or touched them. Two trials of this test were provided for.

Tests III and IV are to be thought of together as an attempt to find out something about the pupil's working knowledge of the numbers from 1 to 10. In Test III, which is called "Number Selection," the examiner placed ten small objects before the child. Suppose these to have been buttons, the examiner began by saying, "Give me five buttons," and followed with other numbers. Three trials of Test III were given.

Test IV is called "Number Identification." Whereas in Test III the child was provided with the name of the number and was required to show its meaning by selecting the appropriate number of objects, in Test IV the examiner exhibited a certain number of objects to the pupil and asked: "How many are there here?" The pupil was supposed to reply 5, 7, or 3, as the case might be, according to the number of objects exhibited. Three trials of Test IV were given.

Considerable thought was expended on the question of number patterns in connection with Test IV. If five objects are presented to a child in a row they constitute, as Lay, Brownell, and others have shown, an appreciably different situation from five objects presented in a quincunx like the arrangement of the spots on a domino, a die, or a playing card. Still other number patterns for small numbers, such as the couplet, the diamond, the quadrant, and the triangle, have been used by Brownell, by Beckmann, and by others. Doubtless each of these patterns affects number perception. Accordingly, the writers have discarded them all, *except as they may arise by chance.* In Test IV the examiner does not place the objects before the pupil. She throws or casts them down on the table, letting them fall as they may. No doubt some of the number patterns are approximated, but when the results from three trials on nine different numbers for each child are taken together, and when 1,356 children are involved, the effect of patterns is obliterated.

In administering Test IV the examiner was cautioned that if one or more of the objects thus presented to the child should escape from the group so that the entire number of objects could not be seen *at a glance,* the presentation should be made again.

It will readily be seen that Tests III and IV are designed to find out whether the children possess what are called 'number concepts,' to a useful and usable degree. Beckmann, in his admirable report already referred to, cautions against inferring that a child possesses

the complete concept of a number upon insufficient evidence. He distinguishes four different criteria or means of telling number concepts. In the first place the child should be able to reproduce 5 in response to some such direction as "Give me 5 buttons." In the second place he should be able to *differentiate* 5 from a number that is not 5, especially from an adjacent number. In other words, he should be able to answer two types of questions—(a) is this or that 5, and (b) is this 4 or 5? In the third place he should be able to *find* 5. In giving this test the examiner presents 5 to the consciousness of the child by some treatment other than naming that number. The direction then is, "Find the same number of buttons." In the fourth place the pupil should be able to *name* 5 when the proper number of objects are presented to him and in answer to the question, "How many buttons are there here?" According to Beckmann's findings the order of difficulty of these functions is as follows: reproduction, differentiation, finding, and naming. The present test, in the interest of brevity, disregards differentiation and finding and requires reproduction and naming—in other words, the easiest and the hardest of the functions.

Although it is not customary to suppose that children know the number combinations when they enter school, Tests V and VI were addressed to the discovery of the extent to which this assumption is true. The method of selecting the combinations for these tests was as follows. First zero combinations and those whose sums exceed 10 were excluded from consideration. The rest were then arranged in the order of their learning difficulty according to the findings of Knight and Behrens.[32] Ten of these combinations were selected, ranging by substantially equal intervals from easy to hard, for use in Test V. Ten other combinations were similarly selected for Test VI.

Test V consisted of ten verbal problems, each requiring for its solution one of the ten selected number facts. These verbal problems were all of one form, namely: If you have——and get (or some other verb of acquiring) ——, how many —— will you have then? Each problem was to be read to the child slowly, and a record of his answer was to be entered on the record blank. The examiner was permitted to read a problem more than once if this seemed to be necessary to secure comprehension.

[32] Knight, F. B., and Behrens, Minnie S. *Learning of the 100 Addition Combinations and the 100 Subtraction Combinations.* New York: Longmans, Green and Company, 1928, pp. 17-20.

In Test VI the examiner sought to ascertain whether the pupil had a functional knowledge of the selected number combinations presented by means of objects. Again suppose the objects to be buttons. The first combination is $2+2$. The examiner shows two buttons to the pupil and asks: "How many buttons have I here?" After the pupil has answered, the two buttons are covered and two more are shown with the same question as before. The examiner now conceals the second group along with the first and asks: "How many are two buttons and two buttons?" If the pupil answers this question correctly, he is held to have satisfied the requirements. He has been able to identify from its components an invisible 4. On the record blank, therefore, his success on this combination is noted by the entry "4" under the heading "Invisible." If, however, the pupil fails to respond correctly, the four buttons are uncovered and the examiner says: "See if you can tell me now how many two buttons and two buttons are." The pupil is now permitted to work this out for himself in any way he can. He may touch the objects, count them, or rearrange them if he chooses. Since, after the objects are uncovered, the pupil's response is made with the objects in view, the entry on the record blank is made under the heading "Visible." Note that the pupil was not called upon to respond to the visible objects unless he failed when they were invisible. Accordingly, if the entry for $2+2$ under "Invisible" on the record blank is 4, there is no entry for $2+2$ under "Visible."

IV. Can Children Count When They Enter Grade One?

1. Rote Counting by Ones

The directions to the teacher read: "Have the pupil count by ones as far as he can. Say something like this, 'Let's see how far you can count.' If the child does not get the idea, count for him as far as three—e.g., 'one, two, three'—telling him to go on from there. Do not count for him beyond three."

While it is true that a child may be able to count by rote without knowing much about the numbers whose names he utters, it is also true that counting is an important basis for number concepts. The function of the series in placing a number is important. Through knowledge of, and practice in, rote counting one secures not only an apprehension of 7 as between 6 and 8, but also of 7 as more than any number which precedes it in the series and less than any number

which follows it. Thus it may be vaguely seen not only that 9 is more than 7, but also that 12 is considerably more than 7.

Again, counting gives drill in the naming of numbers, and as everybody knows, conceptual activity is powerfully supported by the words which name the concepts. No more effective way of learning the names of the numbers has been discovered than learning the names in the serial order—that is, counting. If a child had to learn the names of numbers in a chance and varying order, he would find the task exceedingly difficult.

Counting is basic in the use of number in many other ways. Long before a child knows number facts as such he may be able to respond to the combining of two numbers because of his ability to count. Having three objects and securing two, he may tell you that he has five, not because he knows that 3 and 2 are 5, but because he adds 1, thinking "4" and then adds another 1, thinking "5." We strive hard in school to suppress the counting habit. We do well to do this, but not until after counting has served its purpose. When the child is first practicing his ability to count, when he is counting his fingers, his toes, his blocks, and his books; when he is merely saying, for the pleasure of the rhythm, the number series from 1 to 100, it would be foolish to suppress this activity or to hedge it about in any way. Let the child find out how many pennies he will have if he already has 6 and his father gives him 2. This way of finding out such things is the appropriate way at this time. There comes a time, however, when these less effective ways should be replaced by more effective ways. Probably at no time in the period of school life should a normal child be encouraged or even allowed to get his number facts by counting. Such a method, once indispensable, is by that time infantile.

Table II shows the distribution of 1,290 children according to their success in rote counting by ones. The reader will recall that each pupil was given two trials on this test. These trials, however, were separated from each other by the intervention of the first trial of Test II.

Table II reads: The number of pupils who were not able to count at all (counted to 0) was, at the first trial, 7 or 0.5 percent of the total and, at the second trial, 10 or 0.8 percent of the total; the number of pupils who counted as far as 1, 2, 3, 4, or 5 (1-5) was 35 (2.7 percent) at the first trial and 26 (2.0 percent) at the second trial, and so on.

Another form of statement which may be derived from Table II is the number of pupils who were able to count at least as far as a given number. Thus, of the 1,290 pupils, all but 7 were able to do some counting at the first trial; all but 42 (7 + 35) were able to count at least as far as 11. In other words, 1,126 of these pupils were able at the first trial to count as far as 11. This amounted to 87.3 percent of the pupils. At the second trial, when it may be presumed that the pupils had become a little more accustomed to the examination, 1,150 of them counted at least as far as 11. This amounted to 89.1 percent.

TABLE II.—DISTRIBUTION OF PUPILS ACCORDING TO THEIR ABILITY TO COUNT

Extent of Successful Counting	Trial 1		Trial 2	
	No.	Percent	No.	Percent
0	7	0.5	10	0.8
1– 5	35	2.7	26	2.0
6– 10	122	9.5	104	8.1
11– 15	242	18.8	240	18.6
16– 20	156	12.1	150	11.6
21– 25	36	2.8	44	3.5
26– 30	240	18.6	244	19.1
31– 35	20	1.6	16	1.2
36– 40	154	11.9	142	10.9
41– 45	7	0.5	8	0.6
46– 50	59	4.6	68	5.3
51– 55	1	0.1	1	0.1
56– 60	36	2.8	34	2.6
61– 65	5	0.4	1	0.1
66– 70	24	1.9	20	1.6
71– 75	2	0.1	7	0.5
76– 80	15	1.2	16	1.2
81– 85	2	0.1	1	0.1
86– 90	5	0.4	3	0.2
91– 95	7	0.5	....	....
96–100 and over	115	8.9	155	11.9
Total	1290	100.0	1290	100.0
Median	27.0		27.5	

How many children can count to 10? Certainly 90 percent, since 87 percent of them on one trial and 89 percent of them on the other were able to count at least as far as 11, and the jump from 10 to 11 is relatively hard to make. Similarly, about 60 percent of the children can count to 20; and, as everybody knows, if a child has learned to count to 20, he can learn to count to 100 without much additional effort. The hard spots are in passing from the 9 of one decade to

the even figure of the next decade—from 29 to 30, from 39 to 40, from 49 to 50, and so on. The fact that this is true has a very evident effect upon Table II. Notice the generally serrated character of the figures. They are alternately high and low. Those who can count to 20 seldom fail to reach 29. Those who successfully pass 30 are unlikely to be held up until they reach 40, and so on.

By methods of interpolation it is found from Table II that 75 percent of these children counted successfully to 14 or 15. Finally it is worth noting as an additional item brought out by Table II that an appreciable number of these children can count to 100. Note that, except at the even decades, the frequencies almost disappear until the statistical class "96-100 and over," is reached. In the second trial no children at all failed in the statistical class "90 to 95." It is, therefore, fair to assume that few children, if any, failed throughout the rest of the 90's. Accordingly, it may be said that, at least in the second trial, practically all the 155 pupils recorded in the last statistical class "96-100 and over" could "count to 100." This amounts to 12 percent of the pupils.

Now if it is true that all but about 10 percent of six-year-old pupils upon entering the first grade can count to 10, it does seem foolish to 'teach' such counting two years later when the pupils are eight years old and are entering the third grade. If it is true, as it appears to be, that 75 percent of the children can count to 15, and 60 percent of them to 20 when they are six years old, it serves no good purpose and seems an insult to their intelligence to give them the task of counting the birds in the picture or the girls who are playing 'ring-around-the-rosy.'

There was found to be considerable difference between the children in the different cities in their ability to do rote counting by ones. The median in the case of one city was represented by 13.2 on the first trial and the second trial confirmed the first with a median of 13.5. On the other hand, another city had a median of 60.0 on the first trial and on the second trial the corresponding figure went up to 81.0. In the latter city no child failed to count at least as far as 20, and about half of them counted to 100.

How far should children be able to count? Most courses of study call for counting to 100, and there seems to be good ground for this practice. It affords practice in the naming of all the numbers except those that are combined with the new names *thousand, million,* and

perhaps *billion.* A pupil who can count to 100 has practically acquired all the skill he will ever be called upon to use. On the other hand, anything short of 100 would appear to be incomplete. We appear to have enough need for the first 100 number names to justify us in teaching them in their proper order if they are not known. It is true of course that most of our counting is with small numbers. It may well be that nine-tenths of all our acts of counting do not exceed 15. It would be silly—although it would be quite in line with the principles of some of our dabblers in the curriculum—to advise, on this ground, that the school should abandon all counting beyond 15.

2. Counting by Tens

Supplementary to the counting by ones, the children were tested as to their ability to name the decades in their order—that is, to count by tens. Eleven hundred twenty-one usable returns were received on this topic. Table III gives the results. If the curves for the distributions of Trial 1 and Trial 2 were drawn, they would be almost of the U-type, with high modes at both ends. There is a subordinate mode at 30, owing doubtless in part to the fact that in the directions the teachers were told that if necessary they might start the pupil by saying, "Ten, twenty, thirty." Fully 50 percent of the children were either unable to count by tens at all or were able to count to 100. About 25 percent of them counted to 100. Rather more than 25 percent failed altogether. Fully half of them counted at least as far as 40.

TABLE III.—DISTRIBUTION OF PUPILS ACCORDING TO THEIR ABILITY TO COUNT BY TENS

Extent of Successful Counting	Trial 1		Trial 2	
	No.	Percent	No.	Percent
0	318	28.4	306	27.3
10	53	4.7	52	4.6
20	34	3.0	33	3.0
30	156	13.9	148	13.2
40	56	5.0	44	4.0
50	25	2.2	20	1.8
60	23	2.1	25	2.2
70	13	1.2	11	1.0
80	19	1.7	15	1.3
90	163	14.5	169	15.0
100	261	23.3	298	26.6
Total	1121	100.0	1121	100.0

3. Counting Objects

It has often been pointed out that children may be able to do rote counting, just as they may memorize a rhyme in a foreign language, without knowing the meaning of what they are saying. In order to throw light on this matter, Test II was given. In administering this test, the teacher placed conveniently before the child twenty small objects, such as beads, cubes, pennies, or buttons. He was asked to count as many of these objects as he could, designating the objects as he counted them. If necessary, the teacher was to show him how to do this by counting two or three of the objects. If the pupil became confused or erred in his counting before he reached 20, the number after which this occurred was his score. If he counted all the objects correctly, his score was 20.

Table IV gives the distribution of scores for 1,222 pupils. On the first trial 691, and on the second trial, 728 of the pupils, or 57 and 58

TABLE IV.—DISTRIBUTION OF PUPILS ACCORDING TO THEIR ABILITY TO COUNT OBJECTS

Extent of Successful Counting	Trial 1		Trial 2	
	No.	Percent	No.	Percent
0	18	1.5	19	1.6
1	1	0.1	....	....
2	6	0.5	4	0.3
3	7	0.6	8	0.6
4	14	1.1	10	0.8
5	12	1.0	13	1.1
6	17	1.4	15	1.2
7	10	0.8	15	1.2
8	18	1.5	20	1.6
9	11	0.9	17	1.4
10	47	3.9	37	3.0
11	45	3.7	43	3.5
12	42	3.4	48	4.0
13	54	4.5	49	4.0
14	60	4.9	51	4.2
15	33	2.7	40	3.2
16	30	2.4	30	2.4
17	17	1.4	13	1.1
18	23	1.8	14	1.1
19	66	5.4	48	4.0
20	691	56.5+	728	59.7−
Total	1,222	100.0	1,222	100.0
Median	20		20	

percent, respectively, counted all the objects correctly. Over 90 percent of them counted at least 10 objects correctly, and over 70 percent counted at least 15. Seventy-five percent of the children counted correctly at least through 14.

The evidence from Table IV does not support the view that children are likely to be much more successful in rote counting without objects than they are in counting objects. It will be remembered that, according to Table II, about 60 percent of these children were able to do rote counting to 20 or higher. As already indicated, nearly as many—57 or 58 percent—were able to count objects to 20. Of course, it is not known how much further they would have been able to count objects, and in particular whether their performance would have compared unfavorably with their rote counting in the higher decades. It seems almost certain that it would. Even adults have difficulty in counting a large number of things one by one. Nevertheless it remains true that, so far as Table II and Table IV cover the same ground, almost as large a proportion of the children were successful in counting objects as were successful in rote counting. This statement is further supported by the fact that almost exactly the same proportion of children succeeded as far as 10 in both types of counting—namely about 90 percent.

V. The Usable Number Ideas of Children Entering Grade One

1. Reproducing Numbers

In Test III, as has already been pointed out, the pupil was given three trials—with other tests intervening between trials—in selecting specified numbers of objects from a larger number. He did this in response to the direction, "Give me ——" (the number and kind of objects). It was held that if he could select and hand his teacher five cubes upon being asked to do so, he offered some functional evidence of knowing the meaning of five. This does not by any means indicate that he possesses a complete concept of five. This could only be shown by many kinds of behavior. All that is claimed is that if a child gives you five on being asked to do so, he shows *in that situation* a knowledge of five.

In order to abbreviate the testing, it has been assumed that ability to respond correctly in respect to a given number implies in general the ability so to respond in respect to a smaller number. The teacher began each of the three trials with the number 5. If the pupil suc-

ceeded, he went on to higher numbers and at that trial he was not tested on numbers smaller than 5. At the second and third trials the same procedure was followed. Not infrequently a pupil whose command of the situation was unstable succeeded on 5 at the first trial (and may even have succeeded on one or two higher numbers), only to fail on 5 at the second trial. In the latter case he had no opportunity with the higher numbers but only with those less than 5. On the third trial it was possible that the same child might again succeed with 5 and have an opportunity at the larger numbers.

On this test 1,355 usable returns were received. The success of these 1,355 pupils in reproducing the numbers 5, 6, 7, 8, and 10 is shown in Table V. To the writers this result is remarkable. On the basis of former investigations, one would suppose that the ability of six-year-old children was less than Table V indicates. It is true that the statements of earlier investigators have been more or less general and therefore lacking in reliability. One statement which has been frequently quoted is to the effect that 75 percent of the children who enter school know the numbers 2, 3, and 4. Just what this means no one can say. What is it to *know* a number? Is it to be possessed of a complete psychological concept? If it is, the situation is almost as beclouded as before; for what is a complete psychological concept? The fact is that we an only judge the competence of children in respect to numbers from their *behavior*. In the present test of reproduction one type of behavior has been set up, and according to Table V these six-year-old children give a pretty good account of themselves. They do this on numbers much more difficult than 2, 3,

TABLE V.—THE NUMBER OF PUPILS WHO SUCCEEDED IN REPRODUCING THE NUMBERS 5, 6, 7, 8, and 10

Number of Times Successful in Three Trials	Number of Pupils Who Reproduced				
	5	6	7	8	10
Three times	873	762	727	661	680
Twice only	92	128	148	165	141
Once only	191	198	218	231	217
Total	1156	1088	1093	1057	1038
Failed at all three trials	199	267	262	298	317
Total	1355	1355	1355	1355	1355

and 4. At the same time it surely cannot be asserted that these children possess complete mastery—that is, 100 percent accuracy in all situations—in respect to these numbers. The results of this test of reproduction, however, do indicate that in one very useful way these children possess a not inconsiderable control over the numbers 5, 6, 7, 8, and 10.

How great that control is may now be more fully described. Concerning the number five it was not only found that 1,156 children were able to reproduce it by means of objects, but also that 873 of them did this on all three occasions when the opportunity was presented, and that 92 more did it twice out of three times. In Table VI the figures of Table V have been converted into percents of the total number of children. From Table VI it is noted that the number of children who succeeded in reproducing 5 (at least once) amounted to 85.3 percent of all the children who worked the test. It is also noted that 64.4 percent of these children reproduced 5 on all three trials, while an additional 6.8 percent of them succeeded twice. On how many trials must a child succeed in order to be credited with the ability to reproduce 5? Shall we say that if he does it once out of three times he has, so to speak, passed the test? In the opinion of the writers such a claim cannot be made. If a child is so uncertain of a number that he fails twice out of three times to give you that number of objects, we are bound to say that he has much to learn about the number in question. Shall we say then that if he succeeds twice out of three times he has certainly met the requirements? This is the basis on which some of the items in the Binet test are scored.

Table VI.—The Percent of Pupils Who Succeeded in Reproducing the Numbers 5, 6, 7, 8, and 10

Number of Times Successful in Three Trials	Percent of Pupils Who Reproduced				
	5	6	7	8	10
Three times	64.4	56.2	53.7	48.8	50.2
Twice only	6.8	9.5	10.9	12.2	10.4
Once only	14.1	14.6	16.1	17.0	16.0
Total	85.3	80.3	80.7	78.0	76.6
Failed at all three trials	14.7	19.7	19.3	22.0	23.4
Total	100.0	100.0	100.0	100.0	100.0

For example, a child who successfully counts 13 pennies twice out of three trials is held to have passed that particular requirement of the test. It seems, however, that to meet the requirements of a test standardized in a particular way is one thing, and that to meet the test of competence such as is required in life, whether in school or outside of it, is quite another thing. In school we surely do not feel satisfied when a child qualifies twice out of three times on a fundamental matter. Accordingly, it seems that nothing short of reproduction three times in three trials is fully satisfactory.

At the same time it should be noted that the child who succeeded twice out of three times offers considerable evidence of being on the road to proficiency. A little additional experience—such experience as may quickly be acquired during the first year of school life—will stabilize his reaction and enroll him with the number of those who consistently succeed. Moreover, while it is no doubt true that a few of the children who succeeded but once were only accidentally successful, it is nevertheless probable that most of them are definitely on the way toward the learning of the particular type of behavior required in this test.

It is one thing, therefore, to set up a criterion of success and prescribe definite circumstances under which we shall say a child has qualified, and it is quite another thing to exclude from consideration the meaning of partial success. According to the criterion of three times in three trials it can only be said that a little less than two-thirds of the children can be relied upon to reproduce five. These are the children who have arrived. On the other hand, consideration of the lesser success of those who succeeded either once or twice reveals that 283 more children, or about 30 percent of them, had *some* success. These children are on the way. Only about 15 percent of the children failed altogether in reproducing 5.

Everybody knows that 5 is a much easier number than 6. The most frequent experience a child has with number is in connection with money. Thousands of things are sold at 5 cents each, and only a few at 6 cents. Moreover, we have a coin of the value of 5 cents; we have none whose value is 6 cents. Children easily learn to count by fives, and in fact do so in some of their games. Finally, 6 is a larger number than 5 and this alone would tend to make it more difficult.

Seven hundred sixty-two children out of 1,355, or 56.2 percent, were able to reproduce 6 correctly on all three occasions. An additional 9.5 percent reproduced 6 twice and 14.6 percent succeeded once. The total number of children who had some degree of success amounted to 1088, or 80.3 percent. The latter figure may be compared with 85.3 percent who had some degree of success in reproducing the number 5. The 56.2 percent who had uniform success on 6 may be compared with 64.4 percent who had the same degree of success on 5. Thus the figures bear out the presumed difficulty of 6 in comparison with 5. The writers should, however, have been prepared to find a greater difference. Their speculation would have suggested an appreciably smaller amount of success with the number 6.

Similarly, the pupils did better with 7 than the writers' idea of the difficulty of that number would have suggested. In fact 54 percent of them reproduced it in all three trials. While this was a lower figure than the corresponding percent for the number 6, it was only lower by 2.5 percentage points. Moreover, it was largely offset by the greater partial success of those who gave their teacher the right number once or twice in the three trials. In fact, as Table VI shows, the percent of children who succeeded at least once was quite as great for the number 7 as it was for the number 6.

The number 8 was appreciably more difficult to reproduce than the number 7—especially if we take as the criterion the number and percent of children who were uniformly successful. However, nearly half the children (48.8 percent) had no trouble at all in reproducing 8, and an additional 29 percent proved to be 'on the way.'

The number 10 was a bit easier than the number 8, if we consider total success in reproducing it. The number of children, however, who were partially successful was smaller. On the whole the evidence is that 10 is about as hard as 8.

This leads us to a consideration of the relative difficulty of these five numbers. This may be arrived at by getting a figure for each number which indicates the total amount of success in relation to the possible success. On the number 5, for example, we may multiply 873 by 3, 92 by 2, and 191 by 1. We may then add these results, getting 2,994. The maximum number of successes possible would arise when all the 1,355 children succeeded on all three trials. This would give a maximum of 4,065 successes. We may now express the

percent that the 2,994 successes is of this number. It is 73.7; and this may be taken as the index of difficulty in reproducing the number 5. The indices for all the numbers are as follows:

No.	Index
5	73.7
6	67.4
7	66.3
8	62.6
10	62.5

From these indices it will be seen that 5 is appreciably easier to reproduce than any of these numbers, that 6 and 7 group together, and that 8 and 10 group together.

2. Naming Numbers

It has already been pointed out that Beckmann discusses four different behaviors in respect to a number, and that according to his findings the easiest and hardest of these are reproducing the number and naming the number (including its identification). In the foregoing section the results of the authors' test of reproduction were reported. Consideration will now be given to the naming of the numbers. It is to be understood, however, that this naming includes something more. It includes examination of a group of objects, the determination of how many objects there are in the group, and *then* giving the correct name to this number.

In Test IV the teacher placed before the pupil a certain number of objects and asked him, "How many ——— are there here?" As has already been indicated, the objects were thrown down upon the table so as to avoid, except by chance, the formation of patterns. As in the case of Test III, the teacher began with the number 5 and went up the number series if the child succeeded on 5, and down the series if he failed. Again, as in Test III, three trials were given, with a different activity intervening between the trials.

Tables VII and VIII show the success of the children in naming the numbers 5, 6, 7, 8, and 10. It is at once evident upon comparing these tables with Tables V and VI that the activity with which we are now concerned is more difficult than the activity of reproducing. Nearly all the success scores are lower in Tables VII and VIII than they are in Tables V and VI; and correspondingly, of course, the

figures are higher for failure. While, however, the difference is appreciable and almost uniform, it is not great. Perhaps the best answer to the question is to compare the indices of naming with the indices of reproducing already given. The comparison is shown in Table IX.

TABLE VII.—THE NUMBER OF PUPILS WHO SUCCEEDED IN NAMING THE NUMBERS 5, 6, 7, 8, AND 10

Number of times successful in three trials	Number of Pupils Who Named				
	5	6	7	8	10
Three times	848	704	629	611	571
Twice only	82	134	179	151	187
Once only	175	179	200	217	197
Total	1105	1017	1008	979	955
Failed at all three trials	251	339	348	377	401
Total	1356	1356	1356	1356	1356

TABLE VIII.—THE PERCENT OF PUPILS WHO SUCCEEDED IN NAMING THE NUMBERS 5, 6, 7, 8, AND 10

Number of Times Successful in Three Trials	Percent of Pupils Who Named				
	5	6	7	8	10
Three times	62.5	51.9	46.4	45.1	42.1
Twice only	6.1	9.9	13.2	11.1	13.8
Once only	12.9	13.2	14.7	16.0	14.5
Total	81.5	75.0	74.3	72.2	70.4
Failed at all three trials	18.5	25.0	25.7	27.8	29.6
Total	100.0	100.0	100.0	100.0	100.0

TABLE IX.—COMPARATIVE DIFFICULTY OF REPRODUCING AND NAMING 5, 6, 7, 8, AND 10

Number	Index		Difference
	Reproduction	Naming	
5	73.7	71.1	2.6
6	67.4	62.9	4.5
7	66.3	60.1	6.2
8	62.6	57.9	4.7
10	62.5	56.1	6.4
Mean	66.5	61.6	4.9

This comparison seems to sustain the position that the naming of a number, while it is appreciably harder than reproducing the same number, is not very much harder. The mean difficulty index for reproducing the numbers with which we are here concerned shows that among all the children who took the reproduction tests three times, there were 66.5 percent of successes. On the other hand, the mean difficulty index for naming numbers shows that 61.6 percent of successes were secured among the same children. The difference amounts to this, that if 1,000 children took both tests once, or 500 children twice, or 333 children three times (as here), the expected difference in total number of successes would be 49.

Adopting, as before, the criterion of uniform success as the only satisfactory one, we note that from two to eight percent fewer children qualified in the naming of these numbers than qualified in reproducing them. Nevertheless, the figures are rather impressive for six-year-old children who are just entering the first grade. Since practically the same children who succeeded in naming numbers likewise succeeded in reproducing them, it may be said that something over 60 percent of these children approach an assured competence in handling the number 5. They can make up that number when they are told to do so, and being presented with the appropriate group of objects they can ascertain the fiveness of the group and report it. It is a fair assumption that the other two functional acts distinguished by Beckmann would also be under control—namely "differentiation" and "finding." Similarly, well over 50 percent of the children may be regarded as having this assured competence with respect to 6; and between 40 and 50 percent with respect to 7, 8, and 10.

On the other hand, a great many children are on the way toward competence. This is especially evident with the more difficult numbers such as 7, 8, and 10. In naming numbers, 27.9 percent of the children succeeded once or twice on the number 7; 27.1 percent of the children had the same degree of success on the number 8, and 28.3 percent on the number 10. In fact, upwards of 70 percent of the children had some success even on the most difficult task, namely, identifying and naming 10.

VI. The Ability to Combine Numbers as Shown by Children Entering Grade One

1. Combinations in Verbal Problems

It will be recalled that in Test V the children were examined by means of ten verbal problems as to their knowledge of addition combinations. The following problem is typical of the ten used in this test: "If you have 5 oranges and you get 2 oranges more, how many oranges will you have then?" In this test the following combinations were used in problem form: 5 + 1, 7 + 1, 1 + 9, 4 + 4, 1 + 6, 5 + 2, 8 + 2, 4 + 5, 5 + 3, and 3 + 5. This is the order of difficulty according to Knight and Behrens.

As the teacher read each problem to the child and received his answer, she recorded the answer opposite the combination in question on the score sheet. In other words, the direction to the teacher was to write the child's answer. In practice, however, it was found that some children were simply silent. Whenever a blank occurred after a combination, it was assumed that the child failed to answer and the combination was scored wrong in our tabulations.

Table X shows the success of the children on each of these ten combinations. It will doubtless come somewhat as a surprise to teachers and supervisors—at least it was a surprise to the writers—that so many children were able to handle these number combinations, even when presented in problem form, before they had received any formal instruction in the grades.

It can readily be seen from the way in which the selection was made that the last two or three of the combinations are really difficult.

Table X.—The Number and Percent of Children Who Got Each Combination Right (Test 5, Verbal Problems)

Combination	No. Correct	Percent Correct
5 + 1	970	71.5
7 + 1	866	63.9
1 + 9	657	48.5
4 + 4	500	36.9
1 + 6	658	48.5
5 + 2	594	43.8
8 + 2	591	43.6
4 + 5	282	21.8
5 + 3	431	31.8
3 + 5	365	26.9
Total	5,914	43.6

Yet on these combinations more than 20 percent of the children were able to get the right answer.

That the children, as usual, differed widely in point of ability in the solution of these problems is clearly seen in Table XI. It is there shown that a goodly number of them were quite unable to get any problems right, whereas a considerable number got them all right. The median turns out to be 4.7, from which it may be inferred that nearly half of the children were able to answer at least five problems correctly.

TABLE XI.—DISTRIBUTION OF CHILDREN ACCORDING TO THE NUMBER OF COMBINATIONS RIGHT (TEST 5, VERBAL PROBLEMS)

No. of Combinations Right	No. of Children	Percent of Total
0	143	10.6
1	142	10.5
2	132	9.7
3	160	11.8
4	152	11.2
5	131	9.7
6	134	9.9
7	110	8.1
8	83	6.1
9	78	5.7
10	91	6.7
Total	1,356	100.0
Median	4.7	

2. Combinations with Objects

The last test in the series was Test VI. By means of small objects the children were shown first one and then the other of the terms of an addition combination. As soon as the first term was shown and identified by the child, the objects were concealed; then the same procedure was followed with the second term. The child was then asked for the answer to the combination in this form, "How many are 2 beans and 2 beans?" This part of Test VI will be called the 'Invisible' test, because the objects are concealed from view.

If a child failed on the 'Invisible' test, the objects which had been concealed were brought to view and he was then asked to see if he could now tell "How many are 2 beans and 2 beans?" This part of the test will be called the 'Visible' test.

Table XII gives the results for each combination (1,356 children participating) on the invisible part of the testing with objects. On

exactly half of the combinations 50 percent or more of the children succeeded in giving the right answer without seeing the objects. On no combination did the proportion of successful responses fall much below one third. It is evident that the children found the invisible test slightly easier than the test in the form of verbal problems. The combinations are themselves about as hard as those used in Test V and recorded in Table X. Yet the children did somewhat better with the invisible objects. On no one of the combinations did they fall below an accuracy of 30 percent.

Naturally, if among 1,356 children 895 gave the right answer to $2 + 2$, 461 must have failed to do so. These are the children who took the visible test on that combination. Table XIII shows that on the combination $2 + 2$, of the 461 children who failed in the invisible test, 309 were able to give the right answer when the objects were uncovered. Accordingly, of the children who failed on $2 + 2$, the proportion who were thus able to respond correctly was 67.0 percent. Table XIII gives another figure which has interest in this connection. It shows that on $2 + 2$ the 309 additional correct answers due to showing the objects amounted to 22.8 percent of all the children who took part in Test VI.

Table XIII affords an interesting comment on the extent to which children of this age rely upon concrete data. In the case of every combination, more than half of the children who went wrong when they could not see the objects gave the correct answer when the objects were uncovered. In some cases as many as two-thirds of them did so. Moreover, these additional correct answers increased the proportion of all the children who were able to make correct replies by from 22 to 40 percent (see Table XIII, last column).

The invisible and visible parts of Test VI may now be combined. Table XIV does this. It shows that on $2 + 2$, 1,204 children, or 88.8 percent, obtained the correct answer in one or the other of the two ways allowed by the test. From $2 + 2$ as the easiest combination, the percentages fall off to 71.8 on the hardest combination for these children. This was $4 + 6$. Note, however, that the spread of these percents is not wide. If we leave $2 + 2$ and $3 + 1$ out of account, all the rest lie between 71.8 and 79.8.

The hardest combination (namely, $4 + 6$) is 88th in difficulty among the entire one hundred combinations according to the Knight-Behrens determination. In other words, only 12 of the basic addi-

tion combinations are more difficult. If therefore, we may regard these children as typical, it seems probable that at least one-half and possibly two-thirds of the pupils who enter the first grade can successfully negotiate by means of objects any of the one hundred basic facts of addition.

TABLE XII.—NUMBER AND PERCENT OF CHILDREN WHO GOT EACH COMBINATION RIGHT (TEST 6, INVISIBLE ADDITION)

Combination	No. Right	Percent Right
2 + 2	895	66.0
8 + 1	613	45.2
6 + 1	687	50.7
1 + 7	719	53.0
3 + 1	866	63.9
2 + 4	538	39.7
2 + 8	503	37.1
2 + 6	681	50.2
3 + 7	442	32.6
4 + 6	431	31.8
Total	6,375	47.0

TABLE XIII.—THE SUCCESS IN VISIBLE ADDITION ON THE PART OF CHILDREN WHO HAD ALREADY FAILED ON INVISIBLE ADDITION

Combinations	No. Who Failed on Invisible Addition	No. and Percent of These Correct on Visible Addition		Percent Correct on Total Participants (1356)
		No.	Percent	
2 + 2	461	309	67.0	22.8
8 + 1	743	411	55.3	30.3
6 + 1	669	362	54.1	26.7
1 + 7	637	363	57.0	26.8
3 + 1	490	334	68.2	24.6
2 + 4	818	542	66.3	32.5
2 + 8	853	483	56.6	35.6
2 + 6	675	376	55.7	27.7
3 + 7	914	542	59.3	40.0
4 + 6	925	543	58.7	40.0

The remarkable success of the children in doing this test is further made clear by Table XV. Here the pupils are distributed according to the number of combinations to which they reacted correctly. Even when the objects were invisible, the pupils made a pretty good showing. As has already been suggested by a comparison of Tables X and XII, this type of test proved somewhat easier than the test in

TABLE XIV.—THE SUCCESS OF THE CHILDREN ON ADDING WITH OBJECTS, INVISIBLE AND VISIBLE ADDITION COMBINED

Combinations	Total Number Right	Total Percent Right
2 + 2	1,204	88.8
8 + 1	1,024	75.5
6 + 1	1,049	77.4
1 + 7	1,082	79.8
3 + 1	1,200	88.5
2 + 4	1,080	72.2
2 + 8	986	72.7
2 + 6	1,057	77.9
3 + 7	984	72.6
4 + 6	974	71.8

TABLE XV.—DISTRIBUTION OF CHILDREN ACCORDING TO THE NUMBER OF COMBINATIONS RIGHT (TEST 6, ADDITION WITH OBJECTS)

No. of Combinations Right	Invisible		Invisible and Visible Combined	
	No.	Percent	No.	Percent
0	154	11.4	58	4.3
1	121	8.9	37	2.7
2	125	9.2	30	2.2
3	136	10.0	31	2.3
4	144	10.6	41	3.0
5	140	10.3	37	2.7
6	114	8.4	55	4.1
7	110	8.1	77	5.7
8	108	8.0	101	7.5
9	92	6.8	197	14.5
10	112	8.3	692	51.0
Total	1,356	100.0	1,356	100.0
Median	5.0		10.0	

the form of verbal problems—at least if we may regard each set of combinations as equally difficult. Note that on the invisible test the median number of combinations correct per child was 5.0, whereas the median for verbal problems (Table XI) was 4.7.

When, however, the children see the objects, they prove to be capable of meeting the requirements. Table XV shows that when invisible and visible successes are both counted, more than half the children got all the combinations right.[33]

Of course, the writers do not wish to be understood as regarding a correct response made with the objects visible as an equivalent ac-

[33] This does not mean that two successes were ever counted for any one child. Only those who erred on the invisible test took the visible test.

complishment to a correct response when the objects are concealed. When the child sees the objects, he may count them or group them in one way or another, and this of course is not nearly so efficient as giving the answer without the assistance of the objects. There is, however, a certain significance to the visible test. It shows that something like half the children who were in difficulty were able to extricate themselves. This has a bearing upon the degree to which children in the first grade may be relied upon to help themselves.

VII. Summary of the Primary Investigation

On the basis of individual interviews with 1,356 children in 16 cities and villages and in a few rural districts and to the extent that the data are representative, the writers have found that six-year-old children possess, when they enter grade one, a considerable knowledge of number. The following statements may be made.

1. *Rote counting.* In counting by ones about 90 percent of the children succeeded at least as far as 10 and about 60 percent of them at least as far as 20. The typical (median) child counted to 27 or 28. One in eight of the children counted to 100. Half the children counted by tens at least as far as 40, while one quarter of them counted in this manner to 100.

2. *Counting with objects.* The test used did not require counting of this type beyond 20. The majority of the children (in fact about 60 percent) 'broke the test' by counting as far as they were permitted to go. Seventy-five percent counted at least as far as 14.

3. *Reproducing numbers.* This is one of the writers' two tests of number concepts. Practically all the children 'knew' the numbers from 1 to 4. Eighty-five percent of them reproduced 5 at least once out of three trials, and nearly two-thirds of them did so on all three trials. The numbers 6 and 7 were practically equal in difficulty. Fully 80 percent of the children reproduced them once and about 55 percent three times. The number 8 was of substantially the same difficulty as the number 10. Over 75 percent of the children reproduced each of these numbers once, and about half of them did so three times.

4. *Naming numbers.* This is somewhat more difficult as a test of number concepts than reproducing numbers, and the percents of children who succeeded are from 4 to 8 percentage points less. Yet

even here the children did well. Forty-two percent of them succeeded every time on the hardest number, namely 10. An additional 28 percent of them showed that they were 'on the way' to a reliable understanding of 10 by succeeding either once or twice. Thus a total of 70 percent responded correctly at least once on the hardest number of the series. The corresponding percent for 8 was 72; for 7, 74; for 6, 75; and for 5, 81.

5. *Combinations in verbal problems.* According to the Knight-Behrens difficulty rankings, the addition combinations used in this test ranged from 11th to 81st. Some of the children gave correct answers to all these combinations—in fact 91, or about 7 percent, of them did so. Very nearly half the children got five combinations right, and only 11 percent of them failed to get any right. The combinations ranged in difficulty for these children from $5 + 1$, which 71.5 percent of the children answered correctly, to $4 + 5$, which only 22 percent answered correctly.

6. *Combinations with objects.* The ten addition combinations in this test ranked from 10th to 88th according to the rankings of Knight and Behrens. Half of the children answered at least five of these combinations correctly when the objects were concealed. When the objects were visible, more than half the children answered all the combinations correctly. The 88th combination $(6 + 4)$ was the hardest one for these children, as was to be expected from its placement in the Knight-Behren's list. Yet this combination was answered correctly by nearly one-third of the children (31.8 percent) when the objects were concealed and by 40 percent more of them when the objects were uncovered.

VIII. Supplementary Findings

1. The Cleveland Data

While the foregoing investigation was in progress, a test of kindergarten children was in progress at Cleveland. This test was quite different from the writers', but its object was somewhat the same—namely, to find out how much number ability was possessed by children at an age when the school ordinarily assumes that they have none. Through the courtesy of Director William L. Connor, of the Bureau of Research of the Cleveland Public Schools, a preliminary report of this investigation was made available. As far as the data

permit, a comparison will now be made between the children at Cleveland (hereafter called the C. children) and the children from various parts of Ohio used in the present study (hereafter called the O. children).

1. The C. children were tested in counting in a peculiar way which does not lend itself to comparison with the O. children. They were required to put twenty tacks in a board. Among 1,242 of these kindergarten children 26 percent succeeded in both of two trials, while 22 percent more succeeded once. It is clear that this test has difficulties apart from counting.

2. In counting twenty beads from one tray to another (two trials), 31 percent of the C. children succeeded both times and 20 percent more succeeded once. The O. children succeeded in counting 20 objects in 56.5 percent of their first attempts and in 59.7 percent of their second attempts. But their better success was no doubt largely due to the simplicity of the form of counting. They had *only* 20 objects before them. The C. children were given 56 beads in a tray and were directed to put 20 of them into an empty tray. Moreover, each child was given at the same time a tray containing 25 beads. He was to count 20 of these into a second empty tray. Thus each child had four trays at hand simultaneously. The C. children would probably have been quite equal to the O. children if their task, independent of the counting, had not been so difficult. This opinion is confirmed by the facts shown in the next paragraph.

3. In rote counting by ones, returns were received from 313 C. children, of whom 18 percent counted correctly to 50 or higher. This may be compared with 16 and 18 percent of O. children who counted to 51 or higher on the first trial and second trial, respectively. Seventy-six percent of the C. children counted to 20 or more; 56 and 59 percent of the O. children on the first and second trials, respectively, counted to 21 or more. Ninety-seven percent of the C. children could count at least to 10; about 90 percent of the O. children did so.

4. Of 1,242 children at Cleveland, 32 percent were completely successful in both trials of a test in which they were to reproduce the numbers 5, 7, 9, and 11 by putting the required number of marks in designated spaces on a sheet of paper. According to the marking plan for this test, the highest obtainable score was 10, and the median of all the scores actually made was 8. This is a very creditable performance. The test of reproduction given to the O. children is not

sufficiently like this one to make direct comparison possible.

5. The Cleveland battery likewise included, as did that devised by the writers, a test in recognizing and naming numbers. Again, however, the plan of scoring does not permit comparison with results obtained from the O. children.

6. The children at Cleveland were tested, as the O. children were, with addition combinations in verbal problems. The problem situations, however, seem to be more difficult in some cases than in those devised by the writers, and to vary considerably in difficulty from problem to problem. For example, something considerably more than an ability to respond to 7 + 1 is required in the problem "Bobbie is seven years old. Tom is one year older. How old is Tom?" Problems with invisible units, like years, are rather hard. This undoubtedly accounts for the fact that on three out of four identical combinations the Cleveland results are lower than those for the O. children (see Table XVI).

TABLE XVI.—COMPARISON ON COMBINATIONS IN PROBLEMS

Combination	Percent Correct	
	C. Children	O. Children
8 + 2	36.4	43.6
5 + 1	47.6	71.5
7 + 1	30.1	63.9
4 + 4	37.0	36.9

7. Finally, the Cleveland inquiry contained a test of addition combinations with objects. The 'first trial' of this test is the same in nature as the 'invisible' part of Test VI described in this report. Table XVII shows the results on combinations common to the two tests. The Cleveland testing involved 313 children.

TABLE XVII.—COMPARISON ON COMBINATIONS WITH OBJECTS

Combination	Percent Correct	
	C. Children	O. Children
2 + 2	75.7	66.0
1 + 7	46.6	53.0
2 + 4	37.1	39.7
2 + 8	27.2	37.1
2 + 6	28.8	50.2
Mean	43.1	49.2

It is worth noting that when the C. children who failed on a given combination were given another trial, an additional 7 percent of them (on the average) gave correct responses. In a sense this brings the Cleveland results quite up to those for the O. children.

The results secured by Director Connor and his colleagues at Cleveland may be interpreted in substantially the same way that those of the present investigation may be interpreted. Young children use numbers and possess number ideas to an extent that has not usually been recognized. Commenting on his test results, Mr. Connor has this to say:

> The amount of number used by little children, as revealed by this test, has proved to be a revelation to all of those connected with the teaching and supervision of little children in the Cleveland schools. It seems not at all improbable that the fear that school officers seem to have had of crowding number concepts on little children too soon is wholly without foundation in fact. When one considers the ordinary teaching materials in arithmetic in the light of these findings, fragmentary as they are, it would almost seem as if the course-of-study makers, textbook writers, and teachers of the last generation had conspired to retard by every means possible the natural growth of number concepts which takes place in little children from four to seven years of age.

2. The Cincinnati Data

The data presented in the report of the Primary Investigation indicate the number abilities of 1,356 school entrants in seventeen cities and towns of Ohio.[34] The writers included the city of Cincinnati in their inquiry and received returns from that city on about 1,100 children between the ages of six and six-and-a-half. The performance of these 1,100 Cincinnati children parallels very closely that of the pupils reported in the Primary Investigation; that is to say, all the number abilities were possessed by practically the same proportion of pupils and by precisely the same proportion in several cases. The degree of agreement is shown in Tables XVIII-XXV which follow:

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[34] One of these towns was Port Arthur, Texas; but for brevity the group will be referred to as "17 places in Ohio."

TABLE XVIII.—ROTE COUNTING BY ONES (see Table II)

No. of Pupils Who Counted:	1,356 Entrants in 17 Places in Ohio		1,067 Entrants in Cincinnati	
	Trial		Trial	
	1	2	1	2
	(Percents)	*(Percents)*	*(Percents)*	*(Percents)*
To 11	87.3	89.1	87.7	88.6
To 20	56.4	59.0	61.7	63.3
To 100	8.9	11.9	9.3	10.7

TABLE XIX.—ROTE COUNTING BY TENS (see Table III)

No. of Pupils Who Counted:	1,356 Entrants in 17 Places in Ohio		1,035 Entrants in Cincinnati	
	Trial		Trial	
	1	2	1	2
	(Percents)	*(Percents)*	*(Percents)*	*(Percents)*
To 100	23.3	26.33	21.9	22.3
To 40	28.5	27.3	31.8	28.9
Not at all	50.0	52.5	51.5	54.0

TABLE XX.—COUNTING TWENTY OBJECTS (see Table IV)

Task Achieved	17 Places in Ohio *(Percents)*	Cincinnati (1014) *(Percents)*
Counted all objects	58	62
Counted at least to 10	90	90
Counted at least to 15	70	75
Counted through 14	75	78

TABLE XXI.—NUMBER REPRODUCTION (see TABLE V)

Reproduced	17 Places in Ohio *(Percents)*	Cincinnati (1123) *(Percents)*
5	85.3	83.2
6	80.3	73.4
7	80.7	71.7
8	78.0	67.5
10	76.6	67.0

TABLE XXII.—IDENTIFYING AND NAMING NUMBERS (see TABLES VII–VIII)

Named	17 Places in Ohio *(Percents)*	Cincinnati (1123) *(Percents)*
5	81.5	80.2
6	75.0	68.5
7	74.3	62.9
8	72.2	62.6
10	70.4	60.2

TABLE XXIII.—COMBINATIONS IN PROBLEMS (see TABLE X): PERCENT WHO GOT EACH COMBINATION RIGHT

Combination	17 Places in Ohio (*Percents*)	Cincinnati (*Percents*)
5 + 1	71.5	71.9
7 + 1	63.9	65.0
1 + 9	48.5	53.2
4 + 4	36.9	34.9
1 + 6	48.5	50.8
5 + 2	43.8	42.5
8 + 2	43.6	43.3
4 + 5	21.8	21.6
5 + 3	31.8	34.2
3 + 5	26.9	27.5

TABLE XXIV.—COMBINATIONS WITH OBJECTS (see TABLE XII)

Combinations	17 Places in Ohio (*Percents*)	Cincinnati (1123) (*Percents*)
2 + 2	66.0	70.1
8 + 1	45.2	46.1
6 + 1	50.7	54.5
1 + 7	53.0	53.8
3 + 1	63.9	70.5
2 + 4	39.7	39.3
2 + 8	37.1	36.0
2 + 6	50.2	37.2
3 + 7	32.6	27.4
4 + 6	31.8	27.4

TABLE XXV.—COMBINATIONS WITH OBJECTS (VISIBLE AND INVISIBLE COMBINED): DISTRIBUTION OF PUPILS ACCORDING TO THE NUMBER OF COMBINATIONS CORRECT (see Table XIV)

Combination	17 Places in Ohio (*Percents*)	Cincinnati (1123) (*Percents*)
2 + 2	88.8	90.3
8 + 1	75.5	76.2
6 + 1	77.4	81.4
1 + 7	79.8	81.9
3 + 1	88.5	90.2
2 + 4	72.2	82.2
2 + 8	72.7	73.5
2 + 6	77.9	76.2
3 + 7	72.6	71.9
4 + 6	71.8	72.5

The great deviation of Cincinnati returns from those of other Ohio towns is seen in Table XXI on the ability to reproduce 5, 6, 7, 8, and 10 and in Table XXII on the ability to name these same numbers. In reproducing and naming 7, 8, and 10, about 10 percent

fewer of the Cincinnati children succeeded than in the other centers. In reproducing and naming the number 6, the Cincinnati children fell 7 percent below the other places, while on 5 they fell about 2 percent below the others.

IX. Differences in the Number Abilities of Boys and Girls

Analyzed separately for boys and girls, the returns from Cincinnati show a superiority in favor of the girls. Of 545 boys, three could not count by rote at all, while only one girl among 462 appeared altogether destitute of counting ability. Of the boys, 10.4 percent counted to 100, while 9.3 percent of the girls were able to go through this test to the finish. Including those who counted beyond 50, 19.1 percent of the boys achieved 50, and 27.3 percent of the girls. The median performance for boys was 24.5, while the girls excelled with a median of 29. This was the same test which is shown in the Primary Investigation in Table II of this report. A second trial of this same test resulted in the boys' median advancing to 26.5 and the girls' to 29.7.

In rote counting by tens the boys again fell behind the girls, though the difference was small. For example, on the first trial 32.2 percent of the boys and 31.3 percent of the girls were unable to count at all, while 36.4 percent of the boys were stopped at the first hurdle, namely 10, and 34.3 percent of the girls. For completing the test (first trial) by counting to 100, the percent of boys was 22.0; that of girls, 21.5. The percent of boys who were able to count at least to 50 was 46.5, and that of the girls was 52.5.

For counting twenty objects with enumeration, of 544 boys 10 were unable to start, and 4 out of 473 girls were in the same predicament. Those who ran the whole race were 58.2 percent of the boys and 67.0 percent of the girls. Of the boys 85.3 percent and of the girls 94.1 percent were able to count by objects at least to 10.

Test III on number reproduction is shown for the Preliminary Investigation in Table VII. When the supplementary data at Cincinnati were analyzed separately for boys and girls, the superiority of the girls in reproducing all of these numbers is unquestioned. The facts are shown in Table XXVI.

Table XXVI also shows the ability of 600 boys and 523 girls (Cincinnati data) to identify and name the numbers 5, 6, 7, 8, and 10.

TABLE XXVI.—A COMPARISON BETWEEN BOYS AND GIRLS AS TO NUMBER CONCEPTS

Number	Reproduction		Identifying and Naming	
	Boys (Percent)	Girls (Percent)	Boys (Percent)	Girls (Percent)
5	81.0	85.8	78.6	83.3
6	70.3	77.0	66.6	70.5
7	68.1	75.8	60.5	65.5
8	63.3	72.5	60.0	65.5
10	65.7	68.6	56.6	64.3

For the Preliminary Investigation the results on the same test are shown in Table IX. The superiority of the girls is again apparent.

As to the solution of ten problems involving number combinations, the median performance of the girls at Cincinnati was 5.1 combinations correct out of 10; that of the boys was 4.5. The difference in percents correct on each combination for boys and girls is shown in the first three columns of Table XXVII.

TABLE XXVII.—A COMPARISON BETWEEN BOYS AND GIRLS AS TO SUCCESS ON COMBINATIONS (Figures in Percents)

Problems			With Objects				
			Comb.	Invisible		Visible and Invisible	
Comb.	Boys	Girls		Boys	Girls	Boys	Girls
5 + 1	70.4	73.8	2 + 2	69.2	71.3	90.7	91.8
7 + 1	61.8	68.8	8 + 1	45.6	46.7	72.5	80.8
1 + 9	51.2	55.6	6 + 1	52.5	57.0	78.3	85.1
4 + 4	35.3	34.4	1 + 7	54.0	53.7	80.9	83.2
1 + 6	46.7	55.6	3 + 1	69.3	72.0	88.9	91.6
5 + 2	42.5	42.6	2 + 4	39.3	39.3	78.6	86.3
8 + 2	42.3	44.5	2 + 8	36.2	35.9	71.0	76.3
4 + 5	23.2	19.9	2 + 6	36.2	37.3	75.0	77.6
5 + 3	34.5	34.0	3 + 7	28.5	26.2	68.5	76.0
3 + 5	29.5	25.5	4 + 6	28.0	26.8	71.0	74.3

In Test VI, combinations with objects (invisible), the median performance for boys and girls in Cincinnati was identical with respect to the number of combinations correctly solved, namely, 4.8 out of 10. Percents for each combination are given in Table XXVII. If with the figures just given, there are combined the figures which show the success of the children with the same combinations when

they were permitted to look at the objects, the percents in the last two columns of Table XXVII are obtained.

Concluding, therefore, this section on sex differences as revealed in the returns from Cincinnati, we observe that rather consistently the six-year-old girls surpassed the six-year-old boys. There are, to be sure, exceptions, but these exceptions are not numerous. On the other hand, the differences due to sex may rightly be described as small. There is little if anything in the data to justify a difference in treatment between boys and girls at the time they enter grade one. Any differences of this sort should be on the basis of individuals.

X. The Effect of Kindergarten Training at Cincinnati on the Number Abilities of Pupils

Analyzed separately for those who had been to kindergarten (hereafter called the K. children) and those who had not (hereafter called the NK. children) the returns from Cincinnati show a superiority in favor of those who had received kindergarten training. Of the 636 K. children, 12 could not count by rote at all; while 4 of 371 NK. children failed to the same extent—a result slightly in favor of the NK. children. On the other hand, of the K. group, 13.4 percent and of the NK. group only 4.6 percent were able to count to 100. Again, 28.1 percent of the K. children and only 13.8 percent of the NK. children achieved at least 50.

On the first trial the median performance in rote counting by ones for the K. group was 29.3. The corresponding figure for the NK. children was 19.1. Thus it appears that typically the kindergarten children were able to count ten more than their classmates who had not attended kindergarten.

In Test Ib (counting by tens), considering the first trial only, 28.8 percent of the K. children were unable to start; while the corresponding percent for the NK. children was 37.3. Of the K. children 26.1 percent completed the test by counting to 100; of the NK. children only 14.1 percent did so. The percentages of the children who were able to count at least to 50 were: K. children 54.0; NK. children 37.7.

In Test II, counting twenty objects, the percents in the two groups of those who were unable to start were just the same, namely, 1.1. Of the K. group, however, 70.3 percent completed the test, while only 47.7 of the NK. group did so.

Moreover, 93.7 percent of the K. children, but only 82.3 percent of the NK. children, were able to count by objects at least to 10.

When the Cincinnati results for Tests III and IV on number concepts are analyzed separately for those who went to kindergarten and those who did not, the superiority of the former, in terms of percents correct, is unquestioned (Table XXVIII).

As to the solution of number combinations in problems, the median performance of the K. group was 5.4 combinations correct out of 10; while the median for the NK. group was only 3.5. The comparison by combinations for the two groups is shown in the first three columns of Table XXIX. The differences in favor of the kindergarten children are too large and too consistent to be neglected.

The percents of success for K. and NK. children on combinations with objects are likewise shown in Table XXIX. In respect to 'in-

TABLE XXVIII.—A COMPARISON BETWEEN KINDERGARTEN AND NON-KINDERGARTEN CHILDREN AS TO NUMBER CONCEPTS

Number	Reproduction		Identifying and Naming	
	K Group (*Percent*)	NK Group (*Percent*)	K Group (*Percent*)	NK Group (*Percent*)
5	91.9	73.6	84.3	72.3
6	79.0	63.0	73.5	58.8
7	77.8	60.0	68.8	51.5
8	75.0	53.5	67.8	52.5
10	73.0	55.6	67.6	46.0

TABLE XXIX.—A COMPARISON BETWEEN KINDERGARTEN AND NON-KINDERGARTEN CHILDREN AS TO SUCCESS ON COMBINATIONS (Figures in Percents)

Problems			With Objects				
			Combination	Invisible		Visible and Invisible	
Comb.	K	NK		K	NK	K	NK
5 + 1	78.2	60.0	2 + 2	75.0	61.0	93.4	84.5
7 + 1	72.2	51.5	8 + 1	51.8	35.2	83.8	61.8
1 + 9	58.5	43.1	6 + 1	60.7	42.8	85.8	72.8
4 + 4	39.1	26.8	1 + 7	59.5	42.8	86.3	73.6
1 + 6	55.5	41.8	3 + 1	73.0	65.8	92.7	85.1
5 + 2	48.1	31.9	2 + 4	42.4	33.4	83.2	80.1
8 + 2	48.6	33.2	2 + 8	41.2	26.1	78.6	63.4
4 + 5	24.1	16.9	2 + 6	43.0	26.1	80.7	67.6
5 + 3	38.5	26.6	3 + 7	30.5	21.4	78.1	60.0
3 + 5	29.3	24.3	4 + 6	29.5	23.5	75.8	63.7

visible addition' the differences in favor of the K. group are again consistent and of considerable magnitude. Of course the figures (shown in the last two columns of Table XXIX) which combine invisible and visible addition approach closer to 100. This makes the differences appear smaller than they really are.

In connection with this part of the report it should be noted that the median mental age of the group who had had kindergarten training was 6 years and 6 months and that the median mental age of the children who had not gone to kindergarten was 5 years and 10 months.

This raises a very interesting question as to whether it is the kindergarten or the superior intelligence which produces the consistent superiority of children who had the benefit of kindergarten training. To some there is likewise another interesting question involved. Suppose we were to test these children not only on their knowledge of number but also on other things which they have learned prior to entering the first grade. It is conceivable that we should find the same superiority of the kindergarten children all along the line, but it is generally agreed that a pooling of the products of learning constitutes a rather good measure of intelligence. If this is the case we are confronted by the query, does attendance at kindergarten improve intelligence as intelligence is customarily measured? These questions are left to the speculation of the reader.[35] These two facts are all that the data warrant: first, that among these children of Cincinnati those who had attended kindergarten consistently surpassed in number knowledge those who entered the first grade at the same time without having passed through the kindergarten; and second, that these children from the kindergarten were recorded by their teachers as averaging six years six months of mental age while those who had not attended kindergarten were recorded as averaging only five years ten months of mental age—thus indicating intelligence quotients of approximately 104 and 93, respectively.

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[35] The reader will find material of interest on this point in the *Twenty-Seventh Yearbook* of the Society, particularly in Part I, Chapter XV.—*Editor.*

XI. Samples of Materials

The Pupil's Interview Record and the Teacher's Interview Guide appended here will make clear the details of the Number Test discussed in this chapter.

NUMBER TEST FOR PUPILS ENTERING GRADE ONE—PUPIL'S INTERVIEW RECORD

Name of Pupil.......................**Date of Birth**............**School**............**City**............

Teacher.................**Date of Interview**..........**How many months did pupil attend kindergarten?**......

Test 1. Counting by rote consecutively to:

	(a) By Ones	(b) By Tens
Trial 1..		
Trial 2..		

Test 2. Counting objects with enumeration accurately to:

Trial 1..............

Trial 2..............

Test 3. Number selection (Check when the correct response is given, otherwise leave the space blank.)

	Trial 1	Trial 2	Trial 3
	Those Who Succeed on 5		
5			
7			
9			
8			
10			
6			
	Those Who Fail on 5		
5			
3			
1			
4			
2			

Test 4. Number identification (Check as in Test 3.)

	Trial 1	Trial 2	Trial 3
	Those Who Succeed on 5		
5			
7			
9			
8			
10			
6			
	Those Who Fail on 5		
5			
3			
1			
4			
2			

In Tests 5 and 6 write the child's answer in the brackets.

Test 5.		**Test 6.**	Invisible	Visible
1. 5 + 1	( )	1. 2 + 2	( )	( )
2. 7 + 1	( )	2. 8 + 1	( )	( )
3. 1 + 9	( )	3. 6 + 1	( )	( )
4. 4 + 4	( )	4. 1 + 7	( )	( )
5. 1 + 6	( )	5. 3 + 1	( )	( )
6. 5 + 2	( )	6. 2 + 4	( )	( )
7. 8 + 2	( )	7. 2 + 8	( )	( )
8. 4 + 5	( )	8. 2 + 6	( )	( )
9. 5 + 3	( )	9. 3 + 7	( )	( )
10. 3 + 5	( )	10. 4 + 6	( )	( )

NUMBER TEST FOR PUPILS ENTERING GRADE ONE

B. R. Buckingham
AND
Josephine MacLatchy

TEACHER'S INTERVIEW GUIDE

This number test has been prepared with the hope that it may, in a measure, help to define the number concepts and number abilities which children have when they enter the first grade. This so-called "Teacher's Interview Guide" contains the complete directions for giving the test. In addition, if you are using the test, you will need a copy of the "Pupil's Interview Record" for each pupil examined.

The "Pupil's Interview Record."—This is intended to show the pupil's score on each test or test item. As soon as the pupil has responded to each requirement, make the appropriate entry in the space provided for the purpose. Be careful when giving the test to avoid signs of approval when the child's answer is correct or of disapproval when the answer is wrong.

Selecting the pupils to be tested.—First arrange the names of all your pupils alphabetically according to their last names. Beginning at the top of the list select the first six names of pupils whose birthdays were between March 1, 1922, and September 1, 1922, and who are now entering Grade One for the first time. These are the names of the pupils to be tested. If you wish to test more than six pupils, simply move on further down the list, taking each pupil (without skipping) who qualifies as to date of birth and first entrance to Grade One.

Supplementary record.—The *quality* of the pupil's reactions is often more illuminating than his *quantitative* record. We shall, therefore, be grateful for notes on things of this sort—your impressions of the way the child is thinking and working as he takes the test. Please use the back of the "Pupil's Interview Record" for recording such impressions. If you know the child's home and social environment well enough to comment on the special advantages which he has had or the drawbacks which have confronted him, be sure to include such information. If the home is one in which the child has opportunity to share in the social life of the family, comment upon this. Or, if he comes from a home in which all toys are provided and he is left largely to the care of servants, mention this fact. If the child has attended kindergarten, the teacher of the kindergarten may be able to assist you in this matter.

Test 1. Rote Counting without Objects

(a) Have the pupil count by ones as far as he can. Say something like this, "Let's see how far you can count." If the child does not get the idea, count for him as far as three—e. g., "one, two, three"—telling him to go on from there. Do not count for him beyond three.

(b) Now have the pupil count by tens to 100. Make every reasonable effort to have him understand what is wanted. If necessary start him by

saying, "ten, twenty, thirty," telling him to go on from there. Do not count for him beyond thirty.

TEST 2. COUNTING WITH ENUMERATION

For this purpose provide 20 small objects identical in size, shape, and color. The beads used in kindergarten, pennies, small blocks or cubes, beans, or buttons are suggested. Have the pupil count the objects, either by pointing with his finger or by moving each object aside as he enumerates it, up to 20 if he can. Show him how to do this by counting two or three of the objects. If the pupil becomes confused in his counting, the number after which this occurs is his score. If he counts all the objects correctly, he scores 20.

Repeat Test 1.

Repeat Test 2.

TEST 3. NUMBER SELECTION

The small objects used in Test 2 will serve here. Place ten of them before the child. Then say, "Give me 5 . . . . ." (naming the objects used). If the child gives you five, then ask him to give you the following numbers of objects in succession: 7, 4, 8, 10, and 6. If he *fails* to give you 5 objects, ask him to give you these numbers in succession: 3, 1, 4, and 2.

TEST 4. NUMBER IDENTIFICATION

Placing a certain number of objects before the pupil, say, "How many pennies (or whatever objects are used) are there here?" As you say this throw down before him the number of pennies or other objects specified in the next paragraph.

The first presentation will be of 5 objects. If the pupil answers correctly, then present in turn groups of 7, 4, 8, 10, and 6 objects. If the pupil does not recognize 5 objects correctly, the series of presentations will then be 3, 1, 4, and 2.

Caution.—If 1 or more of the objects escape from the group so that the entire number cannot be seen *at a glance* or if one or more fall from the table, that presentation should not be counted, but should be made again.

Note.—You will recall that supplementary records are requested in the general directions accompanying this test. If you find that the child directly recognizes the number of objects without counting them, a record of this fact is especially desired. For example, if the child "sees four" without stopping to count, please say so. If he can apprehend five directly, make a note of this fact. Give in any case the *largest number* which you feel sure the child recognizes without counting. Use this form of statement: "Apprehends . . . . . without counting." Again, if the child recognizes a smaller group within a larger one and counts the rest of the objects presented—for example, sees 3 within a group of 7 and then counts 4, 5, 6, 7—note this on the back of the child's record sheet. Of course you cannot always tell about this, but if you are fairly sure, enter on the back of the Pupil's Interview Record, "Sees . . . . . in

. . . . ." Any other ways by which the child gets his results should be noted if you are able to recognize them.

Repeat Test 3.
Repeat Test 4.
Repeat Test 3.
Repeat Test 4.

TEST 5. FUNDAMENTAL COMBINATIONS IN PROBLEMS

This test may well be presented as a game—a game of "Finding How Many You have." After the game idea has been established, read each problem slowly and record the child's answer on the Pupil's Interview Record. A problem may be read more than once if necessary for its comprehension.

1. If you have 5 pencils and get 1 more, how many pencils will you have then?
2. If you have 7 apples and get 1 apple more, how many apples will you have then?
3. If you have 1 marble and get 9 more marbles, how many marbles will you have then?
4. If Mary had 4 paper dolls and got 4 more, how many paper dolls did she have then?
5. If you have 1 book and get 6 books more, how many books will you have then?
6. If you have 5 oranges and you get 2 oranges more, how many oranges will you have then?
7. If you have 8 pennies and you get 2 pennies more, how many pennies will you have then?
8. If you have 4 jacks and buy 5 more, how many jacks will you have then?
9. If you have 5 rabbits and buy 3 rabbits more, how many rabbits will you have then?
10. If you have 3 balls and you get 5 balls more, how many balls will you have then?

TEST 6. FUNDAMENTAL COMBINATIONS WITH OBJECTS

Again use 10 small objects—beads, buttons, or beans—which are alike in size, color, and shape. The first combination in this test is $2 + 2$. Show 2 beans to the pupil and ask, "How many beans have I here?" After the pupil has answered (correctly or incorrectly), cover the 2 beans and then show 2 more, asking the same question as before. Concealing the second group along with the first, ask, "How many are 2 beans and 2 beans?" If the pupil answers correctly, go on to the next combination; if he fails, remove the cover and say to him, "See if you can tell me now how many 2 beans and 2 beans are?" Let the pupil work this out for himself if he can. He may be allowed to touch the beans, count them, or rearrange them if he chooses.

On the Pupil's Interview Record, enter after the combination $2 + 2$ and in the *first* brackets the child's answer when the objects were invisible.

Enter in the second brackets his answer after the objects had been uncovered. Of course, if the "invisible" answer was correct, the "visible" answer was not called for.

Present the following combinations in a similar fashion. Question the pupil in the same way and record the results in the same manner.

1. 2 + 2	6. 2 + 4
2. 8 + 1	7. 2 + 8
3. 6 + 1	8. 2 + 6
4. 1 + 7	9. 3 + 7
5. 3 + 1	10. 4 + 6

CHAPTER V

A CRITICAL EVALUATION OF METHODS OF ANALYZING PRACTICE IN FRACTIONS

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Recent developments in the techniques for analyzing and constructing instructional materials in arithmetic include two methods related to procedures for studying the content of drill in fractions. These methods make it possible to construct practice exercises in fractions according to specifications that can be set in advance, as can now be done in the work with whole numbers. It is the purpose of this paper to point out the fundamental difference between the two methods that have been proposed by Knight and his co-workers and by Brueckner for analyzing the content of drill materials in fractions. The discussion will be limited to a consideration of the procedures as applied to the subtraction of fractions.

The method of analyzing practice in fractions proposed by Knight and his co-workers consists in determining the frequency with which the "unit skills," into which each of the processes can be analyzed, occur in a set of practice materials. In Table I is given the list of skills involved in the subtraction of fractions.

TABLE I.—ANALYSIS OF SUBTRACTION OF FRACTIONS IN TERMS OF THE LEARNING PROCESS[1]

	Unit Skill Number
I. As to the Form of Stating the Example	
A. When the numbers are written with figures	
1. Indicated subtraction, as ½ — ¼..................	1
2. Column subtraction..........................	2
3. Words as "minus," "less," "subtract —— from ——," "from —— subtract ——," "from —— take ——," "take —— from ——," as "Take ⅓ from 5/12"	3
B. When numbers are written in words	
1. Indicated subtraction, as two-sevenths — one-sixth....	4
2. Words, as I. A. 3 above, as seven and two-thirds minus one-fourth................................	5

[1] Knight, F. B., Luse, E. M., and Ruch, G. M. *Problems in the Teaching of Arithmetic.* Iowa City, Iowa: Iowa Supply Store. 1925.

(Table I continued.)

	Unit Skill Number
II. As to Procedure	
A. Nature of the minuends and subtrahends	
1. When both are proper fractions	
a. Similar fractions, as $\frac{7}{9}$ — $\frac{2}{9}$	6
b. When one given denominator is common denominator, as $\frac{8}{9}$ — $\frac{2}{3}$	7
c. When denominators have no common factor, as $\frac{3}{4}$ — $\frac{2}{7}$	8
d. When denominators have a common factor but the common denominator is not present, as $\frac{5}{6}$ — $\frac{4}{9}$	9
2. Minuend a whole number and subtrahend a proper fraction, as 8 — $\frac{3}{4}$	10
3. Minuend a mixed number and subtrahend a proper fraction	
a. Similar fractions, as $1\frac{5}{7}$ — $\frac{4}{7}$	11
b. When one given denominator is common denominator, as $4\frac{5}{12}$ — $\frac{1}{3}$	12
c. When denominators have no common factor, as $3\frac{4}{5}$ — $\frac{1}{4}$	13
d. When denominators have a common factor but the common denominator is not present as $4\frac{5}{6}$ — $\frac{3}{4}$	14
4. Minuend a whole number and subtrahend a mixed number, as 8 — $6\frac{1}{6}$	15
5. Minuend a mixed number and subtrahend a whole number, as $9\frac{4}{13}$ — 4	16
6. Minuend a mixed number and subtrahend a mixed number	
a. Similar fractions, as $4\frac{7}{8}$ — $2\frac{3}{8}$	17
b. When one given denominator is the common denominator, as $5\frac{3}{4}$ — $2\frac{5}{8}$	18
c. When denominators have no common factor, as $4\frac{1}{4}$ — $3\frac{2}{7}$	19
d. When denominators have a common factor but the common denominator is not present, as $4\frac{7}{8}$ — $1\frac{1}{6}$	20
B. Subtraction of Numerators of Similar Fractions	21
C. Change of Fractions to Similar Fractions when	
1. One given denominator is common denominator, as $\frac{3}{4}$ — $\frac{1}{2}$	22
2. Given denominators have no common factor, as $\frac{3}{4}$ — $\frac{2}{5}$	23
3. Denominators have a common factor but the common denominator is not present, as $\frac{7}{12}$ — $\frac{3}{8}$	24
D. If mixed numbers are present, the subtraction of the fraction (by borrowing if necessary) and then the integers	25
E. Borrowing	
1. Not necessary	
a. Proper fractions, no borrowing, as $\frac{3}{4}$ — $\frac{1}{2}$	26
b. Mixed numbers used, no borrowing, as $2\frac{7}{8}$ — $1\frac{1}{3}$	27
c. Whole number from mixed number, as $9\frac{1}{2}$ — 3	28
2. Necessary	
a. When a mixed number is subtracted from a whole number, as 18 — $13\frac{3}{4}$	29

(Table I continued.)

	Unit Skill Number
b. When a fraction is subtracted from a whole number, as $6 - \frac{1}{4}$	30
c. When the fraction in the subtrahend is greater than the fraction in the minuend	
(1) Similar fractions, as $12\frac{1}{4} - 3\frac{3}{4}$	31
(2) When one given denominator is common denominator, as $7\frac{1}{6} - 4\frac{1}{3}$	32
(3) When denominators have no common factor, as $3\frac{1}{8} - \frac{4}{5}$	33
(4) When the denominators have a common factor but the common denominator is not present, as $8\frac{3}{4} - 4\frac{5}{6}$	34
d. When a mixed number with a larger fraction is subtracted from a mixed number with a smaller fraction making the integral difference zero, as $3\frac{1}{2} - 2\frac{3}{4}$	35
3. Analysis of remainder	
a. Zero in results	
(1) Fractional remainder zero, as $5\frac{3}{4} - 2\frac{3}{4}$	36
(2) Integral remainder zero, as $4\frac{1}{2} - 4\frac{1}{4}$	37
(3) Whole remainder zero, as $13\frac{1}{2} - 13\frac{1}{2}$	38
b. Whole number, as $7\frac{1}{4} - 3\frac{1}{4}$	39
c. A proper fraction	
(1) Irreducible, as $\frac{5}{7} - \frac{1}{2} = \frac{3}{14}$	40
(2) Reducible, as $\frac{9}{10} - \frac{3}{10} = \frac{6}{10}$ or $\frac{3}{5}$	41
d. A mixed number	
(1) Fraction irreducible, as $6\frac{3}{8} - 2 = 4\frac{3}{8}$	42
(2) Fraction reducible, as $10\frac{7}{15} - 6\frac{4}{15} = 4\frac{3}{15}$ or $4\frac{1}{5}$	43

As can be seen, there are listed 43 unit skills. One series of skills, the first five, is concerned with the form of stating the example. Another series is concerned with the procedure involved in working the example. Under "procedure" are listed the unit skills related to the nature of minuends and subtrahends, the subtraction of similar and dissimilar fractions and mixed numbers, borrowing, and the analysis of the answer.

To apply these skills in the construction or analysis of drill materials Knight's procedure requires the determination of the frequency with which each of these 43 unit skills occurs in the material being analyzed. A report published in the *Third Yearbook of the Department of Superintendence* (page 73) shows the distribution of the unit skills in division of fractions in drill materials found in five arithmetic textbooks and illustrates the procedure to be used in analyzing

the content of exercises in fractions according to the "unit skill" method.

Without a doubt the analyses of unit skills in each process by Knight and his co-workers are a most valuable contribution. It is as basic for work in fractions as is the list of number combinations in whole numbers. Drill materials which omit any of the unit skills listed for the various processes must, of course, be adjudged to be deficient. The drill ought to be distributed according to the difficulty and importance of each of the unit skills.

Certain investigations by the writer have suggested that the determination of the frequency with which these unit skills occur in exercises in fractions is not an adequate method of determining the completeness of the drill material and the extent to which the 'area' of the process under consideration has been covered. The 'area' of a process can readily be determined by preparing a set of examples in which these unit skills occur in all of the various possible combinations. As will be demonstrated, this can easily be done (see Table IV).

A set of examples in the subtraction of fractions which contain all of the unit skills listed by Knight is given in Table II.

Table II contains an analysis of the unit skills found in each of the examples. The numbers of the unit skills refer to the corresponding numbers in Table I; for example, type $5/8 - 1/8 = 1/2$, contains unit skills 3, 6, 21, 26, 41, listed in Table I. As is pointed out in the note in Table II, the list of 16 examples which contain all of the unit skills listed in Table I can be reduced to 13 in number if unit skills are not listed separately for the finding of a common denominator for dissimilar fractions, such as $1/3$ and $1/4$, in which the common denominator is the product of given denominators, and for $1/4$ and $1/6$, in which the common denominator is the product of factors of the given denominators. Both of these types of denominators can be listed as one unit skill—denominator a product of given denominators.

In Table III is given the distribution of practice on each of the unit skills in the first thirteen examples in Table II. As can be seen, the number of times each of these skills occurs varies from one to nine. This shows that it would be possible to prepare sets of similar examples duplicating the skills found in the examples in Table II, which would give practice in each of the unit skills listed in Table I as often as the person constructing the exercises would estimate was

TABLE II.—SIXTEEN TYPE SUBTRACTION EXAMPLES CONTAINING ALL OF KNIGHT'S UNIT SKILLS

Type	Type Number (See Table IV)	Unit Skills Occurring
1. $\frac{5}{8}$ minus $\frac{1}{8}$ = $\frac{1}{2}$	4	3, 6, 21, 26, 41
2. One-half minus one-fourth = $\frac{1}{4}$	28	5, 7, 22, 26, 40
3. $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$	45	1, 8, 23, 26, 40
4. $9 - \frac{1}{2} = 8\frac{1}{2}$	17	1, 10, 30, 42
5. $4\frac{2}{3} - \frac{2}{3} = 4$	8	1, 11, 21, 27, 36, 39
6. $3\frac{3}{4} - \frac{1}{2} = 2\frac{1}{4}$	30	1, 12, 22, 27, 42
7. $2\frac{3}{8} - \frac{1}{3} = 2\frac{1}{24}$	47	1, 13, 23, 27, 42
8. $5 - 1\frac{1}{3} = 3\frac{2}{3}$	18	1, 15, 25, 29, 42
9. $5\frac{1}{4} - 3 = 2\frac{1}{4}$	9	1, 16, 25, 28, 42
10. $4\frac{1}{8} - 1\frac{7}{8} = 2\frac{1}{4}$	26	1, 17, 21, 31, 43
11. $9\frac{1}{10} - 8\frac{1}{2} = \frac{3}{5}$	44	1, 18, 22, 32, 35, 37, 41
12. $\begin{array}{r} 3\frac{1}{4} \\ -1\frac{2}{3} \\ \hline 1\frac{7}{12} \end{array}$	52	2, 19, 23, 33, 42
13. Two and one-half − two and one-half = 0	15	4, 17, 21, 27, 38
14. $\frac{5}{6} - \frac{4}{9} = \frac{7}{18}$		1, 9, 24, 26, 40
15. $4\frac{5}{6} - \frac{3}{4} = 4\frac{1}{12}$		1, 14, 24, 27, 40
16. $11\frac{1}{8} - 1\frac{5}{6} = 9\frac{7}{24}$		1, 20, 24, 34, 42

Note: In Brueckner's analysis of unit skills unlike fractions of all kinds are regarded as presenting the same difficulties when neither of the denominators is the common denominator; for example, $\frac{1}{3} - \frac{1}{4}$ and $\frac{1}{6} - \frac{1}{8}$ are considered to involve the same unit skill in finding common denominators. Hence, skills 9, 14, 20, 24, and 34 in Knight's list of unit skills are not specifically represented in the first 13 examples in Table II and were not used in the tests given to determine the difficulty of standard types of examples. The last three examples do introduce the missing unit skills.

TABLE III.—NUMBER OF TIMES UNIT SKILLS OCCUR IN THE FIRST THIRTEEN TYPES GIVEN IN TABLE II

Number of Times Skills Occur	Skills by Number as Listed in Table II
1	2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 15, 16, 18, 19, 28, 29, 30, 31, 32, 33, 35, 36, 37, 38, 39, 43
2	17, 25, 40, 41
3	22, 23, 26
4	21, 27
5	42
9	1
0	9, 14, 20, 24, 34*

* These five skills consider denominators such as 4 and 6 separately from such denominators as 3 and 4. The three additional examples, 14, 15, and 16, would provide for these skills.

necessary. At first glance it would therefore seem that the drill content of such exercises would make adequate provision for work in subtracting fractions, since all unit skills would occur with a certain frequency, estimated to be satisfactory.

It is easy to demonstrate the fallacy of such an assumption. This can be done by making examples which contain combinations of these skills which are quite different from those given in the examples in Table II. Such a series of examples is given in Table IV where there are listed 53 types of examples in subtraction of fractions, each of which contains a different set of unit skills. The number of the unit skills found in each type is given in the table. By referring directly to Table I, the nature of these skills can be determined. This list of examples constitutes a suggested standard list of types of examples in the subtraction of fractions.

A percent of error for each type as determined by tests given to 775 sixth-grade pupils is also given in the table. The range in the percents of error is from 2.3 on type 3 to 40.2 on type 8. Many difficult types of examples are included in the test which are not given in Table II. It is conceivable that many of these types would be missing in a set of examples built upon the skill basis alone. The reader can readily discover interesting facts concerning the relative difficulty of the various types of examples. Such information is of great importance in the preparation of practice exercises in subtraction of fractions. The writers have similar data regarding standard lists of types of examples in each of the processes of addition, multiplication, and division.

In Table IV all freak types of examples, such as $9/5 — 9/8$, $6\,3/6 — 4\,2/4$, $11\,8/12 — 4/5$, etc., are omitted. It is believed that the list of examples

TABLE IV.—STANDARD ANALYSIS OF 53 TYPES OF SUBTRACTION EXAMPLES IN FRACTIONS

(All given with numbers and sign in horizontal form)

Type	Skills Occurring	Percents of Error of 775 Pupils in Grade VI-B
1. $5/4 - 3/4 = 1/2$	44,† 21, 26a,† 41	5.0
2. $8/8 - 3/8 = 5/8$	44, 21, 26a, 40	3.0
3. $2/3 - 1/3 = 1/3$	6, 21, 26, 40	2.3
4.* $5/8 - 1/8 = 1/2$	6, 21, 26, 41	6.9
5. $1/2 - 1/2 = 0$	6, 21, 26, 38	29.0
6. $4\,2/3 - 1/3 = 4\,1/3$	11, 21, 27, 42	21.5
7. $7\,5/6 - 1/6 = 7\,2/3$	11, 21, 27, 43	29.2

TABLE IV.—(Continued)

Type	Skills Occurring	Percents of Error of 775 Pupils in Grade VI-B
8.* $4\frac{2}{3} - \frac{2}{3} = 4$	11, 21, 27, 36, 39	40.2
9.* $5\frac{1}{4} - 3 = 2\frac{1}{4}$	16, 25, 28, 42	8.2
10. $10\frac{1}{2} - 10 = \frac{1}{2}$	16, 25, 28, 37, 40	15.4
11. $4\frac{3}{5} - 2\frac{1}{5} = 2\frac{2}{5}$	17, 21, 27, 42	6.7
12. $3\frac{3}{4} - 2\frac{3}{4} = 1$	17, 21, 27, 36, 39	29.9
13. $7\frac{7}{8} - 2\frac{1}{8} = 5\frac{3}{4}$	17, 21, 27, 43	19.9
14. $1\frac{5}{8} - 1\frac{1}{8} = \frac{1}{2}$	17, 21, 27, 37, 41	14.8
15.* $2\frac{1}{2} - 2\frac{1}{2} = 0$	17, 21, 27, 38	32.5
16. $1 - \frac{1}{2} = \frac{1}{2}$	10, 30, 40	26.4
17.* $9 - \frac{1}{2} = 8\frac{1}{2}$	10, 30, 42	26.7
18.* $5 - 1\frac{1}{3} = 3\frac{2}{3}$	15, 25, 29, 42	29.4
19. $3 - 2\frac{5}{12} = \frac{7}{12}$	15, 25, 29, 37, 40	38.0
20. $1\frac{1}{3} - \frac{2}{3} = \frac{2}{3}$	11, 21, 31, 37, 40	29.2
21. $6\frac{1}{3} - \frac{2}{3} = 5\frac{2}{3}$	11, 21, 31, 42	23.4
22. $7\frac{1}{8} - \frac{5}{8} = 6\frac{1}{2}$	11, 21, 31, 43	31.8
23. $1\frac{1}{8} - \frac{5}{8} = \frac{1}{2}$	11, 21, 31, 37, 41	26.5
24. $10\frac{1}{3} - 4\frac{2}{3} = 5\frac{2}{3}$	17, 21, 31, 42	26.3
25. $9\frac{3}{5} - 8\frac{4}{5} = \frac{4}{5}$	17, 21, 31, 35, 37	33.7
26.* $4\frac{1}{8} - 1\frac{7}{8} = 2\frac{1}{4}$	17, 21, 31, 43	33.9
27. $2\frac{1}{4} - 1\frac{3}{4} = \frac{1}{2}$	17, 21, 31, 37, 41	27.3
28.* $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$	7, 22, 26, 40	7.8
29. $\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$	7, 22, 26, 41	13.2
30.* $3\frac{3}{4} - \frac{1}{2} = 3\frac{1}{4}$	12, 22, 27, 42	14.9
31. $4\frac{5}{6} - \frac{1}{3} = 4\frac{1}{2}$	12, 22, 27, 43	26.5
32. $3\frac{3}{4} - 1\frac{1}{2} = 2\frac{1}{4}$	18, 22, 27, 42	10.7
33. $7\frac{3}{5} - 1\frac{1}{10} = 6\frac{1}{2}$	18, 22, 27, 43	25.4
34. $1\frac{3}{4} - 1\frac{1}{8} = \frac{5}{8}$	18, 22, 27, 37, 40	21.4
35. $1\frac{5}{6} - 1\frac{1}{2} = \frac{1}{3}$	18, 22, 27, 37, 41,	21.4
36. $14\frac{3}{4} - 14\frac{1}{2} = \frac{1}{4}$	12, 22, 32, 42	27.8
38. $2\frac{1}{6} - \frac{2}{3} = 1\frac{1}{2}$	12, 22, 32, 43	37.2
39. $1\frac{1}{10} - \frac{4}{5} = \frac{3}{10}$	12, 22, 32, 37, 40	28.6
40. $1\frac{1}{2} - \frac{5}{6} = \frac{2}{3}$	12, 22, 32, 37, 41	30.1
41. $7\frac{1}{2} - 2\frac{3}{4} = 4\frac{3}{4}$	18, 22, 32, 42	30.1
42. $3\frac{1}{6} - 1\frac{2}{3} = 1\frac{1}{2}$	18, 22, 32, 43	37.1
43. $3\frac{1}{4} - 2\frac{5}{8} = \frac{5}{8}$	18, 22, 32, 35, 37, 40	33.5
44.* $9\frac{1}{10} - 8\frac{1}{2} = \frac{3}{5}$	18, 22, 32, 35, 37, 41	27.3
45.* $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$	8, 23, 26, 40	8.6
46. $\frac{9}{15} - \frac{1}{2} = \frac{1}{10}$	8, 23, 26, 41	31.6
47.* $2\frac{3}{8} - \frac{1}{3} = 2\frac{1}{24}$	13, 23, 27, 42	29.2
48. $4\frac{5}{6} - 1\frac{2}{5} = 3\frac{13}{30}$	19, 23, 27, 42	23.7
49. $6\frac{1}{2} - 6\frac{2}{5} = \frac{1}{10}$	19, 23, 27, 33, 37, 40.	22.5
50. $8\frac{1}{2} - \frac{2}{3} = 7\frac{5}{6}$	13, 23, 33, 42	29.2
51. $1\frac{1}{3} - \frac{1}{2} = \frac{5}{6}$	13, 23, 33, 38, 40	30.8
52.* $3\frac{1}{4} - 1\frac{2}{3} = 1\frac{7}{12}$	19, 23, 33, 42	28.5
53. $6\frac{1}{4} - 5\frac{2}{3} = \frac{7}{12}$	19, 23, 33, 37, 40	33.9

* The examples marked with the asterisk are the thirteen identical examples listed in Table II. All examples in Table IV that are not starred represent types of examples not found in Table II.

†Unit skill 44 is not included in Table I. It is listed in the (unpublished) analysis as: Subtracting proper fractions from improper fractions.

Unit skill 26a is also not listed in Table I. This unit skill shows that no carrying was necessary.

in Table IV represents all of the possible 'legitimate' combinations of skills that occur in the subtraction of fractions, as far as procedure is concerned. For the sake of space all examples are given with numbers written as figures and in horizontal form; hence, skills 1 to 5, concerned with the various forms in which examples can be stated, are not listed. Each example involves, therefore, unit skill 1 in addition to those listed. Variations in the form of writing the example need not be considered, it seems to the writers, in preparing a possible standard list of types representing various combinations of standard unit skills with respect to procedure.

The method of using this standard list of types of examples to check the content of instructional materials has been illustrated in a study recently published by one of the writers[2] in which an analysis was made of the distribution of practice on each of 58 types of examples in the subtraction of fractions found in ten fifth-grade arithmetic textbooks. The list of 58 types used in that study contained freak types which probably ought not to occur in drill materials. By omitting them, the list has been reduced to the 53 types given in Table IV. The technique of analysis proposed by Brueckner consisted of determining the frequency with which each of these types of examples occurred in the drill exercises. If all of the types occur, it is obvious that all unit skills must occur (see Table V). The report showed that very inadequate attention had been given in all textbooks to the inclusion of many important types of examples in subtraction of fractions, especially those involving zero difficulties. Many of the important types were missing entirely. For example, in one text 46 of the 58 types listed were never found in the drill exercises. It is conceivable that all of the unit skills might have been present in these drills—and hence the drills might have been rated as satisfactory from the point of view of the number of times each of the unit skills occurred in the drill materials—even though many important combinations of skills were missing entirely. It is evident, therefore, that an analysis of the content of drill materials in fractions, made on the basis of the determination of the frequency with which unit skills occur, might lead to a conclusion quite at variance from one arrived at on the basis of an analysis of the occurrence of

[2]Brueckner, L. J. "A technique for analyzing the distribution of drill in fractions." *Jour. of Educ. Method,* VII, 352-358.

various types of examples containing different combinations of unit skills.[3]

The data mentioned suggest that a basic need in the construction of drill materials in fractions must be the preparation of a list of unit skills which constitute the complete process. An analysis of drill materials to determine the frequency with which these unit skills occur will yield important information as to the care with which instructional materials have been prepared. However, as has been shown, these data will give an inadequate analysis of the drill content, and the mere tabulation of the frequency with which each of a series of unit skills occurs will give no indication of the extent to which attention has been given to the provision for practice on various combinations of these skills. An analysis of the frequency of occurrence of unit skills alone would be like an analysis of instructional materials in multiplication of whole numbers in terms of the frequency with which each multiplication fact occurred without paying any attention at all to the types of examples in which these

TABLE V.—FREQUENCY OF OCCURRENCE OF UNIT SUBTRACTION SKILLS IN THE 53 TYPE EXAMPLES GIVEN IN TABLE IV

(All the examples were expressed horizontally with figures given and sign indicated, thus eliminating skills 2–5)

Unit Skill	Frequency	Unit Skill	Frequency
6	3	26	7
7	2	26a†	2
8	2	27	17
9*	....	28	2
10	2	29	2
11	7	30	2
12	6	31	8
13	3	32	8
14*	....	33	5
15	2	34*	....
16	2	35	3
17	9	36	2
18	8	37	15
19	4	38	3
20†	....	39	2
21	21	40	14
22	16	41	10
23	9	42	15
24*	....	43	8
25	4	44†	2

(*) See note to Table III. (†) Added skills not given by Knight.

[3] In this connection see Knight's brief comment elsewhere in this Yearbook on the central problem involved.

number facts occur. Certainly from the point of view of the learner the occurrence of various types of examples is as vital a factor as the occurrence of unit skills.

With a little practice an analysis of the types of examples in a set of practice materials can easily be made; for example, one can analyze the materials provided for subtraction of fractions in a typical text in less than two hours. Standard lists of types of examples in each of the other processes in fractions have been prepared by a method similar to that used in developing the standard list of types in subtraction of fractions given in Table V.

The analysis of instructional materials by means of these standard lists of types of examples shows as serious a situation in each of the other processes in fractions as was found in subtraction of fractions.

CHAPTER VI

MIXED VERSUS ISOLATED DRILL ORGANIZATION[1]

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I. THE PROBLEM

In planning and prosecuting a maintenance program one impt tant choice pertains to the content of each drill unit. These units may each concern themselves with a single process such as Unit I on "Addition of Whole Numbers," Unit II on "Subtraction of Whole Numbers," etc. On the other hand, each review unit may be a mixed drill in which examples of several or many processes may appear.[2] With these two extreme types in mind it is easily seen that an almost endless variety of types of content is possible. For experimental purposes it is convenient to test isolated drill as such against mixed drill as such, although it is theoretically possible that some adaptation of either principle would be better than either the pure isolated or the pure mixed type. That the issue between mixed and isolated drill presents a clear-cut technical problem is evident. But that the issue is an important one is not to be taken for granted until experimental data upon it are available. Until recently practically all drill material for maintenance purposes was built on the assumption that the isolated type was superior, although several recent drill pads and other review devices are at least flirting with the mixed type of drill organization. Some maintenance material is built on a frank and sincere acceptance of the mixed drill theory.

II. THE EXPERIMENTAL MATERIALS

With the purpose of conducting an experiment on the merits of mixed versus isolated drill two sets of drill materials were constructed,

[1] The data here reported were gathered by the writer while he was working at the College of Education, State University of Iowa, in 1928, under the direction of F. B. Knight.

[2] It is understood, of course, that in constructing mixed drills one would not insert examples of a process which had not been learned by the pupils at the time of taking a given unit.

each consisting of twenty-six twenty-minute drills. The content of the first set was identical with the second except that in the one set each drill dealt with one topic, such as addition of fractions, while in the second set each drill was mixed in nature; that is, the same list of examples was arranged in radically different ways.

Tables I-III will give a picture of the content of these two sets of drills and their organization.

1. The Contents of the Drills

Table I is merely a list of the gross functions, or blocks of skills, which were drilled upon and tested during the course of the experiment. Each of these blocks of skills is numbered and is referred to by number in many cases hereafter. Tables II and III assume reference to Table I.

All the examples of any process were so built that when a pupil had finished them he would have received properly distributed practice with systematically recurring emphasis on the harder combinations. Such distribution of practice is defended by Dr. Luse's [3] experiment.

2. The Organization on the 'Isolated' Theory

The purpose of Table II is to show the frequency of drill on gross functions practiced in the isolated drills. It is to be read as follows:

TABLE I.—GROSS SKILLS PRACTICED IN DRILL TESTS

Skill No.	Skill	Number of Examples in Drills
1	Addition of Whole Numbers	61
2	Subtraction of Whole Numbers	64
3	Multiplication of Whole Numbers	63
4	Division of Whole Numbers	68
5	Addition of Fractions	63
6	Subtraction of Fractions	62
7	Multiplication of Fractions	64
8	Division of Fractions	61
9	Addition of Denominate Numbers *	5
10	Subtraction of Denominate Numbers *	6
11	Division of Denominate Numbers *	7
12	Multiplication of Denominate Numbers *	8

* Denominate numbers were slighted since instruction in them does not come until late in the fifth grade.

[3] Luse, Eva May. *Transfer Within Narrow Mental Functions*, University of Iowa Monographs in Education, No. 5.

Addition of whole numbers (listed in Table II as skill number 1) was practiced 20 times the first week of the experiment, 20 times the tenth week, and 21 times the nineteenth week. Subtraction of fractions (listed as skill number 5) received 20 practices the fifth week, 23 practices the fourteenth week, and 20 practices the twenty-third week. In all cases the numbers at the top of the table refer to the skills according to their respective numbers listed in Table I; the numbers at the left of the table refer to the consecutive drill numbers (or to the week number since these drills were given once a week) of the drill program. Drills were given in consecutive order; Drill I was given the first week of the experiment and consecutive drills continued for 26 weeks; that is, the numbers on the left of Table II may be read as drill numbers or as the number of the week of the experiment.

3. The Organization on the 'Mixed' Theory

Table III shows the distribution of practice with respect to gross total skills as it occurred in the mixed drills. The table is to be read in exactly the same fashion as Table II. For instance, addition of whole numbers received drill 2 times the first week, 2 times the second week, 3 times the third week, 3 times the fourth week, etc.

The reader should note the contrast between distribution of practice in Table II (isolated drills) and Table III (mixed drills). The inherent characteristic of each type of drill is shown vividly by these two tables. Isolated drill means 'bunched' practice, coming seldom. Mixed drill means small amounts of practice coming frequently.

4. Typical Detail of Content

Space forbids the spreading out of a detailed analysis of the content of the examples used in this experiment. However, the data in Table IV for addition are typical of all the material used.[4]

Table IV shows the composite distribution of practices given on each of the basic addition facts in the addition examples and problems appearing in the drill materials used. By composite distribution of practice is meant the practice given to each unit of skill in adding up the columns, for the determination of an answer, and then adding down the column to check the answer or vice versa.

[4] For a complete analysis of the nature of the examples used see the writer's *Mixed vs. Isolated Drills.* (Doctor's Dissertation, State University of Iowa, Iowa City, Iowa, 1928.)

TABLE II.—DISTRIBUTION OF GROSS SKILLS: GROUP I (ISOLATED DRILL)

	SKILL											
	1	2	3	4	5	6	7	8	9	10	11	12
1......	20											
2......		21										
3......			20									
4......				22								
5......					20							
6......						20						
7......							22					
8......								20				
9......									5	5		
10......	20											
11......		21										
12......			21									
13......				21								
14......					23							
15......						22						
16......							21					
17......								22				
18										1	7	8
19......	21											
20......		22										
21......			22									
22......				25								
23......					20							
24......						20						
25......							21					
26......								19				

TABLE III.—DISTRIBUTION OF GROSS SKILLS: GROUP M (MIXED DRILL)

	SKILL											
	1	2	3	4	5	6	7	8	9	10	11	12
1......	2	3	3	3	3	2	2	3			1	
2......	2	2	3	3	2	2	2	2			1	1
3	3	3	2	3	2	2	2	2				
4......	3	2	3	2	2	2	1	2			2	
5......	3	2	2	3	2	3	2	2		1		
6......	2	3	2	3	3	2	1	3		1		
7......	2	2	2	3	2	3	3	2		1		
8......	2	3	4	2	2	2	2	2	1			
9......	3	2	2	2	4	2	3	2				1
10......	2	3	1	3	3	3	2	2				
11......	2	2	3	2	3	2	3	2		1		
12......	3	3	2	2	2	3	2	2			1	
13......	3	3	3	3	2	2	2	3				
14......	2	2	1	2	2	2	2	4	1	1		1
15......	2	2	1	3	2	3	2	3				1
16......	2	3	1	2	2	3	5	3				
17......	2	3	3	2	3	2	1	3				1
18......	2	4	1	2	3	2	3	3				
19......	2	3	2	3	2	3	2	2		1		1
20......	2	2	2	4	2	3	3	2			1	
21......	2	2	2	3	2	3	3	3			1	
22......	3	2	3	4	2	4	3	2				
23......	2	3	2	2	1	3	3	3				
24......	1	2	2	2	3	2	4	3		1		
25......	3	2	2	3	3	2	3	3	2			1
26......	5	2	3	4	2	2	2	2	2	1	1	

TABLE IV.—ADDITION OF WHOLE NUMBERS: COMPOSITE DISTRIBUTION OF PRACTICES

Number Added	Number Added To									
	0	1	2	3	4	5	6	7	8	9
0	10	15	12	8	16	14	19	14	12	15
1	10	19	21	18	23	16	22	28	27	23
2	6	15	16	28	22	29	21	28	26	31
3	12	24	18	19	16	19	24	25	20	21
4	9	13	15	12	17	17	26	18	21	29
5	12	14	11	13	21	21	20	21	22	18
6	11	8	14	17	9	10	27	13	21	22
7	9	13	15	16	18	10	23	15	20	10
8	8	15	14	14	15	21	18	16	19	18
9	12	9	12	13	24	20	15	25	23	24
Carrying		74	96	58	20	14	2			

III. PROCEDURE AND SUBJECTS

Six hundred and thirteen pupils in twenty-four different classes were used for the experiment. All of these pupils were given an initial and final test.

The tests used for both the initial and final test were of the mixed type and genuinely exhaustive in nature. The testing time in each case was 150 minutes. Every process practiced in the drills was covered adequately in the tests. Tests of unusual length and of careful construction were used to avoid any unreliability of the measures and to make possible analysis of gains by processes as well as in arithmetic in general. The coefficient of reliability for the tests was .97; P. E., .006.

Two groups were formed by pairing on the basis of performance on the initial test. These groups were kept intact throughout the twenty-six weeks of the experiment. One of these groups (Group I) was given the isolated drill; the other group (Group M) was given the mixed drill.

The average C. A. for Group M pupils was twelve years, one month; for Group I, twelve years, two months.

The drills were given on consecutive Fridays. The rest of the week each class proceeded as usual. Directions for remedial work following the corrections of the drills were uniform in both groups. In this way conditions under which maintenance programs have to operate in ordinary circumstances were approximated.

All testing, drilling, and other management of the classes was conducted by the classroom teachers. The experimenter did not visit any class at any time. All contact with the teachers was by correspondence (letters were sent weekly to teachers in both groups). Before the experiment began the superintendents and teachers involved were told that no comparisons between schools would be made. In this way it was felt that extraneous factors of rivalry might at least be lessened, if not entirely avoided.

IV. Gross Results

1. Comparison of the Groups as Wholes

The first comparison which may be of interest is between the two groups as wholes with pupils unpaired. There were 267 pupils in Group M who took both the pretest and the final test, and 263 in Group I. The data of this general comparison are shown in Table V.

Table V.—General Comparison of Groups M and I

	Mean of Right Answers Group M	Mean of Right Answers Group I	Diff.
Pre-Test	93.97	92.75	.82 ± 1.673
Final Test	150.89	133.51	17.37 ± 2.332

A second comparison deals with the performance of both groups throughout the twenty-six drill periods. The scores of all pupils in all classes throughout the experiment were available for study, but a tabular report of them is too space-consuming for reproduction here. A fair verbal summary of the progress of classes through the drill weeks is as follows:

The scores of classes in Group M started comparatively low and then gradually worked up to high ground at the end. On the other hand, Group I started with much higher scores at the beginning of the experiment, fluctuated much, and finished with scores which were high, yet somewhat lower than those earned by Group M.

2. Comparison of the Groups with Pairing

Table VI shows the comparison between Group M and Group I (470 cases) on accomplishment in terms of right answers to examples in the pretest and final test, when all exact pairs of pretests are used. The data in the table are so clearly identified that no detailed explanation is needed.

Attention is directed especially to the last column in the final test section showing the significance of the differences between the performances of the two groups on the final tests. All statistically significant differences are starred in Tables VI, VII, VIII, and IX. The group using mixed drills was consistently superior on every comparison made, and in Table VI, in 9 out of 12 comparisons the differences were overwhelmingly significant and in favor of Group M. On the composite for the test the difference was more than nine times the P.E. (diff.). Statistically such differences are unquestionably significant.[5]

The data reported in Table VI were recast in terms of comparative percents. A quantitative report of percent gains is not inserted here, but a fair verbal description is as follows: Both groups gained. The percent of gain for Group M exceeded that of Group I in all instances. The Group M excess gain ranged from 9 percent (division of whole numbers) to 69 percent (denominate numbers). Average excess gain for all processes as listed in Table VI was 23 percent. Each group improved in arithmetical skill, but pupils using mixed drill consistently showed gains in excess of those produced by pupils using isolated drill. On the whole, pupils using mixed drill gained 65 percent in ability as contrasted with a gain of 42 percent for pupils using isolated drill—a difference of 23 percent.

3. Gains for Different Levels of Ability

Since it is quite possible that the gains in favor of the mixed type of drill were made by pupils of some particular grade of ability, it is

[5] The differences could be reported greater, and justifiably so, by using a formula for determining the PE of difference between the gains of the two groups, in which the ordinary formula PE (diff.) $= \sqrt{PE_1^2 + PE_2^2}$ is displaced by the formula PE (diff.) $= \sqrt{PE_1^2 + PE_1^2 - 2rPE_1 \cdot PE_2}$. Applying this second formula to the addition of whole numbers in the present study the difference divided by the PE (diff.) becomes 1.30, an increase of 8 percent. However, as differences were so marked without the use of the longer formula, the writer used the usual PE (diff.) formula.

TABLE VI.—COMPARISON OF PERFORMANCES IN TERMS OF RIGHT ANSWERS, GROUP M VERSUS GROUP I, EXACTLY PAIRED
(Asterisks in the last column indicate significant differences.)

Operations	Mean Rights		S.D. of Dist.		P.E. of Means		Difference	P.E.(diff.)	Diff. ÷ P.E.(diff)
	Group M	Group I	Group M	Group I	Group M	Group I			
Pretest									
Addition of Whole Numbers...	8.67	8.67	2.050	2.050	.096	.096	....	....	....
Subtraction of Whole Numbers	27.16	27.16	4.310	4.310	.197	.197	....	....	....
Multiplication of Whole Nos...	8.73	8.73	2.526	2.526	.118	.118	....	....	....
Division of Whole Numbers...	2.84	2.84	2.653	2.653	.126	.126	....	....	....
Composite of Whole Numbers.	47.38	47.38	9.460	9.460	.441	.441	....	....	....
Addition of Fractions.........	7.63	7.63	5.454	5.454	.254	.254	....	....	....
Subtraction of Fractions......	11.59	11.59	7.398	7.398	.344	.344	....	....	....
Multiplication of Fractions....	8.14	8.14	6.704	6.704	.318	.318	....	....	....
Division of Fractions.........	12.00	12.00	8.782	8.782	.433	.433	....	....	....
Composite of Fractions.......	40.31	40.31	23.097	23.097	1.107	1.107	....	....	....
Denominate Numbers.........	5.69	5.69	4.482	4.482	.219	.219	....	....	....
Composite of Test...........	92.41	92.41	32.400	32.400	1.425	1.425	....	....	....
Final Test									
Addition of Whole Numbers...	10.43	9.36	1.881	2.178	.882	.102	1.06	.08	1.20 M
Subtraction of Whole Numbers	28.38	27.48	3.392	4.524	.155	.207	.90	.25	3.60 M
Multiplication of Whole Nos...	11.09	10.03	2.649	3.101	.124	.146	1.06	.19	5.57*M
Division of Whole Numbers...	4.87	4.30	2.995	2.998	.142	.142	.57	.20	2.85 M
Composite of Whole Numbers.	54.51	50.34	8.965	10.735	.418	.501	4.17	.66	6.31*M
Addition of Fractions.........	17.19	13.30	6.308	7.656	.294	.357	3.89	.45	8.64*M
Subtraction of Fractions......	20.22	16.90	5.438	7.040	.252	.327	3.32	.41	8.09*M
Multiplication of Fractions....	20.69	17.50	5.342	6.368	.253	.302	3.19	.39	8.17*M
Division of Fractions.........	20.11	17.63	5.650	6.768	.278	.333	2.48	.43	5.76*M
Composite of Fractions.......	77.46	66.34	20.814	23.757	.997	1.138	11.12	1.51	7.36*M
Denominate Numbers.........	19.26	15.31	9.129	9.618	.447	.471	3.95	.64	6.17*M
Composite of Test...........	153.05	131.94	33.700	39.800	1.482	1.751	21.11	2.29	9.22*M

profitable to recompute the data to locate, if possible, the ability level or levels which react most favorably to mixed drill in contrast to isolated drill and also to locate an ability level which reacts more favorably to isolated drill if there be such. All pupils were classed in three levels of ability on the basis of the number of right answers on the initial tests (tests which consumed 150 minutes of actual testing time and hence provided a good measure of arithmetical ability). Table VII shows the comparative effects of mixed and isolated drills on pupils of different levels of ability.

a. Highest Level.—In Table VII the same type of data is given for pupils in each third of the two experimental groups as was given for the undivided groups in Table VI. Originally Table VII was in form identical with Table VI. For purposes of economy, only the differences and their P.E.'s of final test scores are reproduced here. There were no differences on the pretests, since the exact pairings produced complete equality of the two groups. It will be seen that Group M is again consistently superior to Group I, and that in six contrasts out of twelve the differences are very significant. On the whole test, Group M is superior to Group I, as shown by the difference in the means on the composite, which is 3.85 times its P.E. Mixed drills, then, within the reliability and validity of this experiment, provide a better maintenance technique for pupils superior in arithmetic than do the same examples organized in isolated drills.

b. Middle Level.—Table VII shows that in the middle thirds of the two groups the pupils using mixed drill were significantly superior in nine comparisons of the twelve made. On the composite of the final test, Group M had a margin over Group I that was 7.35 times its P.E.

c. Lowest Ability Level.—It is to be noted that the lowest third of Group M pupils is also in every case superior to the lowest third of Group I pupils judged by the results achieved on the final tests, although they were of course, like the others, exactly paired by identical scores on the pretests.

From Tables VI and VII it appears, then, that pupils who were equal in ability in arithmetic at the beginning of the experiment were not at all equal in ability after following different types of drill programs. Pupils using mixed drills were superior to pupils using isolated drills. This superiority exists for the three levels of ability. All of the forty-eight comparisons showed Group M pupils to be

TABLE VII.—COMPARISON OF PERFORMANCES IN TERMS OF RIGHT ANSWERS: HIGHEST, MIDDLE, AND LOWEST THIRDS OF GROUP M VERSUS HIGHEST, MIDDLE, AND LOWEST THIRDS OF GROUP I, EXACTLY PAIRED IN THE TEST

	Highest Thirds			Middle Thirds			Lowest Thirds		
	Difference†	P.E.(diff.)	Diff. / P.E.(diff.)	Difference†	P.E.(diff.)	Diff. / P.E.(diff.)	Difference†	P.E.(diff.)	Diff. / P.E.(diff.)
Addition of Whole Numbers...	1.01	.18	5.61*	1.13	.20	5.65*	.70	.24	2.91
Subtraction of Whole Numbers.	1.03	.42	2.45	1.70	.43	3.95*	3.21	.50	6.42*
Multiplication of Whole Nos....	1.15	.27	4.25*	.41	.30	1.36	1.66	.33	5.03*
Division of Whole Numbers. ..	.20	.29	.68	1.13	.29	3.89*	.68	.24	2.83
Composite of Whole Numbers	3.18	.78	4.07*	3.31	.91	3.63	6.02	1.14	5.28*
Addition of Fractions.........	1.77	.70	2.52	3.24	.68	4.76*	7.38	.71	10.39*
Subtraction of Fractions.......	1.17	.47	2.48	2.05	.44	4.65*	6.78	.69	9.82*
Multiplication of Fractions....	2.54	.46	5.52*	2.54	.63	4.03*	4.39	.66	6.65*
Division of Fractions..........	1.04	.52	2.00	2.20	.67	3.28	3.69	.77	4.79*
Composite of Fractions........	11.10	1.58	7.02*	8.45	1.97	4.28*	19.82	2.30	8.61*
Denominate Numbers.........	3.74	.92	4.06*	5.38	1.03	5.22*	3.67	1.22	3.00
Composite of Test............	11.00	2.85	3.85	21.54	2.93	7.35*	29.88	3.31	9.00*

*A significant difference.
†The difference is the mean of Group M minus the mean of Group I.

superior to Group I after twenty-six weeks of drills, and in a large majority of these instances the superiority was unquestionably statistically significant.

4. Other Comparisons

a. On the Basis of Attempts.—The pairing of pupils on a basis of number of examples attempted, in contrast to number of examples correct on the initial tests, would provide another basis for comparison. Such comparisons were made, but since they fail to show any differences from the general picture provided by a study of Tables VI and VII they need not be reproduced here.

b. On the Basis of Gain in Speed.—It is of at least minor interest to study the gain in speed of work by the two groups involved. Space cannot be taken for reporting the data, but the main facts on these points are: When we pair two groups exactly on the basis of right answers on the initial test, we find that the isolated group was a bit more rapid in work on the initial, but the mixed group was more rapid in the final test. Significant differences in speed on the final test for the mixed group lay in the processes of subtraction, multiplication, and division of whole numbers, the composite for whole numbers in general, addition and subtraction of fractions, and the composite for fractions in general. In no process was the isolated group significantly more rapid, and for the tests as a whole the mixed group was more rapid but not significantly so.

c. On the Basis of Percent of Accuracy.—Both groups gained in power over the processes drilled upon when gain is measured either by increase of speed or increase of accuracy. Obviously, the common element of systematic drill was profitable. The excess gain in accuracy of one group over the other was computed. On this basis the mixed type of drill yielded the greater gain. The excess of percent gain in accuracy varied from an excess of 3 in subtraction of whole numbers to one of 69 in denominate numbers. A 23 percent excess gain on the total composite score in favor of Group M is perhaps the single fact of greatest importance.

d. Movement within Quarters.—In Table VIII comparisons between the two groups (470 cases) are made in an entirely different manner. The contribution of each of the experimental groups to each of the quarters of the total group in terms of percents is shown. In the pre-test section of the table it will be noted that both groups con-

TABLE VIII.—COMPARISON OF THE CONTRIBUTIONS TO QUARTERS IN TERMS OF PERCENTS: GROUP M VERSUS GROUP I, EXACTLY PAIRED

	First Quarter		Second Quarter		Third Quarter		Fourth Quarter	
	Group M	Group I	Group M	Group I	Group M	Group I	Group M	Group I
Pretest								
For All Twelve Measures	50	50	50	50	50	50	50	50
Final Test								
Addition of Whole Numbers	38	62	43	57	58	42	60	40
Subtraction of Whole Numbers	37	63	49	51	51	49	63	37
Multiplication of Whole Numbers	35	65	49	51	60	40	55	45
Division of Whole Numbers	38	62	51	49	55	45	56	44
Composite of Whole Numbers	28	72	51	49	68	32	68	32
Addition of Fractions	33	67	55	45	55	45	59	41
Subtraction of Fractions	34	66	49	51	61	39	57	43
Multiplication of Fractions	31	69	49	51	57	43	62	38
Division of Fractions	27	73	44	56	60	40	69	31
Composite of Fractions	28	72	47	53	56	44	59	41
Denominate Numbers	34	66	45	55	57	43	65	35
Composite of Test	27	73	49	51	62	38	61	39

tributed equally (50 percent each), to each of the quarters of the total group—necessarily because of the pairing of cases on pre-test scores for purposes of valid comparison on final test scores. However, it is very evident from a study of the table that Group M pupils consistently move out of the lower quarters in competition with Group I pupils, while Group I pupils consequently move down into the lower quarters. Table VIII shows in a rather striking fashion the effect of mixed versus isolated drill on the maintenance of skills dealt with in this investigation.

The movement within quarters of the total group of pupils using mixed drills in comparison with those using isolated drills as reported in Table VIII will be to many the most satisfactory comparison offered. This is summarized more clearly in Table IX.

TABLE IX.—NET MOVEMENT WITHIN QUARTERS FROM SCORES ON PRETEST TO SCORES ON FINAL TEST

	Pre-Test				Final Test			
Quarter	1	2	3	4	1	2	3	4
M—Group	50	50	50	50	27%	49%	62%	61%
I—Group	50	50	50	50	73%	51%	38%	39%

V. SUMMARY OF RESULTS

Out of 192 comparisons reported in the foregoing tables, and others deleted for space considerations, 186 were heavily in favor of Group M, five were slightly in favor of Group I, while in one case the two groups were tied. In this statement speed of work, as measured by the number of examples attempted, is not considered. These 192 comparisons are exhibited in Table X.

TABLE X.—SUMMARY OF COMPARISON OF MIXED VERSUS ISOLATED TYPES OF MAINTENANCE DRILL ORGANIZATION

Comparison in Terms of	Number	In Favor of		
		Mixed Drill	Isolated Drill	Tied
Right Answers	48	48	0	0
Gain Percents	48	48	0	0
Accuracy	48	45	2	1
Contribution to Quarters	48	45	3	0
Total	192	186	5	1

VI. Conclusions

Within reasonable limits the results of this study justify the following conclusions:

1. All pupils profited by use of drills furnished them, but those using mixed drills showed 23 percent greater gain than those using isolated drills.

2. The pupils making lowest scores on the pre-test showed the greatest relative gain.

3. Mixed drill is significantly superior to isolated drill in the maintenance of skills in arithmetic considered in this study. When differences in gains are consolidated and reported as a single figure, the total difference in favor of the mixed type of drill is over nine times the probable error of that difference.

4. Those functions most recently taught showed most gain in both groups, but, as in other operations, mixed drill produced the greater benefit.

5. Pupils using mixed drills learned to work more accurately and more rapidly than pupils using isolated drills.

6. The weekly drill scores of pupils using mixed drills fluctuated much less widely than the scores of pupils using isolated drills, and pupils using the mixed drills were reported by teachers to sustain interest at a higher level, presumably for this reason.

7. Of all the skills dealt with whose pre-test status would admit of reasonable gain, division of whole numbers was the least affected by the drill program considered from the angle of absolute gains. This would indicate that the process had been poorly mastered in the beginning.

In this study the number of subjects was large, the reliability of the test used was high, and the length of the drill program was long enough to support the contention that the results obtained were reliable. Experiments of a similar nature on similar problems are needed in other grades. Only through such experimental channels can many of the practices now based on mere opinion be vindicated or condemned.

CHAPTER VII

THE LEARNING OF THE ONE HUNDRED MULTIPLICATION COMBINATIONS[1]

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The report which follows contains a statement of the problem, a brief description of the methods used, data on difficulty and on errors, an analysis of difficulty, some statistical comparisons, and a summary.

I. THE PROBLEM

This report gives the findings of a rather exact study of the learning of the one hundred multiplication combinations by twenty-five third-grade children. The study aimed to supply useful data on several aspects of multiplication; for example:

1. To determine the amounts of practice needed by typical third-grade children to learn each of the one hundred basic multiplication combinations. (Such knowledge is useful in curriculum construction and in appraising classroom material and technique.)

2. To determine the amount of practice needed by the children to maintain the combinations after learning.

3. To determine the relative difficulty of the one hundred combinations by means of analyses of attempts to learn them (by recording and summarizing all the responses and the types of responses made by the children to the various combinations).

4. To increase our knowledge (by means of an analysis) of the errors occurring during learning.

5. To throw light upon the nature of the combinations representing various difficulty levels.

6. To increase our knowledge, through various analyses of data,

[1] Mr. Norem's work on this study was done at the State University of Iowa during 1927-28.

of the general problem of difficulty of the one hundred combinations.[2]

II. The Method

The experiment was conducted with twenty-five pupils in the third grade of the Elementary School of the University of Iowa during the school year 1927-1928. It was carried out by means of an individual drill and test program involving the initial learning of the one hundred fundamental multiplication combinations. Strict individual work was essential because all practice was to be recorded. To prevent unknown amounts of practice at home, letters were sent to the parents or guardians of the children asking them to refrain from giving assistance in drill work in multiplication at home. The children were instructed to practice only when working in school.

The multiplication process was explained to these pupils by the additive method. They were directed, however, not to count up to the answers, but to memorize them in their relation to the combinations. Occasional reports secured from the children enabled the experimenter to learn approximately how nearly they followed this direction.

Examples of these reports are:

"I just look at them a little bit and they come to me."

"I get them by writing them in my mind. I do not try to figure any out but try to remember them."

"If I have had them before, I just remember what it was."

"They just come to my mind, never figure anything out unless I just have to.

"Just seem to remember it, I don't know why it is."

"You remember the answer in your mind when you are told it. I get it best that way. I don't try to figure them out but you can do it."

"After they are learned, I look at it closely and when I see the numbers the answer just comes to me."

The drill and test program was opened by two preliminary tests given a week apart on the one hundred combinations to determine which of them were already known by the children.

The combinations not responded to correctly in both of these tests were the object of a learning and drill program until mastered. A child was tested on a mastered combination once a week for a period of six weeks, and then once a month to a limit of three months. This

[2] Reviews of related studies that were made in this connection are omitted in the present report because adequate descriptions of them are available elsewhere.

was to see if one practice a week was sufficient for maintenance. At the close of the experiment each child's response time to every combination was taken by means of a polygraph. Individual record charts were kept for each child. The charts exhibited all practices, all errors, and all undue hesitations which were interpreted as evidence of incomplete learning. These form the basis for the data in this report. The combinations were presented and learned in random order, but starting with the smaller combinations and working up. Tables were not taught, since it is fairly well agreed that the multiplication tables facilitate the so-called 'table-errors.'

It is needless to repeat here the minutiae of the experimental procedure. Briefly, the procedure was to count the number of practices that the children needed to learn each combination well enough to remember it over a period of two to three months if aided by but one practice a week in the maintenance, or review, program. Throughout the work no unusual motivation or technique was used. An earnest attempt was made to do nothing which could not be easily done in any typical third-grade classroom.

Two features of the learning in this study were not typical of classroom learning. First, each child worked only when an observer could easily count his practice. The children, however, soon became used to this observation and showed no serious signs of nervous tension because of the presence of an observer. Second, all errors were immediately corrected. It is doubtless true that in many learning situations progress is impeded, perhaps significantly, by practice in error. However, drill and learning devices are available on a commercial basis which are so constructed that practice in error is certainly reduced to a bare minimum. Hence this aspect of the experiment after all approximates a good learning situation quite possible in a classroom.

At the close of the experimental part of the study each child's response time to every combination was taken by means of a Jacquet polygraph. By 'response time,' as used here, is meant the time that elapsed from the moment the child's attention was directed to a given combination until the answer was written out. A final response time record was prepared as is illustrated in Table I.

TABLE I.—TYPICAL RESPONSE TIMES OF SIX PUPILS, RECORDED IN SECONDS

Combination	Pupils					
	A	B	C	D	E	F
2 × 2..........	3.0	1.6	1.4	1.0	1.6	3.0
3 × 4..........	2.8	1.4	1.0	1.2	3.2	2.6
5 × 3..........	4.0	8.4	2.0	1.4	5.6	2.2
0 × 1..........	3.4	2.2	2.4	2.2	7.2	2.4

The significance of time data is based on the assumption that hesitation is evidence of incomplete learning. Also, it is possible that after as well as during learning, variations in the promptness of response are useful in determining true difficulty of the combinations.[3]

III. THE RESULTS

The data obtained in the course of this experiment were carefully surveyed, summarized, and finally organized into a form which the experimenters judged would set the results out in a serviceable fashion.

1. Data on Difficulty Rankings

Table II presents significant data relating to the one hundred multiplication combinations. The legends for the column headings in Table II will facilitate a comprehension of it if given here in full.

Column A: General difficulty rank
Column B: Composite scores [4]
Column C: Average number of learning responses
Column D: Average number of errors
Column E: Average response time in seconds
Column F: Percent knowing combinations at beginning
Column G: Average number of weekly tests.[5]

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[3] For a full description of the experimental procedure see: Norem, Grant M.—*The Learning of the One Hundred Multiplication Combinations.* An unpublished investigation available from the Librarian, State University of Iowa, Iowa City, Iowa.

[4] These scores are obtained by weighing and pooling the various aspects of difficulty—the number of learning responses, the number of erroneous responses, and the response time.

[5] This shows the extent to which the investigation was carried into the field of maintenance.

TABLE II.—DATA ON THE ONE HUNDRED MULTIPLICATION COMBINATIONS

	A General Rank	B Composite Scores	C Av. No. Responses	D Av. No. Errors	E Av. Response Time	F Percent Knowing	G Av. No. Tests
6 × 9	100	456	84.20	10.52	4.13	....	4.16
7 × 8	99	435	68.32	10.16	7.88	4	2.80
7 × 6	98	429	77.96	9.20	5.63	4	3.72
8 × 6	97	419	73.88	9.48	5.38	4	3.24
4 × 8	96	412	76.64	9.28	4.00	8	4.28
4 × 9	95	382	74.56	8.88	1.95	....	5.08
7 × 9	94	377	67.36	7.76	6.00	4	3.16
7 × 7	93	370	69.84	8.16	3.50	....	4.44
6 × 7	92	344	57.40	7.00	7.00	8	3.68
4 × 7	91	328	59.12	7.24	4.05	8	5.76
7 × 5	90	327	63.28	7.28	2.40	4	4.56
5 × 9	89	314	56.60	6.84	4.10	8	5.20
8 × 8	88	312	52.84	6.88	5.13	....	3.00
9 × 7	87	272	40.32	4.68	9.00	4	3.44
8 × 5	86	271	48.84	6.40	2.65	4	5.24
9 × 6	85	260	41.16	4.32	7.88	4	3.00
3 × 9	84	259	47.36	5.76	2.95	8	5.32
8 × 7	83	256	33.88	4.08	10.50	4	3.64
4 × 6	82	227	41.24	4.12	4.35	4	6.32
5 × 6	81	207	38.24	3.64	4.00	4	5.36
9 × 4	80	206	32.56	3.56	6.00	8	4.84
5 × 8	79	206	34.88	3.12	6.00	8	5.20
3 × 8	78	205	41.12	2.96	4.00	12	6.12
6 × 8	77	200	28.96	3.52	6.63	4	3.80
7 × 4	76	198	36.16	3.64	3.63	8	4.56
6 × 6	75	196	29.52	3.20	6.58	4	5.04
9 × 5	74	191	31.64	3.44	4.83	20	5.28
8 × 9	73	191	27.84	2.64	7.67	4	4.80
3 × 6	72	172	33.28	2.60	3.55	6	6.76
8 × 3	71	168	28.68	2.20	5.50	....	5.72
4 × 4	70	168	29.24	3.40	3.00	16	7.00
8 × 4	69	158	24.00	1.92	6.50	4	5.08
9 × 8	68	141	19.92	1.32	7.13	....	4.56
4 × 3	67	137	18.48	3.52	2.95	8	7.36
5 × 7	66	136	20.16	1.56	6.00	4	4.80
9 × 3	65	125	18.52	1.72	4.88	8	5.48
4 × 5	64	123	20.20	2.60	2.48	16	7.56
3 × 7	63	119	21.48	1.50	3.45	8	7.24
6 × 4	62	119	26.28	1.28	2.25	16	6.72
6 × 3	61	116	23.48	1.60	2.25	4	7.28

TABLE II.—(Continued)

	A	B	C	D	E	F	G
6 × 5	60	98	13.24	1.08	4.88	....	5.64
3 × 4	59	95	14.64	1.56	3.13	8	8.12
5 × 5	58	92	15.52	1.64	2.25	28	7.08
9 × 9	57	88	12.20	1.76	4.58	4	6.08
5 × 3	56	87	14.44	1.36	2.60	20	8.04
5 × 4	55	85	14.76	1.36	2.25	16	7.84
0 × 1	54	73	8.00	1.56	2.80	20	8.52
7 × 3	53	68	13.56	.76	1.80	12	7.40
5 × 2	52	62	9.12	.88	2.45	52	8.48
6 × 0	51	62	7.96	1.24	2.15	28	8.40
7 × 2	50	69	7.56	.72	3.00	44	8.28
3 × 5	49	55	9.44	.24	2.67	44	8.08
8 × 2	48	49	7.16	.56	2.13	28	7.92
2 × 9	47	47	4.64	.16	3.60	44	8.16
0 × 4	46	46	3.04	.20	3.95	24	7.72
4 × 2	45	46	4.44	.36	3.15	56	8.44
0 × 6	44	46	3.92	.20	3.63	20	7.28
1 × 1	43	45	5.48	.68	2.05	48	8.12
0 × 7	42	45	5.00	.60	2.35	36	8.24
2 × 4	41	44	4.80	.24	3.05	52	8.44
3 × 3	40	42	5.36	.24	2.60	24	8.20
2 × 3	39	42	3.92	.36	2.85	32	8.32
9 × 2	38	41	5.20	.24	2.55	28	8.04
6 × 2	37	40	4.28	.24	2.75	52	8.56
3 × 0	36	39	4.00	.56	2.13	28	8.64
2 × 7	35	37	4.16	.08	2.75	44	8.40
1 × 0	34	36	2.60	.28	2.80	36	8.40
8 × 0	33	36	3.24	.12	2.85	32	8.00
3 × 2	32	35	4.56	.16	2.15	40	8.28
2 × 8	31	35	3.00	.08	2.80	40	8.52
0 × 2	30	34	3.44	.24	2.35	24	8.00
0 × 9	29	33	3.52	.32	2.00	32	7.96
2 × 1	28	32	2.12	.04	2.95	52	8.52
2 × 2	27	30	2.80	.28	2.07	64	8.48
1 × 2	26	30	1.72	....	2.93	44	8.48
0 × 5	25	30	3.40	.20	1.95	32	8.08
2 × 0	24	30	2.36	.04	2.60	28	7.80
1 × 7	23	29	1.52	.20	2.48	52	8.64
3 × 1	22	29	2.00	.04	2.60	40	8.24
7 × 0	21	28	2.56	.08	2.55	52	8.00
1 × 6	20	28	1.96	.08	2.40	40	8.56
1 × 5	19	28	.68	....	3.00	64	8.88
2 × 5	18	27	2.28	.08	2.27	60	8.68
7 × 1	17	27	.88	.16	2.60	60	8.76
8 × 1	16	27	2.24	.04	2.28	40	8.68

TABLE II.—(CONTINUED)

	A	B	C	D	E	F	G
1 × 4	15	26	.76	.04	2.68	52	8.76
5 × 0	14	26	3.16	.16	1.60	28	8.12
0 × 8	13	25	2.16	.04	2.15	36	8.04
0 × 3	12	25	2.40	.20	1.73	36	8.12
4 × 0	11	25	2.52	.08	1.87	32	8.20
4 × 1	10	23	1.32	.20	1.89	48	8.80
2 × 6	9	23	1.40	.08	2.05	64	8.80
0 × 0	8	23	.12	....	2.65	40	8.96
1 × 9	7	22	1.00	.08	2.05	56	8.72
1 × 3	6	22	.96	.04	2.13	68	8.76
5 × 1	5	22	1.36	....	2.05	64	8.56
6 × 1	4	21	1.32	.04	1.95	60	8.36
9 × 0	3	20	2.24	.12	1.29	28	8.12
9 × 1	2	18	1.40	.04	1.60	60	8.44
1 × 8	1	18	.80	.08	1.73	64	8.60

a. Comments on Table II: Column A.—Column A of Table II may be considered first. The nature of some of the items given here calls for further comment. The three factors, practice, error, and speed, and the weights attached to them, which were used to make up the general difficulty rank set forth in Column A of Table II, were selected entirely on a common-sense basis. The Pearson product-moment correlation between average number of responses and average number of errors is .95 ± .006.

The question arises as to how closely related the difficulty rankings are to the general order in which the combinations were learned. The coefficient of correlation between these two factors was .51 ± .05 when obtained by the Spearman rank-difference method.

It is evident that sequence of learning somewhat affects the difficulty of the items learned. However, combinations have to be learned in some order, no matter how random the learning may be. An examination of Table II will show that 0 × 1 is the hardest of the zero combinations. This is due, in all probability, to the fact that it was the first zero combination attempted. In general, it seems safe to assume that the sequence of the combinations in learning (when the table order is avoided) affects, but by no means dominates, the learning difficulty of the combinations.

In a later section of this report a comparison will be made between the rankings presented here and rankings based on frequency of errors made by upper-grade children. Here it is enough to state that

Column A—general difficulty rankings—may be used to guide decisions when the relative emphasis on the various combinations is an important consideration. Probably its most important use is in connection with judging the merits of drill provisions in text and supplementary materials and in the construction of detailed suggestions to teachers on the teaching of the multiplication combinations.

b. Column C.—From a study of Column C we gain an idea of the amount of practice needed by the average child for learning the combinations when the learning situation approximates typical conditions. Under unusually effective conditions the amounts of practice needed may be somewhat less than those reported here. Conversely, under serious handicaps, the learning may require substantially more practice than suggested in Column C. The averages of less than one response appearing in this column (and in others) are due to the fact that some of the combinations were known to a large proportion of the children at the beginning of this study.

It is important to know the reliability of these data. Table III reports sample probable errors of the data of Column C, Table II, which data are averages. Because of lack of space it is impossible to do more than summarize the facts. The probability is that the

TABLE III.—SAMPLES OF RELIABILITY MEASURES OF THE AVERAGE NUMBER OF LEARNING RESPONSES

Combinations	Average No. of Learning Responses	P.E. (aver.)	Average Difference in No. of Learning Responses within Three Ranks of Combinations	Ratio of Average Difference to P.E. (aver.)*
6 × 9......	84.20	7.27	3.21	.44
4 × 7......	59.12	5.19	2.08	.40
5 × 6......	38.24	3.56	1.21	.34
8 × 3......	28.68	3.25	.92	.28
4 × 3......	18.48	2.55	.92	.36
0 × 1......	8.00	1.07	.84	.79
3 × 2......	4.56	1.09	.16	.15
8 × 0......	3.24	.47	.09	.19
2 × 5......	2.28	.55	.06	.11
9 × 1......	1.40	.38	.07	.18

*This column shows what part the "average difference in the number of learning responses between the various ranks" is of the probable error of the average. If the ratio was 1.00, then the probability would be that the rank of the combination would be correct to within one rank; if it was .25 the probability would be that the rank would be correct to within four ranks, etc.

TABLE IV.—SAMPLES OF RELIABILITY MEASURES OF THE AVERAGE NUMBER OF ERRORS

Combinations	Average No. of Errors	P.E. (aver.)	Average Difference in No. of Errors within Three Ranks of Combinations	Ratio of Average Difference to P.E. (aver.)
6 × 9......	10.52	.60	.41	.69
4 × 7......	7.24	.82	.22	.27
7 × 4......	3.64	.56	.13	.24
8 × 9......	2.64	.38	.17	.43
3 × 4......	1.56	.34	.04	.12
7 × 3......	.76	.19	.08	.42
1 × 0......	.28	.07	.02	.28
4 × 1......	.20	.07	.03	.40
2 × 7......	.08	.04	.01	.18
9 × 1......	.04	.03	.01	.26

TABLE V.—SAMPLES OF RELIABILITY MEASURES OF THE MEDIAN RESPONSE TIME

Combinations	Median Response Time in Seconds	P.E. (mdn.)	Average Difference in Response Time within Three Ranks of Combinations	Ratio of Average Difference to P.E. (mdn.)
9 × 9......	4.58	.75	.13	.17
0 × 4......	3.95	.50	.07	.13
1 × 0......	2.80	.41	.02	.04
0 × 7......	2.35	.45	.03	.06
6 × 1......	1.95	.33	.03	.08

most reliable 'average response' rankings in Column C are correct to within three ranks, that the least reliable ones are correct to within fifteen ranks, and that the ranks of the more difficult combinations are the most reliable.

c. Column D.—Sample reliabilities of the number of errors made during learning as reported in Column D are reported in Table IV.

d. Column E.—Sample reliabilities of the measures of the response time are reported in Table V.

2. Individual Differences

Table VI contains items of interest relative to the type of children who took part in this experiment and their individual variations in learning and retention. The legends for the column headings in this table are as follows:

"C. A." is the chronological age.

"I. Q." is the intelligence quotient.

"Total Combinations Passed in Pretests" indicates the total number of combinations which were known to the child at the beginning of this study.

"Total Errors in Responses" shows the total number of errors made by each child.

"Total Learning Responses" gives the total number of learning responses and includes all the responses made previous to Maintenance Tests.

"Total Responses in Maintenance" presents the total number of responses which were carried in the review work for at least six weeks with but one practice per week.

In no case was a test response which was erroneous or followed by a wrong response entered in the column headed "Total Responses in Maintenance Work." All such test responses were included in

TABLE VI.—INDIVIDUAL DIFFERENCES IN THE CHILDREN USED IN THIS EXPERIMENT

Child	C. A.	I.Q.	Total Combinations Passed in Pretests	Total Errors in Responses	Total Learning Responses	Total Responses in Maintenance Work
A	7–11	146	48	54	532	807
B	9– 6	103	5	275	2,565	657
C	8– 8	119	34	207	2,008	712
D	8– 9	107	36	320	2,079	711
E	8– 6	108	43	317	2,225	712
F	8–11	126	65	17	516	845
G	8– 7	117	35	79	1,078	756
H	8– 7	121	35	86	1,293	772
I	8– 7	104	19	237	2,219	664
J	9– 4	136	36	268	2,758	661
K	8– 3	145	40	51	1,253	783
L	9– 0	126	17	288	3,271	568
M	10– 0	102	37	179	1,129	688
N	9– 2	98	25	221	2,128	692
O	9– 2	148	42	267	2,316	722
P	9– 4	111	16	237	2,671	624
Q	8– 1	143	1	108	1,426	677
R	9– 3	117	25	517	3,105	647
S	8– 8	134	42	305	2,874	613
T	8– 9	140	27	340	2,114	713
U	8– 9	110	18	220	2,180	600
V	8– 4	128	29	143	2,461	665
W	9– 7	108	49	204	1,782	763
X	8– 6	110	24	271	2,181	668
Y	8– 9	126	44	154	1,645	625
Aver.	8–10	121.3	32	215	1,992	694

the column headed "Total Learning Responses." The last column in Table VI, like the last column in Table II, is indicative of the extent to which the combinations were carried in the maintenance test series. The great variation between individuals in a learning situation is very apparent from this table. The desirability of devising methods of learning which allow flexibility in amounts of practice within a group is also clear. It is interesting to note in connection with Table VI that the Pearson product-moment coefficient of correlation between the intelligence quotients and the total number of learning responses made by each child is—.09 $\pm$ .13.

3. Data on Errors

Table VII shows the most prevalent errors and throws some light on the nature of the errors made by the children who acted as subjects in this investigation. Of the 5,365 errors recorded in the course of the study, the 3,895 most common ones are presented in this table. The errors appearing here are those made at least five times in connection with the forty combinations having the greatest frequency of errors. The table should be read thus: in learning 8×9 the response 56 was given 5 times; 63, 16 times; 64, 7 times; and 81, 6 times. There were also given 21 miscellaneous responses (Column M), making a total (Column T) of 55 erroneous responses to the combination 8×9.

The C's have been placed in the body of the table to locate the correct response readily.

Comments on Table VII.—Most of the erroneous responses presented in Table VII (91 percent of them, in fact) are the correct responses to some other of the hundred multiplication combinations. The figures appearing in the column headed M at the right of the table indicate the remaining 9 percent. Further, 67 percent of all the errors indicated are the correct responses of closely allied combinations. By the 'closely allied combinations' is here meant combinations whose multipliers and multiplicands are not over one digit above or one digit below those of the combination in question; thus the combinations closely allied to 7×8, as here defined, are 8×8, 6×8, 7×9, 7×7, 6×9, 8×7, 8×9, 6×7.

One naturally concludes that the zero-combination errors, such as $6 \times 0 = 6$, are due to processes carried over from addition as $6 + 0 = 6$. The large number of errors appearing in the responses to 6×0 and 0×1 as compared with the other zero combinations

RESPONSES

COMBINATIONS	1	6	8	9	12	15	16	18	20	21	24	25	27	28	30	32	35	36	40	42	45	48	49	54	56	63	64	72	81	M	T
8×9																									5	16	7	C	6	21	55
8×8																						12	5	8	23	33	C	36	5	10	132
9×7																								17	20	C	34	23			94
7×9																		8		6		5	11	33	28	C	40	20	5	18	174
8×7																				7		12		32	C	15	7			6	79
7×8																				32		23	5	67	C	23	19	12		32	213
9×6																					8	8		C	52	15	8				91
6×9																		8	8	20	32	13	8	C	48	42	7			37	223
7×7														7					5	44	13	29	C	14	5	10	15			15	157
8×6																16		13	20	38	10	C	7	23	26		7			36	196
6×8																		8		18		C		22	22		6				76
9×5																	7	11	14		C		7	12							51
5×9																		28	29	11	C		10	13						8	99
7×6															12	36	17	11	15	C	16	29	19	16	10					21	202
6×7															11	24	8	4	12	C	6	31	5	24	10						135
8×5											7				20	24	31		C	18	25	6									131
5×8															7		10	8	C	8	19	5									57
9×4													9	8	6	20	10	C			14									5	72
4×9													17	19	5	27	34	C	5		24			7						54	192
6×6																8	7	C				13		7	5		5				45
7×5												13			14	37	C	22	20	10	29									6	151
5×7															7	5	C	6			6										24
8×4											9		5	6		C		18		6		7									51
4×8											35		8	16	17	C	8	32	21	21	5	14	5				5			31	218
5×6															C	8	20	10	22	10											70
7×4										12	13		13	C		14		7		5											64
4×7										27	34	6	34	C		18	13			6										15	153
3×9								8			28	5	C	36		6		14												21	118
8×3							5	13			C		14	9																	41
3×8							7	9	5	22	C		15			6														7	71
6×4										5	C			5	13	9															32
4×6					7		6	25	8	19	C			5		10															80
3×7									10	C	13			7																	30
4×5					5	6			C	6		15																			32
6×3							16	C		6	11																				33
3×6						6	11	C	7	6	25																				55
4×4			9		19		C	37	5		10																				80
4×3			6	10	C	8	9																								33
6×0		43																													43
0×1	42																														42
	42	43	15	10	31	20	54	92	35	103	185	39	115	118	112	268	165	208	171	260	207	207	82	295	254	154	160	91	16	343	

TABLE VII.—THE MOST COMMON ERRORS MADE WITH THE FORTY MOST DIFFICULT COMBINATIONS

can be explained by the fact that they were first zero combinations to be learned. When the children understood the nature of these combinations they had little trouble, as is indicated by the difficulty rankings in Table II. Some products, such as 36 and 42, seemed

to be easily connected with a wide range of combinations; while others, such as 49, seemed less susceptible to promiscuous connection-forming. The multiple appearing most frequently as an error is 54. Commonly the responses which appear most often as errors are the correct answers to the combinations requiring the greatest amount of practice. Thus a response which is the correct one most often also gets to be used as an incorrect one most often.

When we consider the delicacy of adjustment required in the correct formation of the combinations involved, we become quite patient with the young learner. So many wrong responses are almost correct and are so closely related to correct ones. From Table VII take, for example, characteristic wrong responses to 6×7 and see how excusable they are:

24 wrong responses of 32—but that is rather like 42
11 wrong responses of 30—but 6×5 does equal 30
8 wrong responses of 35—but 5×7 does equal 35
12 wrong responses of 40—but 5×8 does equal 40
31 wrong responses of 48—but 6×8 does equal 48
5 wrong responses of 49—but 7×7 does equal 49

No error like 43 or 44 appears frequently. On the contrary most of the 'wrongs' seem to be 'right' associations attached to the 'wrong' stimulus. For example, 4×3 may be tied to 9 (3×3) or to 15 (5×3) or to 16 (4×4), but it is not tied to 7 $(4 + 3)$ or to 1 $(4 - 3)$ or to some correct response to a 'distant' combination.

4. An Analysis of Difficulty

The question arises as to why combinations vary in difficulty. Though no final answer can be given to the inquiry, the nature of the data collected suggests certain possible causes of the relative difficulty of the combinations. These will be taken up in turn and evaluated.

a. The 3×6 Versus the 6×3 Type.—The order of magnitude of the two factors in a combination naturally suggests itself as one condition of degree of difficulty. It is of interest, therefore, to compare the general difficulty of the combinations when the multiplier is smaller than the multiplicand (as 3×6 or 5×8) with the difficulty of the same combinations when the multiplier is larger than the multiplicand (as 6×3 or 8×5). In order to secure a convenient and reliable basis, a comparison was made of the number of errors in the learning responses of the two classes of combinations. About 11 per-

cent of the learning responses of the 3×6 type of combinations and 10 percent of those of the 6×3 type were erroneous. The difference of 1 percent in the difficulty of the two situations is not large enough to be significant. Hence no source of difference in difficulty can be found here. These facts support what common sense would assume and what has been shown to be true in addition and subtraction.[5]

b. Difficulty as Indicated by Learning Responses.—If the average numbers of learning responses required by the subjects of this experiment to learn each of the one hundred multiplication combinations are arranged in tabular form such that multipliers lie along the left-hand margin of the table and the multiplicands along the upper margin, it will be noticed that a considerable relationship exists between the size of the product of a combination and the number of learning responses needed for mastering it. This outcome agrees with common sense and with similar findings in addition and subtraction. Statistically, the correlation between the size of the product and the general difficulty rankings gave a coefficient of $.86 \pm .03$ by the Spearman rank-difference method.

Obviously, other factors operate to make one combination more difficult than another. Some of these factors might profitably be sought. Here we may point out merely that the products correct for several different combinations and closely resembling others seem to be the tangled centers of error.

c. Relation between Difficulties in Multiplication, Addition, and Subtraction.—The question might profitably be raised as to whether there is any relationship between the difficulty ranking of the one hundred multiplication combinations and that of the corresponding one hundred addition combinations. The averages of the ranks of the forty-five multiplication combinations and their reverses were correlated with the averages of the ranks of the corresponding addition combinations and their reverses, as ranked by Knight and Behrens. The coefficient of $.27 \pm .08$ secured by the Spearman rank-differencee method would indicate little relationship between difficulty in addition and difficulty in multiplication. The correlation between multiplication and subtraction obtained in the same way was $-.07 \pm .10$.

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[5] Knight, F. B., and Behrens, M. O., *Learning the 100 Addition and the 100 Subtraction Combinations*, p. 100.

Further inquiries into the relations existing between addition, subtraction, and multiplication will probably prove valuable, especially in attempts to discover facilitations and interferences which may exist between combinations in the several processes. But such inquiries are beyond the scope of the present data.

One topic of interest is the precise relative difficulty of the combinations in addition, subtraction, and multiplication. An index of difficulty in terms of amounts of practice needed to learn may be obtained by comparing the amounts needed for the multiplication combinations as reported in Table VIII with similar amounts reported by Knight and Behrens [6] for addition and subtraction.

These comparisons (for amounts of practice for average children learning addition and subtraction in the second grade and multiplication in the third grade) show that: (a) 37 addition combinations are relatively more difficult than corresponding subtraction or multiplication combinations; (b) 32 subtraction combinations are relatively the most difficult, and (c) 31 multiplication combinations require more practice than do the corresponding combinations in addition or subtraction. From the averages of the averages, subtraction seems a bit the easiest process. In these investigations subtraction was taught as a unit after addition had been learned; if it had been taught by sprinkling it throughout the addition, it might very well have been of even greater relative ease.

d. Comparison with the Clapp Difficulty Rankings.—There is evidently little relationship between the difficulty ranking presented by this investigation and that by Clapp in his well-known study dealing with upper-grade children. The correlation of the rankings of these two studies by the Spearman rank-difference method gives a coefficient of $.32 \pm .06$. This lack of agreement in rankings is of considerable importance because these studies represent two very different types of investigation. Clapp's ranking of the one hundred combinations was based on the number of errors made by upper-grade pupils whose previous practice with each combination was unknown. How much Clapp's rankings are due to inherent difficulty and how much to inequalities in practice received is not clear.

A general comparison of the two difficulty rankings can easily be made by observation of Table VIII, in which the multiplicands

[6] *Op. cit.*

TABLE VIII.—DIFFICULTY RANKINGS OF ONE HUNDRED MULTIPLICATION COMBINATIONS AS REPORTED BY KNIGHT-NOREM AND BY CLAPP

(Knight-Norem rankings are the upper and Clapp rankings the lower in each cell)

	Multiplier	Multiplicand 0	1	2	3	4	5	6	7	8	9	Total	
Knight-Norem Quadrant Average 33	0	8 1	54 97	30 94	12 88	46 93	25 99	44 91	42 98	13 95	29 86	303 842	Knight-Norem Quadrant Average 46
	1	34 82	42 72	26 39	6 4	15 6	19 10	20 11	23 2	1 15	7 31	193 272	
	2	24 92	28 12	27 9	39 24	41 45	18 5	9 37	35 25	31 7	47 13	299 269	
Clapp Average 47	3	36 85	22 17	32 18	40 28	59 46	49 33	72 36	63 54	78 59	84 56	535 432	Clapp Average 47
	4	11 96	10 23	45 44	67 30	70 22	64 47	82 58	91 73	96 60	95 74	631 527	
Knight-Norem Quadrant Average 41	5	14 83	5 3	52 21	56 43	55 14	58 29	81 62	66 53	79 57	89 52	555 417	Knight-Norem Quadrant Average 82
	6	51 87	4 16	37 34	61 41	62 40	60 38	75 42	92 61	77 76	100 79	619 514	
	7	21 100	17 20	50 26	53 50	76 69	90 48	98 68	93 55	99 77	94 78	691 591	
Clapp Average 46	8	33 89	16 32	48 19	71 35	69 65	86 66	97 75	83 81	88 67	73 71	664 600	Clapp Average 63
	9	3 90	2 27	38 8	65 63	80 70	74 51	85 80	87 84	68 64	57 49	559 586	
Total K-N......... Total Clapp........		235 805	200 319	385 312	470 406	573 470	543 426	663 560	675 586	630 577	675 589		

are arranged along the top and the multipliers are placed along the left. The Knight-Norem rankings are in the upper, the Clapp rankings in the lower half of each cell.

The reader interested in detailed comparisons of rankings based on upper-grade difficulty (a product of inherent difficulty and unknown but unequal amounts of practice) with those based on original learning difficulty will find Table VIII the source of useful data. For example, he will find that Clapp reports spurious difficulty for the zero combinations.

The following correlations present information of interest: The coefficient of correlation between the difficulty ranks of the 3×6 type of combination with those of the 6×3 type (combinations and their reverses) in the Clapp study was $.91 \pm .02$ by the Spearman rank-difference method. It will be recalled that in this investigation the correlation was $.76 \pm .04$ by the Pearson product-moment method. Apparently, as children mature, meaningful relationships increase.

The coefficient of correlation between the difficulty ranking and the size of the product in the Clapp investigation was $.61 \pm .07$. In this study the coefficient of correlation was $.86 \pm .03$. The Spearman rank-difference method was used in computing these correlations. The disagreement, while not violent, is noteworthy.

This series of comparisons with the Clapp difficulty ranking leads one to conclude that there is considerable difference between the findings of the two investigations. The following questions arise: Would these differences be understood if the learning conditions behind the Clapp study were known? Are the differences due to the fact that some combinations are more difficult for beginners to learn than for older children to retain? Should not the difficulties which arise in the learning of the one hundred multiplication combinations be analysed when the process of learning is in progress instead of afterwards, when it must be done in terms of results, the causes unknown? The facts seem to indicate that these questions should be answered in the affirmative.

IV. Summary

The following statements are in summary of this study of multiplication:

1. An approximation of the amounts of practice needed to learn the one hundred multiplication combinations is given in Table II. Significant facts about the subjects used are reported in Table VI.

2. When mastery has been well established, one practice a week is sufficient for maintenance. That, however, is often insufficient practice for maintaining the combinations during the first two weeks following the initial learning of them. How much less than one practice a week would satisfactorily maintain skill is unknown.

3. The relative difficulty of the one hundred combinations as found in this study is shown in Table II.

4. The internal consistency of the data upon which the Knight-Norem rankings are based is high. When difficulty is measured in various ways, high agreement exists. Difficulty, measured by number of responses to learn, correlates $+.95 \pm .01$ with an independent measure of difficulty (number errors made during learning).

The 80th percentile child needs far more practice than the 20th percentile child, but a correlation of amounts of practice, combination by combination, of $.83 \pm .02$ is reported.[7]

5. Facts about errors:

a. About 91 percent of multiplication errors are the correct answers to some other of the one hundred combinations.

b. About 67 percent of the errors are the correct answers to very closely allied combinations, even when combinations are not taught in tables.

c. When one combination has become susceptible to error, combinations very closely allied to it become more susceptible to error.

d. In general, a response which really is the correct one most often also appears most often as an incorrect one.

6. Facts about difficulty:

a. There is considerable correspondence between the difficulty of a given combination, like 2×6, and its reverse, 6×2.

b. The size of the product of a combination is a partial index to its difficulty. Other factors affecting difficulty may be order of presentation, delicate interrelationships, and a series of unknowns.

c. Combinations which are more susceptible to an interchange of multiples are more difficult.

7. Rankings based on learning data do not agree with rankings based on upper-grade errors with previous experience unknown. The Knight-Norem rankings correlate with those of Clapp only $.32 \pm .06$. While both rankings are fraught with error, it is argued that a determination based upon difficulty in original learning is the more valuable.

[7] Data on amounts of practice needed for the 20th and the 80th percentile child have been omitted for economy of space.

CHAPTER VIII

A MEASUREMENT OF TRANSFER IN THE LEARNING OF NUMBER COMBINATIONS

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I. THE PROBLEM

An experimental study was recently performed by the authors in the public schools of St. Paul to determine to what extent the teaching of a fundamental number combination in the direct order transfers to the reverse order of that combination; thus, if the combination $7+5=12$ has been learned, to what degree has the combination $5+7=12$ been established?

Three possibilities may be considered: (1) The two functions may be entirely independent, so that the teaching of the combinations in one order results in no transfer of learning to the same combinations of digits when given in the reverse order; (2) it is possible that a partial transfer may take place, so that if combinations are studied in one order, some are learned in the other order, but not as many as are learned in the order studied; (3) it is possible that transfer is complete, and that practice on the combinations in one order causes the child to learn them equally well in both orders.

A study of texts published over a long period of time revealed that, although the authors have differed in theory as to the necessity of teaching both forms of a combination, there has not been very great difference in practice. Some texts teach what are referred to as the forty-five fundamental addition combinations; others hold that there are one hundred such combinations. A text published in 1847 provided drill for the 100 combinations; a text published in 1895 provided for study of both forms but omitted the zero combinations. Texts published in 1916 and 1918 gave the 45 combinations in their addition tables with all these combinations in the form of larger digits

plus smaller digits. Drill was provided on the reverse combinations but little attention was given to the zero combinations. The most recent texts stress the teaching of the 100 combinations, but the distribution of drill on the direct and reverse forms has not been determined on the basis of experiment. An examination of four arithmetic texts in common use shows that combinations where the larger digit appears first occur with nearly twice the frequency of the combinations in which the smaller digit appears first.

As the result of an extensive investigation Clapp [1] listed all combinations in the order of their difficulty determined by the number of times a combination was missed for each grade and for all grades. The results show that in some cases the same combinations of digits occurring in opposite orders are very different in degree of difficulty. From these results Clapp concluded that combinations of the same digits stated in opposite orders are no more alike than are other combinations. In other words, a pupil may learn one form of a combination, but that knowledge does not carry over to the other form of that combination.

The authors' experience is that in grades above the third a pupil will miss different combinations on successive identical tests. The class as a whole will miss nearly the same number of combinations, but the results for individuals vary considerably. It is apparent that errors on addition combinations above the third grade are quite accidental; that is, errors are due to reasons other than purely mathematical difficulties. It is doubtful whether combining these accidental errors will give a measure of the relative difficulty of learning combinations. This is adequately demonstrated by the results of the recent study of Knight and Behrens.[2]

II. Preliminary Experimentation to Develop Methods

In order to determine the best technique to use and the validity of the results obtained in the experimental solution of the problem, a number of tests were given to hundreds of pupils in various grades before the beginning of the final experiment. Since space will not

[1] Clapp, F. L. *The Number Combinations: Their Relative Difficulty and Frequency of Their Appearance in Textbooks.* Bureau of Educational Research Bulletin No. 1. Madison, Wisconsin: University of Wisconsin, 1924; pp. 20.

[2] Knight, F. B., and Behrens, M. S. *The Learning of the 100 Addition Combinations and 100 Subtraction Combinations.* New York: Longmans, Green and Company. 1928.

permit their extended discussion here, only the conclusions reached by the authors are presented.

1. In general, single tests are unsatisfactory for obtaining reliable data with reference to the pupils' knowledge of elementary number combinations in addition above the third grade.

2. Factors other than that of the knowledge of combinations play a large part in the results obtained by testing pupils above the third grade; *e. g.*, lapses in attention, carelessness, etc.

3. Results in testing are more valid in the lower grades than in the higher grades because the pupils are more likely not to know a number fact and hence the percent of accidental error is less.

4. It may be assumed that the results from tests given to secure a measure of the difficulty of number combinations will be most reliable when the tests are given before and after a period of study to pupils who are studying the combinations for the first time.

5. The study of transfer contemplated in this investigation can best be performed with pupils just entering the 2B grade, who already know a varying proportion of the basic addition facts as a result of incidental experiences both in and out of school.

III. Procedure in the Final Experiment

1. Subjects of the Experimental Study

Three 2B grades were chosen for the final experiment. The classes were designated by the letters L, G, and M, referring to the school in which the classes were located. School L is located in what is considered the best residential district in the city of St. Paul; School G is situated in an average residential community; School M is a downtown school. The enrollment in Class L was 32 pupils, in Class G 33 pupils, in Class M 28 pupils. The ages and I.Q.'s of the pupils in the three grades were about normal for that grade. The pupils had been with their present teachers for two weeks in the new semester before the beginning of the experiment and had therefore become adapted to them. The pupils had been given very little work in oral addition and practically no work in written problems. A large proportion of the pupils were not able even to write all of the ten digits at the beginning of the experiment. Whatever combinations they knew had been learned entirely incidentally, yet in some cases a surprisingly large number were already known.

2. Preliminary Training of Pupils

One week was spent in each class in preparation for the experiment. On Monday each child was given a sheet containing eighteen simple addition examples on the upper half of the paper. The lower half contained the same examples, but to these the answers were given. The answers contained numbers from 2 to 19 inclusive. The first part of the period was spent in writing the answers in the upper half of the addition sheet as they were given in the lower half. This gave practice in placing answers and in writing numbers from 0 to 19. The balance of the period was given to drill in oral recognition of the numbers written.

For the remainder of the week the pupils studied the eighteen combinations and noted their own progress by individual graphs. Full directions were given to the teacher as to the kind of work to be done, so that the procedure in the classes concerned was, so far as possible, identical. At the end of the week the teachers reported that they were satisfied that the pupils could write and recognize all numbers less than twenty, and that they were able to write the answers to the printed examples in their proper places. The children were taught to skip the combinations to which they could not give automatic responses.

3. Organization of Materials and Grouping of Combinations

The 36 reversible elementary addition combinations of significant numbers, not including zero combinations, were divided into three groups of approximately equal difficulty, as determined by the study of Smith.[3] This was done as follows: The 36 reversible combinations of significant numbers were arranged in order of their difficulty as obtained by Smith's study. These were divided into three equal groups containing 12 combinations each. One part contained the combinations that were most difficult; another the combinations that were average in difficulty; the third, the combinations that were the least difficult. One-third of the combinations in each of these parts were placed on one card. Similarly, two other cards were constructed from the remaining combinations, making in all three cards with 12 combinations on each ranging from very easy to very difficult.

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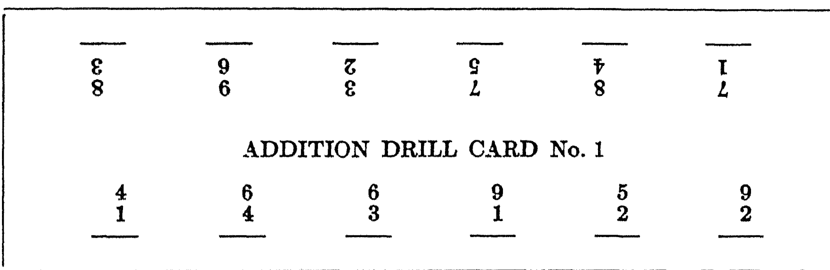
[3] Smith, J. H. "Individual variations in arithmetic." *Elem. Sch. Jour.*, 21: 1921, 762-770.

The combinations were printed on three separate cards in the manner shown in Fig. I. It will be seen that the examples without answers are given on one side of each card, and that the same combinations with answers are given on the reverse side. One of these cards was used for a week for drill and test during each of the daily practice periods.

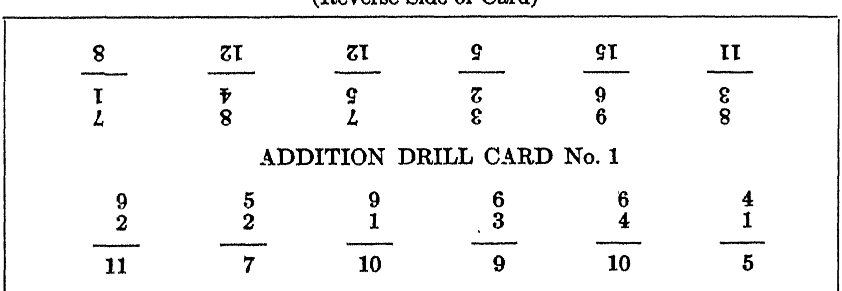
4. Experimental Procedure Used in Studying Combinations

The pupils studied the combinations on the side on which the answers were given and then tested themselves by writing on a sep-

(Front Side of Card)



(Reverse Side of Card)



Combinations on Card II

6	3	7	6	5	4	7	9	9	7	5	8
5	1	6	2	1	2	4	7	8	2	4	5

Combinations on Card III

8	4	2	8	7	6	9	8	9	9	5	8
2	3	1	7	3	1	3	6	5	4	3	1

FIG. I.—FORM OF CARD USED FOR STUDY

arate paper placed below the card the answers to the examples given on the side which did not contain the answers. In case the pupils did not know certain combinations they could find the answers on the reverse side. This method of studying the combinations is referred to in this study as the "drill-card method."

The set of combinations on the respective cards will be referred to as "Card No. 1," "Card No. 2," and "Card No. 3." A combination will be designated as "direct" when it consists of a larger digit plus a smaller digit. The opposite form, a smaller digit plus a larger digit, will be designated as "reverse." As explained above, the order of the combinations on the cards is direct; if the digits are interchanged the combination becomes reverse. A problem such as $\begin{array}{c}7\\ \underline{5}\end{array}$ is read $7 + 5$; *i.e.*, direct order; $\begin{array}{c}5\\ \underline{7}\end{array}$ is read $5 + 7$; *i.e.*, reverse order. The utmost care was taken to have children add from the top down so as to have consistency in the type of combinations.

5. The Methods of Teaching in the Experiment

The final experiment continued for three weeks. Each class studied one card for a full week, beginning Monday and ending Friday, twenty minutes each day. No home work was assigned. In order to secure uniformity in drill, the methods to be used on the different cards were specified in detail. The methods commonly employed in the St. Paul schools and the drill-card method just explained were decided upon in order that the results obtained might be independent of any particular method. The combinations on Card No. 1 were taught by a combination of the usual methods of the teacher and the drill-card method. Card No. 2 was taught by the usual methods of the teacher (flash cards, games, etc.). Card No. 3 was taught exclusively by the drill-card method. Teachers were given full directions for the procedure in each case, so that the instruction in the various classes was as uniform as possible.

6. Order in Which Cards Were Given

The combinations on each card were studied for one week. Class G studied Card No. 1 the first week, Card No. 2 the second week, and Card No. 3 the third week. Class L studied Card No. 2 the first week, Card No. 3 the second week, and Card No. 1 the third week. Class M

studied Card No. 3 the first week, Card No. 1 the second week, and Card No. 2 the third week. This was done so as to get more reliable data on related problems, such as the difficulty of combinations and the relative efficiency of each method of instruction. The distribution of practice is shown clearly in Fig. II.

	Class G	Class L	Class M
First Week......	Card No. 1 Usual Methods Drill-Card Method	Card No. 2 Usual Methods Only	Card No. 3 Drill-Card Method Only
Second Week....	Card No. 2 Usual Methods Only	Card No. 3 Drill-Card Method Only	Card No. 1 Usual Methods Drill-Card Method
Third Week.....	Card No. 3 Drill-Card Method Only	Card No. 1 Usual Methods Drill-Card Method	Card No. 2 Usual Methods Only

FIG. II.—SHOWING THE CARD STUDIED EACH WEEK BY CLASSES G, L, AND M FOR THREE WEEKS, AND THE METHODS USED IN TEACHING

In all work, including drill cards, flash cards, games, board work, and concrete presentations, the larger number of a combination was always written or given first; $\begin{array}{r}7\\ \underline{6}\end{array}$ was read 7 plus 6, never 6 plus 7. In flash cards, the larger number always appeared on top, and in the concrete problems the larger number was given first. In other words, as far as was humanly possible all combinations were taught in the direct order.

7. The Tests Used to Measure the Results

With each card there were two test sheets. Test A contained the identical combinations found on the card in the upper half and the reverse forms of the same combinations in the lower half. These tests are referred to as Test A1, A2, or A3 (the numeral designating the card to which they belonged). The upper part of the test was first given. Before writing the answers of reverse combinations on the lower part of the test, the pupils folded the papers so that the answers in the upper half could not be seen. This test on direct and reverse combinations was given at the beginning of the period on Monday as a pretest and again at the end of the period on Friday to discover growth in knowledge of the combinations.

Test B, a daily test, contained all direct combinations as given on the card, but not the reverse forms. This test was given at the beginning and at the end of each daily period, except on Monday at the beginning and on Friday at the end of the class period, when Test A was given. This test included Test B1, Test B2, and Test B3. B1 contained the combinations on Card No. 1, B2 those on Card No. 2, and B3 those on Card No. 3. The schedule of tests is indicated in Fig. III.

	Monday		Tuesday		Wednesday		Thursday		Friday	
	Beginning	End	Beginning	End	Beginning	End	Beginning	End	Beginning	End
Card No. 1	A1	B1	B1	B1	B1	B1	B1	B1	B1	A1
Card No. 2	A2	B2	B2	B2	B2	B2	B2	B2	B2	A2
Card No. 3	A3	B3	B3	B3	B3	B3	B3	B3	B3	A3

FIGURE III.—DIAGRAM SHOWING SCHEDULE OF TESTS FOR EACH CARD IN FINAL EXPERIMENT

8. Supervision of the Work

There was a personal conference with each teacher before the experiment started. One of the writers explained the purpose of the experiment and the use of materials. Several teachers were interviewed, and those were chosen who in the judgment of the writers would be best fitted for the work. One of the writers was in daily communication with the teachers and conferred with them personally at least once a week. Permission and full coöperation were obtained from the superintendent of schools, the principals and the teachers of each school in which the work was done.

III. Results in the Final Experiment

Tables were devised to show the number of combinations missed in either direct or reverse order in the initial and final test for the week, and those missed in both orders in the Pretest A and in the Final Test A, and also the number of combinations in the direct order missed each day in Test B.

Table I shows the results for Class L, 32 pupils. It should be read as follows: On Pretest A1, given on Monday, 63 combinations were missed in the direct order, 54 in the reverse order, and in 42 of these

cases the same combination was missed by a pupil in both orders. At the close of the period Monday, in Test B1, 40 combinations were missed in the direct order, this test containing no combination in the reverse order; on Tuesday in Test B1, given at the beginning of the period, 31 combinations in the direct order were missed, and at the close 21; similarly, results for Wednesday and Thursday are shown; on Friday at the beginning of the period, 18 combinations were missed in Test B1; at the close of the period in Final Test A1, 7 combinations were missed in the direct order and 4 in the reverse order, and in 2 of these cases the same combination was missed by a pupil in both orders. The results for Tests A2, B2, and A3, B3 are shown in the same way in Tables II and III, respectively. The results for the tests are summarized in totals for each class. The results of the B tests, while not important in this study, are given because they are interesting in showing the rate of learning.

To obtain an expression of the amount of growth in learning, the number of errors in the final test was subtracted from the corresponding number in the pretest, and the remainder was divided by the total number of errors on the first test. Thus, in Table I, 63 combinations in the direct order were missed in Test A1 on Monday, and 7 of the same type were missed in the Final Test A1 on Friday. This shows a reduction of 56 in the number of errors, which is 88.8 percent of the possible gain and may be taken to signify the percent of possible improvement actually achieved. The other percentages in the column are calculated in the same way. Tables II and III show the information, in the same form, for Classes L and M.

From the results in Table I it would appear that there was slightly greater gain in learning the reverse form, which was not studied, than in learning the direct forms which were studied. This is also indicated in the results shown in Tables II and III. The differences in the number of direct and reverse combinations learned is very small, but what slight differences obtain are in the same direction in each class studied. Without further study it is impossible to state conclusively the reason for this result, but we may conclude that in the case studied the learning was at least as great for the reverse order, which was not studied, as for the direct order which was studied.

In Table IV the totals for Tables I, II, and III are combined in a single table, and the totals are given. The percentage of possible gain for all classes is also calculated.

TABLE I.—SHOWING THE NUMBER OF COMBINATIONS MISSED BY 2B-GRADE PUPILS IN EITHER DIRECT OR REVERSE ORDER AND IN BOTH ORDERS ON TEST A, THE NUMBER OF DIRECT COMBINATIONS MISSED EACH DAY ON TEST B, AND THE PERCENTS OF POSSIBLE GAIN

(Class L, 32 Pupils)

Card	Order	Monday			Tuesday		Wednesday		Thursday		Friday			Percent Gain	
		Test A	Both	Test B	Test B		Test B		Test B		Test B	Test A	Both	One Order	Mean
					Beginning	End	Beginning	End	Beginning	End					
1	Direct.....	63	42	40	31	21	17	9	20	16	18	7	2	88.8	90.6
	Reverse....	54			(No	tests	in reve	rse giv	en)			4		92.5	
2	Direct.....	90	61	80	77	40	59	52	40	33	18	23	19	74.4	73.7
	Reverse....	100										27		73.0	
3	Direct.....	95	80	83	63	65	67	65	55	46	55	46	38	51.5	55.7
	Reverse....	110										44		60.0	
Total	Direct.....	248	183	203	171	126	143	126	115	95	91	76	59	Mean 71.5	73.3
	Reverse....	264										75		75.1	

TABLE II.—TABLE SHOWING THE NUMBER OF COMBINATIONS MISSED BY 2B-GRADE PUPILS IN EITHER DIRECT OR REVERSE ORDER AND IN BOTH ORDERS ON TEST A, THE NUMBER OF DIRECT COMBINATIONS MISSED EACH DAY ON TEST B, AND THE PERCENT OF POSSIBLE GAIN

(Class G, 33 Pupils)

Card	Order	Monday			Tuesday		Wednesday		Thursday		Friday			Percent Gain	
					Test B		Test B		Test B						
		Test A	Both	Test B	Beginning	End	Beginning	End	Beginning	End	Test B	Test A	Both	Order	Mean
1	Direct.....	17	9	15	28	11	3	4	10	5	3	4	3	76.4	77.4
	Reverse....	28			(No	reverse	tests)					6		78.5	
2	Direct.....	66	52	31	35	20	13	7	13	9	18	13	11	80.3	78.2
	Reverse....	63										15		76.1	
3	Direct.....	37	21	28	33	17	19	16	21	8	6	7	2	81.0	86.9
	Reverse....	57										4		92.9	
Total	Direct.....	120	82	74	96	48	35	27	44	22	27	24	16	Mean 78.2	80.8
	Reverse....	148										25		82.5	

TABLE III.—SHOWING THE NUMBER OF COMBINATIONS MISSED BY 2B-GRADE PUPILS IN EITHER DIRECT OR REVERSE ORDER AND IN BOTH ORDERS ON TEST A, THE NUMBER OF DIRECT COMBINATIONS MISSED EACH DAY ON TEST B, AND THE PERCENTS OF POSSIBLE GAIN

(Class M, 28 Pupils)

Card	Order	Monday			Tuesday		Wednesday		Thursday		Friday			Percent Gain	
		Test A	Both	Test B	Test B		Test B		Test B		Test B	Test A	Both	One Order	Mean
					Beginning	End	Beginning	End	Beginning	End					
1	Direct.....	39	29	5	8	2	1	0	0	0	0	0	0	100.0	100.0
	Reverse....	45			(No	revers	e tests)					0		100.0	
2	Direct.....	59	50	36	42	31	27	16	14	8	3	2	2	96.6	95.5
	Reverse....	54										3		94.4	
3	Direct.....	40	39	33	28	12	27	6	13	6	5	4	2	90.0	92.6
	Reverse....	42										2		95.2	
Total	Direct.....	138	118	74	78	45	55	22	27	14	8	6	4	Mean 95.5	96.0
	Reverse....	141										5		96.5	

TABLE IV.—TOTALS OF COMBINATIONS MISSED BY 2B-GRADE PUPILS AND PERCENT OF COMBINATIONS, DIRECT AND REVERSE (93 Pupils)

Class	Order	Monday			Tuesday		Wednesday		Thursday		Friday			Percent Gain	
					Test B		Test B		Test B						
		Test A	Both	Test B	Beginning	End	Beginning	End	Beginning	End	Test B	After	Test A	One Order	Mean
L	Direct.....	248	203	203	171	126	143	126	115	95	91	76	59	71.5	73.3
	Reverse....	264										75		75.1	
G	Direct.....	120	82	74	96	48	35	27	44	22	27	24	16	79.2	80.8
	Reverse....	148										25		82.5	
M	Direct.....	138	118	74	78	45	55	22	27	14	8	6	4	95.5	96.0
	Reverse....	141										5		96.5	
Total	Direct.....	506	403	351	345	219	233	175	186	131	126	106	79	Mean 82.0	83.3
	Reverse....	553										105		84.7	

Table VI shows that the total number of direct combinations missed on the pretests was 506 and in the reverse order 553. In 403 of these cases the same combination was missed by a pupil in both orders. If the direct and reverse forms of combinations are of identical difficulty, then the combinations missed in one order should be missed in the other form also, provided the tests are absolutely reliable. It will be seen that this is true in 403 of the cases. The unreliability of the tests can be accounted for by factors such as counting, lapse of attention, etc. The difference between 506 and 351, which is 155, indicates the number of combinations learned the first day by all pupils. Between Monday and Tuesday there was a gain of 6; on Tuesday during class the gain was 126; between Tuesday and Wednesday there was a loss of 14; Wednesday during class there was a gain of 58; between Wednesday and Thursday there was a loss of 11; Thursday's class netted 55 combinations; between Thursday and Friday there was a gain of 5 and on Friday a gain of 20 in the direct combinations. It will be seen that the total gain for direct combinations was 82 percent of the possible gain. In the reverse combinations, which were not studied, there was a decrease in the number of incorrect answers from 553 to 105, or 448 combinations, or a gain of 84.7 percent of the possible gain. There was a mean gain of 83 percent in both forms, although only one was studied. It will be noted that what is true of the totals is very nearly true of totals for each class and for each card, as shown in Tables I, II, and III.

The expected happened in regard to the direct form which was studied, in that the greatest number was learned the first day and fewer each succeeding day. This fact indicates that the experiment was apparently well conducted and that unknown or accidental factors entering into the results obtained by testing were reduced to a minimum. The surprising thing is that, although the reverse forms were never mentioned during the study, except in the pretest and final test, the pupils in some manner made the adjustment to reverse combinations to such an extent that a very large percentage was learned in reverse combinations. This result was consistent throughout all classes and for the different methods of presentation of the combinations.

Table V, adapted from Tables I, II, and III shows the percent of gain for different methods of presentation. The table shows that for Card No. 1, where the method of teaching was a combination of the methods of the teacher and the drill-card method, the percent

TABLE V.—TABLE SHOWING PERCENT OF GAIN FOR EACH METHOD USED IN INSTRUCTION

Method	Order of Combination	Class L	Class G	Class M	Mean Totals
Card No. 1 .	Direct......	88.8	76.4	100.0	88.4
	Reverse....	92.5	78.5	100.0	90.3
Card No. 2 .	Direct......	74.4	80.3	96.6	83.8
	Reverse....	73.0	76.1	94.4	81.2
Card No. 3 .	Direct......	51.5	81.0	90.9	74.4
	Reverse....	60.0	92.9	95.2	82.7

of growth on the reverse combinations was slightly greater than in the direct forms; for Card No. 2, when the usual methods of the teacher were employed as in Card No. 2, a slightly greater gain was shown for the direct forms; and for Card No. 3, where the drill-card method was used exclusively, a somewhat greater percentage was gained for the reverse forms of combinations. It may be concluded that different methods of presentation of combinations have relatively little effect upon the extent of transfer.

An interesting and somewhat valuable by-product of the main study was the information gained as to teaching methods. This was made possible by the fact that the cards were carefully constructed so as to contain combinations of equal difficulty, and so that each card was taught by a specified method or combination of methods. Any difference in rate of learning among the cards may then be assumed to be due to the variable factor—*i. e.*, the method employed.

Comparing the general gains, Table V indicates that the method used for Card No. 1 is the most efficient, that the method for No. 2 ranks second, and that the method used for No. 3 is the least efficient, as measured by the amount of improvement.

In order to compare the amount of transfer taking place in groups of pupils of different intelligence ratings according to I.Q.'s, tables were constructed similar to Tables I, II, and III. Pupils having I.Q.'s above 120 were placed in one group; those having I.Q.'s between 100 and 120 in another; and those having I.Q.'s under 100 in a third group. Tables VI, VII, and VIII contain the results of these groupings. The value of these data is somewhat limited, since the I.Q.'s of only 52 of the pupils were available.

TABLE VI.—SHOWING THE NUMBER OF COMBINATIONS MISSED BY 2B-GRADE PUPILS WHOSE I.Q.'S RANGED FROM 120 TO 156, IN EITHER DIRECT OR REVERSE ORDER, AND IN BOTH ORDERS IN TEST A; THE NUMBER OF DIRECT COMBINATIONS MISSED EACH DAY ON TEST B, AND THE PERCENTS OF POSSIBLE GAIN
(16 Pupils)

Card	Order	Monday			Tuesday		Wednesday		Thursday		Friday			Percent Gain	
		Test A	Both	Test B	Test B		Test B		Test B		Test B	Test A	Both	One Order	Mean
					Beginning	End	Beginning	End	Beginning	End					
1	Direct.....	19	9	3	6	5	3	3	4	3	1	2	2	89.4	86.3
	Reverse....	17										3		82.3	
2	Direct.....	41	33	18	10	5	7	4	3	4	2	2	0	95.1	92.8
	Reverse....	42										4		90.5	
3	Direct.....	31	23	37	29	17	19	17	24	16	17	12	11	61.3	66.9
	Reverse....	40										11		72.5	
Total	Direct.....	91	65	58	45	27	29	24	31	23	20	16	13	82.4	82.1
	Reverse....	99										18		81.8	

TABLE VII.—SHOWING THE NUMBER OF COMBINATIONS MISSED BY 2B-GRADE PUPILS WHOSE I.Q.'S RANGED FROM 100 TO 120 IN EITHER DIRECT OR REVERSE ORDER, AND IN BOTH ORDERS IN TEST A; THE NUMBER OF DIRECT COMBINATIONS MISSED EACH DAY ON TEST B, AND THE PERCENTS OF POSSIBLE GAIN

(27 Pupils)

Card	Order	Monday			Tuesday		Wednesday		Thursday		Friday			Percent Gain	
		Test A	Both	Test B	Test B		Test B		Test B		Test B	Test A	Both	One Order	Mean
					Beginning	End	Beginning	End	Beginning	End					
1	Direct.....	30	20	20	16	10	8	4	13	7	12	1	1	96.6	93.9
	Reverse....	34										3		91.2	
2	Direct.....	61	41	41	39	22	23	22	20	17	10	7	5	88.5	89.4
	Reverse....	62										6		90.3	
3	Direct.....	46	38	38	21	27	30	26	31	17	18	24	18	50.0	55.0
	Reverse....	55										22		60.0	
Total	Direct.....	137	99	99	76	59	61	52	64	41	40	32	24	76.6	78.0
	Reverse....	151										31		79.5	

TABLE VIII.—SHOWING THE NUMBER OF COMBINATIONS MISSED BY 2B-GRADE PUPILS WHOSE I.Q.'S RANGED LESS THAN 100, IN EITHER DIRECT OR REVERSE ORDER, AND IN BOTH ORDERS IN TEST A; THE NUMBER OF DIRECT COMBINATIONS MISSED EACH DAY ON TEST B, AND THE PERCENTS OF POSSIBLE GAIN

(9 Pupils)

Card	Order	Monday			Tuesday		Wednesday		Thursday		Friday			Percent Gain	
		Test A	Both	Test B	Test B		Test B		Test B		Test B	Test A	Both	One Order	Mean
					Beginning	End	Beginning	End	Beginning	End					
1	Direct.....	13	8	9	5	0	0	0	2	1	1	0	0	100.0	95.0
	Reverse....	10										1		90.0	
2	Direct.....	27	27	17	18	9	18	15	14	4	4	10	10	63.0	62 5
	Reverse....	29										11		62.0	
3	Direct.....	24	22	14	8	12	7	9	3	3	5	4	4	83.3	79.1
	Reverse....	24										6		75.0	
Total	Direct.....	64	57	40	31	21	25	24	19	8	10	14	14	78.1	83.7
	Reverse....	63										18		87.3	

The results from this grouping, according to mentality, show very little difference from the results of the original grouping by classes. The pupils having the highest I.Q.'s according to Table VI show a total gain of 82.4 percent in the direct order and 81.8 percent in the reverse form; the pupils having I.Q.'s ranging from 100 to 120 on the average show a gain of 76.6 percent in the direct order and 79.5 percent in the reverse, according to Table VII; the group having the lowest I.Q's, whose record is shown in Table VIII, gained 76.1 in the direct combinations and 87.3 percent in the reverse form, which was not studied.

According to the most widely accepted theories on the transfer of training, transfer takes place only to the extent that the pupils generalize and comprehend the application of the identical elements in the unfamiliar subject. This ability to generalize has always been associated with our ideas of intelligence; hence we would expect to find the greatest transfer taking place within the most intelligent group, as measured by I.Q.'s. In this experiment the greatest transfer was found in the inferior group. This investigation should be extended to a larger number of pupils.

IV. Conclusions

Based upon the data presented in this study, the following conclusions are drawn:

1. When pupils of any mental level are taught only the direct form of an addition combination such as $\begin{array}{c}7\\ \underline{4}\end{array}$ as nearly as can be, the reverse form, $\begin{array}{c}4\\ \underline{7}\end{array}$, is learned concomitantly at least as completely as the direct form.

2. The bond formed in learning the direct form of an addition combination carries over almost completely to the reverse form. The amount of carry over is influenced very little by the method of presentation.

3. Of the methods used in this experiment, the most efficient presentation of combinations to 2B-grade pupils is that employed on Card No. 1 where the combination of the teacher's usual method and the drill cards used for study was the basis of the teaching.

CHAPTER IX

AN EXPERIMENTAL STUDY IN IMPROVING ABILITY TO REASON IN ARITHMETIC

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I. Purpose and Method

The purpose of this study was to determine the value of certain diagnostic and practice tests as means of improving ability to reason in arithmetic. The method was that of equivalent groups as outlined by McCall in his *How to Experiment in Education.* The experimental factor under investigation was the Stone Diagnostic and Practice Tests. The subjects were public-school pupils, mainly of Spokane, Washington. Other cities represented were Tacoma, Millwood, and Pullman, Washington; and Bend, Oregon.

Precautions were taken to isolate the experimental factor. The chief precaution was that of using final scores from equivalent pupils only. Pupils were matched as equivalent on the basis of four factors: namely, initial score in Survey I, mental age, chronological age, and school grade. Not more than one year's difference was allowed in any matching factor. For example, Mary, who made a score of 6 on the Survey I test,[1] had a mental age of 11-9, a chronological age of 12-0, and belonged to the sixth grade, could be matched with Walter, who had a score of 7 on the Survey I test, a mental age of 12-3, a chronological age of 13-0, and belonged to the sixth grade. Differences in matching factors are ordinarily well under one year. Pairings were checked to make certain that any slight advantage which accrued to either the experimental or the control group in one matching was offset in other matchings.

As a further means of isolating the experimental factor, pupils were paired for the most part from schools in the same system; and it is believed that schools were divided into experimental and control groups so that teaching ability was kept practically equivalent.

[1] Survey tests were scored on the basis of correct answers, one score representing approximately a year in arithmetical reasoning age or grade, according to nation-wide returns.

Teachers and principals in both experimental and control schools understood that the measurements were to be repeated at the close of the experimental trial and teachers and principals in control schools agreed to refrain from mentioning the Survey tests to their pupils or in any way modifying their regular arithmetic work during the period of experimentation.

II. The Tests Used

As used in the experiment, the Stone tests served three purposes: (1) survey, (2) diagnosis, and (3) practice.

The *survey tests* were the original Stone Reasoning Tests in Arithmetic. Their purpose is to afford a measure of each pupil's ability to reason in written arithmetic problems. Their use is fully described and illustrated in the published manual.[2]

The *diagnostic tests* were designed to accompany the survey tests. Their purpose is to afford more precise means of locating each pupil's difficulties in arithmetical reasoning. They enable each pupil to think, by graduated steps, into and through his individual difficulty.

The *practice tests* were designed to follow the diagnostic tests. Their purpose is to afford needed practice on specific difficulties as located by survey and diagnostic tests. They enable each pupil to rethink the reasoning involved in his individual difficulty.

III. Determinations Desired

Since the diagnostic and practice tests [3] had been devised to correct weaknesses in the reasoning problems of Survey I, it was expected that their use would produce superior gains in the solution of the problems of Survey I. This seemed a safe assumption. The real question was: How valuable are such gains? The answer to the question necessitates four determinations: (1) the value of the Survey I problems as compared with the arithmetic work which the experiment replaced, (2) the degree of transfer, (3) the relative degree to which

[2] *Standardized Reasoning Tests in Arithmetic and How to Utilize Them.* Bureau of Publications, Teachers College, New York City. The survey tests have recently been improved and extended and the manual revised.

[3] A trial use of the tests was made possible through the coöperation of Mr. Charles Henry, then Superintendent of Schools, Pullman, Washington. An experimental edition was printed through the coöperation of Superintendent O. C. Pratt, Spokane, Washington.

the diagnostic and practice tests made provision for individual differences, and (4) the permanence of gains.

The first of these items presents two aspects: (1) the amount of regular work replaced by the Survey I problems, and (2) the relative validity of these problems and of the work they replaced. As to the amount of work replaced, that probably was not great. School Number 1 secured its results by using approximately 10 minutes a day, three times a week, for six weeks. Other schools in the preliminary trial used more time, some as high as 30 minutes a day for five weeks. As to the relative validity of Survey I problems and the regular work replaced, that remains undetermined. There are, however, a number of bases for an opinion: (1) Diagnostic and practice work was confined to the better problems of Survey I. (2) A comparison can be made of the thinking required in the survey, diagnostic, and practice test problems with that usually required in regular arithmetic work. (3) The problems of the tests portray buying and selling situations for the most part, and are stated in terms of whole numbers, United States money, and the most frequently used fractions.

The effects of the diagnostic and practice tests were measured by using survey tests before and after experimentation.

Transfer was measured in scores from problems on which there was no diagnostic and practice work.

The relation between gains and the degree of native ability was studied by use of intelligence quotients and the scores made on the survey tests.

Permanency of gains was measured by giving survey tests to a few classes one year after the initial survey.

The evidence of the value of the diagnostic and practice tests is based on the results from the school systems in which the trial was completed. Usable returns were received from twenty-three schools in three cities, with approximately seventy classes represented.

The results of the trials were measured by three Stone Survey Tests, Survey I, Survey II, and Survey III. The problems of Survey I were the basis of the diagnostic and practice problems used as the experimental factor. Hence the experimental pupils had specific diagnostic and practice help in the problems of Survey I. The problems of Survey II are practically equivalent in difficulty to those of Survey I, but they are different in both content and reasoning. The problems of Survey III are, with the exception of numbers three

and nine, practically equivalent in difficulty to those of Survey I; the two tests are different in content but similar in reasoning. Results for Survey II and Survey III have a special bearing on transfer.

IV. The Chief Results

1. The Preliminary Trial

A preliminary trial was made during the first semester of the experiment. Table I shows the results as measured by Survey Test I.

Table I.—Results from Paired Spokane Pupils as Measured by Survey I for Initial Scores, and by Survey I Repeated for Final Scores
(Grades 5A to 7B inclusive. Time-limit 15 minutes.)

Schools as Matched	Number of Pairs	Mean Gains (Survey I Repeated, over Survey I Initial)		Differences in Favor of Exper. (diagnostic and practice tests)	Experimental Coefficient
		Exper.	Control		
Experimental No. 1 with Control 2...........	58	8.51	1.29	7.22	5.78
Experimental No. 1 with Controls 3 and 4.....	59	8.53	1.26	7.27	6.16
Experimental No. 5 with Control 6...........	58	7.68	0.98	6.70	5.40

It will be seen in Table I that 58 paired pupils from Experimental School No. 1 made a mean gain of 8.51 Survey I scores, while the pupils from Control School 2 made a mean gain of 1.29 Survey I scores. This gives a difference in favor of the Experimental School of 7.22, with an experimental coefficient of 5.78.[4] The other comparisons are read in the same way.

The results from these 175 pairs of equivalent pupils obviously indicate that pupils who used the diagnostic and practice tests developed much more speed and accuracy in the solution of Survey I problems than did pupils who were taught by the traditional methods and materials. Though the amount was a surprise, it was expected

[4] An experimental coefficient of 1 would indicate practical certainty that there is a difference in favor of the diagnostic and practice tests. An experimental coefficient of 5.78 indicates 5.78 × practical certainty that there is a difference in favor of the diagnostic and practice tests. See McCall, W. A., *How to Experiment in Education*, pp. 154-156.

that there would be superior gains by the experimental group, since the weakness revealed in the problems of Survey I are the basis of the diagnostic and practice test work.

The transfer required for this success is obviously direct and comparatively small in amount. For the purpose of securing data on its extent, Survey II was given to both experimental and control pupils at the close of the preliminary trial. The results were these: (1) The 58 pupils of Experimental School 1 averaged 7.17 on Survey II, while the paired equivalent pupils averaged 5.71. This is a difference of 1.46 in favor of the Experimental Group. (2) When School I was matched with Schools 3 and 4, the difference in favor of the Experimental Group was 1.40 Survey II scores. (3) When School 5 was matched with School 6, the difference in favor of the Experimental Group was 3.63. The significance of these differences is more fully realized when it is recalled that the equivalence of these groups was carefully established at the beginning of the experimental period and that, according to nation-wide returns, one Stone Survey score represents approximately one year or one grade of progress in arithmetical reasoning.

Since the problems of Survey II are different in both content and reasoning procedure, the figures just given afford substantial evidence of superior transfer effect for the diagnostic and practice tests.

2. The Main Trial [5]

The purpose of the main trial was to secure further evidence as to the value of the diagnostic and practice materials as compared with the regular arithmetic work.

Conditions were the same for the main trial as for the preliminary, except that pupils were allowed as much time as they needed to do their best in the survey tests. Three measures of the experimental differences were secured in this trial. Scores were secured at the beginning and at the end through the use of Survey I, Survey II, and Survey III.

a. Results of Main Trial as Measured by Survey I.—Table II shows gains as measured by Survey I in the main trial. According

[5] The data were gathered and the computations made for the main trial by Mr. Eugene Giles, now Superintendent of Schools, Sharon, Washington. See Giles, Eugene, *An Evaluation of the Stone Diagnostic and Practice Tests,* Master of Arts Thesis, Library, State College of Washington.

TABLE II.—RESULTS FROM PAIRED SPOKANE AND BEND PUPILS AS MEASURED BY SURVEY I FOR INITIAL SCORES, AND SURVEY I REPEATED FOR FINAL SCORES

Grade	Schools as Matched	Number of Pairs	Mean Gains		Differences in Favor of Exper. Group	E. C.
			Exper.	Control		
5 A	Experimental School A with Control Schools B, C, and D........	33	5.21	0.83	4.38	1.63
5 A and 6 B	Experimental P with Control D...........	12	3.45	1.17	2.28	1.12
5 A	Experimental S with Controls B, C, and D	17	1.87	0.89	0.98	0.74
5 B	Experimental R with Controls B, C, and D	16	2.16	1.33	0.83	0.51
5 A	Experimental M with Controls B, C, and D	30	5.52	1.47	4.05	2.29
7 A	Experimental O with Controls B, C, and D	14	5.54	−0.24	5.78	2.10
6 A	Experimental N with Controls B, C, and D	20	6.49	2.60	3.89	1.93
8 A	Experimental Y with Controls B and C....	30	7.39	2.42	4.97	1.92

to it, the experimental pupils in every group gained more than did the equivalent control pupils with whom they were paired. The gain ranged from 0.83 Survey I scores in School R to 5.78 in School 0; the experimental coefficient from .51 to 2.29. These coefficients are large enough to indicate high probability of a true difference.

The results in Table II, together with those in Table I, appear to be convincing evidence that the Stone Diagnostic and Practice Tests were more potent than the regular arithmetic work in developing ability to solve problems on which there had been specific diagnostic and practice work.

b. Results of Main Trial as Measured by Survey II.—The data in Table III are the measures of relative success in solving problems in which neither experimental nor control pupils had had specific teaching. They are measures of *transfer*.

As may be observed, Table III shows appreciable differences in favor of the use of diagnostic and practice tests. In six of the seven

TABLE III.—RESULTS FROM SPOKANE AND BEND PAIRED PUPILS AS MEASURED BY SURVEY II FOR INITIAL SCORES AND SURVEY II REPEATED FOR FINAL SCORES
(Pairings the same as in Table II)

Grade	Schools as Matched	Number of Pairs	Mean Gains		Differences in Favor of Exper. Group	E. C.
			Exper.	Control		
5 A	Experimental A with Controls B, C, and D	33	3.06	0.07	2.99	1.55
5 A and 6 B	Experimental P with Control D.	12	1.40	0.88	0.52	0.22
5 A	Experimental S with Controls B, C, and D	17	1.05	0.44	0.61	0.45
5 B	Experimental R with Controls B, C, and D	16	0.26	0.43	−0.17	0.36
5 A	Experimental M with Controls B, C, and D	29	3.42	0.65	2.77	1.47
7 A	Experimental O with Controls B, C, and D	14	2.00	1.03	0.97	0.35
6 A	Experimental N with Controls B, C, and D	20	3.60	1.39	2.21	0.87

matchings the excess gain of experimental over control pupils was distinctive, varying from .52 score (approximately half a year) to 2.99 scores (approximately three years) and the experimental coefficients are sufficiently large to warrant the conclusion that there is a true difference in favor of the diagnostic and practice tests.

Smaller differences in favor of the experimental work were expected to appear in Table III because the latter shows the results of using diagnostic and practice problems bearing directly on the problems of Survey I,[6] whereas no diagnostic and practice work was given on the problems of Survey II. The superior gains made on the part of the experimental pupils shown in Table III may therefore be regarded as evidence of transfer; the diagnostic and practice tests afforded more help in learning how to solve new problems than did the regular arithmetic work.

c. Results of Main Trial as Measured by Survey III.—In the classes from which the following results were secured the diagnostic

[6] The failure of Experimental Group R to gain may be due to a change of teachers during the experiment.

and practice tests were substituted for all the regular work of the Experimental Group instead of being used in conjunction with it. The time devoted was forty minutes a day, five days a week, for five weeks. Survey I was used at the beginning, and Survey III at the close of the experiment. The problems in Survey III are different in content from those in Survey I and are somewhat more difficult. They are for the most part equivalent in type of reasoning. Therefore the use of Survey III afforded additional data as to the degree to which diagnostic and practice tests are superior to regular work as a means of securing improvement in solving problems. (Problems here are different, though equivalent.) These results, as Table IV shows,

TABLE IV.—RESULTS FROM PAIRED PUPILS FROM TACOMA SCHOOLS, USING SURVEY I AS THE INITIAL AND SURVEY III AS THE FINAL MEASURE
(Grades 5A and 6B. No time limit.)

Schools as Matched	Number of Pairs	Mean Gains		Differences in Favor of Exper. Group	E. C.
		Exper.	Control		
Experimental T and W with Controls I, H, M, and J	29	3.35	−0.23	3.58	1.92

are further evidence of superior transfer from the special training.

Still another source of data with reference to transfer was secured by using scores from all Tacoma pupils in the experiment. Intelligence records were not available for certain pupils, but it was assumed that in this large number of cases intelligence was equivalent in the groups compared, without resorting to actual pairing. There were 205 Experimental Pupils and 102 Control Pupils. As measured by Survey III, at the close of the trial the Experimental Pupils had an average of 3.95 scores more than the average made on Survey I at the beginning of the trial, while the Control Pupils had an average of 0.55 of a score less. These figures add emphasis to those presented as to the superior transfer effect secured through the use of diagnostic and practice tests.

d. Summary of Gains.—By way of showing results in summary form, the gains of all the matched pairs of the main trial were combined. The result showed that the Experimental Pupils had a difference of 4.47 Survey I scores in their favor—E. C., 5.68. With such a high E. C. there is certainty that the Experimental Group

excelled in the problems of the Survey I Tests at the end of the experiment.

In another comparison 140 pupils of the experimental schools gained 2.38 Survey II scores while the 140 controls were gaining 0.85 Survey II scores. The difference, 1.53 Survey II scores, represents over a year's growth in ability to reason. This superior gain was apparently secured through transfer of training from the diagnostic and practice tests.

e. Permanence of Gains.—A small amount of evidence was secured regarding the permanence of gains made by experimental and control pupils. Schools 5 and 6 were remeasured by using Survey I one year after they had been given the initial test. Forty-four pupils in the experimental school took Survey I both times and made an average gain of 2.82 Survey I scores over their previous year's performance. Sixty-five cases in the control school made an average gain of 1.62 Survey I scores. The difference, 1.20 Survey I scores, is the equivalent of a little more than one year's work in favor of the experimental pupils. Compared with the difference in gains found at the end of the experimental trial, this difference seems rather small (see Table I), and probably indicates the need of a short period of practice to reëstablish the habits which were operating at the end of the training period. Retention is the final criterion of usefulness of any teaching materials. The fact that the experimental pupils did retain a decided superiority is another indication of the value of the diagnostic and practice tests.

f. Evidence of Provision for Individual Differences.—One of the purposes in constructing the diagnostic and practice tests was to provide for individual differences. The tests are so organized and administered as to provide opportunity for each pupil to make his best individual progress. The degree to which this purpose was realized was studied by assembling the intelligence quotients and reasoning gains from a random selection of pupils. One hundred ninety-four pupils were included from the experimental group and 184 from the control group. The pupils from each group were divided into three subgroups on the basis of I.Q.'s. The divisions were so made that the pupils of lower subgroups were those with I.Q.'s below the 18th percentile, the higher groups those with I.Q.'s above the 83rd percentile, and the middle groups those between the 17th and 84th percentiles. The upper limit of the lowest subgroup was I.Q. 85 and

the lower limit of the highest group was 115, leaving a range of 86-114 in the middle group.

The mean gains for each subgroup are shown in Table V.

TABLE V.—MEAN GAINS IN SURVEY I OF SUBGROUPS CLASSIFIED ON THE BASIS OF I.Q.

	Lowest Sixth	Middle Two-Thirds	Highest Sixth
Experimental Schools (194 pupils)	3.13	3.52	6.11
Control Schools (184 pupils)	1.13	1.20	1.14

According to Table V all three subgroups of experimental pupils gained more than corresponding groups of control pupils. The outstanding feature is the very large gain made by the highest experimental subgroup. It appears that the diagnostic and practice tests afford notably effective help in furthering the arithmetical reasoning of the pupils with high I.Q's.

The transfer secured from the diagnostic and practice tests was also studied with reference to individual differences. The method of securing subgroups was the same as that just stated. Table VI contains the results.

TABLE VI.—MEAN GAINS IN SURVEY II OF SUBGROUPS CLASSIFIED ON THE BASIS OF I.Q.

	Lowest Sixth	Middle Two-Thirds	Highest Sixth
Experimental Schools (194 pupils)	1.60	2.05	3.92
Control Schools (184 pupils)	0.36	0.70	1.15

Table VI is also especially noteworthy because of the large gain made by the most intelligent experimental subgroup. Evidently the diagnostic and practice tests are especially effective in transfer with pupils of the highest I.Q.'s.

V. SUMMARY

This study affords data for tentative conclusions regarding the value of the Stone Diagnostic and Practice Tests.

(1) Comparative gains made by paired equivalent pupils show that the use of these tests produces greater gains in ability to reason

in arithmetic than does the regular work in arithmetic that the tests may displace in classroom use.

(2) The evidence shows that the gain in reasoning ability secured by these tests transfers to reasoning demanded by other problems of different content, though of similar nature.

(3) This transfer is greater than the transfer secured by an equivalent amount of regular arithmetic work.

(4) The gains appear to persist (limited evidence).

(5) The greatest gains and the largest transfers are made by pupils possessing the highest I.Q.'s.

This study suggests, of course, the need of further experimental work. Scientific data should be secured for establishing such optimums as grade readiness, length of daily practice period, number of practices per week, and so forth.

CHAPTER X

A TEST IN ARITHMETIC FOR MEASURING GENERAL ABILITY OF PUPILS IN THE FIRST SIX GRADES

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I. Character and Purpose of a General Test

The test described in this article is the result of a study that has been in progress a number of years. It is of the nature of a general test which, in spite of the many kinds of tests that have recently been developed, still forms one of the most useful types.

A general test has for its specific function that of enabling a teacher to compare the work of her pupils with what is typical in the accomplishment of other pupils in the same grade and of the same age. In the manner of a general intelligence test it may be composed of samples of work that have proved efficient in measuring ability of given kinds.

Such a test should cover a wider field of content than a test intended to measure speed and accuracy in computing or one meant to measure reasoning ability in solving problems. It should be more selective than an inventory test, the main purpose of which is to find what pupils have accomplished in some specific line of work. It should also be more easily administered than a laboratory test, which is intended primarily to analyze from a psychological standpoint the complex character of arithmetical processes. Moreover, a general test should hold before teachers and pupils standards which are not only broad and inclusive, but which are also within the range of a wholesome normal development of a child's mind.

II. Aims in View in Making the Progress Tests

When the authors undertook the composition of a series of general tests, they felt that some of the tests then current were narrowing the work of the schoolroom to a few types of problems and forcing

upon young children mechanical types of work far too difficult for them, and that others of the tests were calling for obsolete types of problems. They therefore resolved:

(1) To compose a series of tests that would be in keeping with modern aims in teaching arithmetic;

(2) To select a form of test which would measure the achievement of pupils in a broad field of numbers and which would make the tests distinct from those intended to measure speed and accuracy and other limited aspects of the work;

(3) To include work covering basic concepts as well as the four processes and their applications;

(4) To select only those problems that belong to useful and well-defined types and to exclude both obsolete and unusual types;

(5) To give prominence to work in narrative problems and to make this section of the series a test in applying numbers to everyday practical situations and not merely a test in logic;

(6) To make the tests diagnostic, so that difficulties met with by pupils might be discovered and overcome;

(7) To cover such difficulties as those that occur in handling the zero, in using the decimal point in problems based on United States money, and in computing with common business fractions;

(8) To make the tests difficult enough so that they could be used to differentiate pupils according to ability;

(9) To grade the work carefully and to make some of it simple enough so that it could be used to measure ability of untrained children in the first grade as well as to test pupils in the second, third, fourth, fifth, and sixth grades; and

(10) To compose tests that would achieve the desired results with economy of time and energy.

III. Determination of the Form of the Tests

In order to make the tests effective in differentiating pupils according to arithmetical age and general ability, the authors decided to select problems that pupils need to know how to solve and to arrange the problems according to difficulty. This required a graded series of problems which should, as far as possible, do equal justice to pupils in the first grade, as well as to those in the second, third, fourth, fifth, and sixth grades. In this selection their endeavor was to find problems that were useful in measuring the accomplishment in each

grade of pupils in the lowest quarter, of those in the quarters at either side of the median, and of those in the upper quarter. The pupils were to use these series of graded problems not only to see how far they could go on each 'ladder,' but also to see what perfect records they could make. Each pupil's total score was then to be compared with standards arranged according to age and grade.

IV. Preliminary Studies

As preliminary work, the authors gave three series of inventory tests. One of these, called 'Number Puzzles,' consisted of four tests covering work (1) in counting and in various concepts of numbers, (2) in handling coins and in solving problems based on United States money, (3) in telling time and solving problems based on time, and (4) in measuring with linear measure. This series of tests, which included seventy-seven exercises, was given to 2,200 pupils in Grades I, II, III, and IV. The second series, called 'Number Ladders,' was made up of series of problems in addition, subtraction, multiplication, and division. This series, which included, all told, forty-five problems, was given to 2,087 pupils in Grades II, III, and IV. The third series of tests was composed of five sections, one dealing with narrative problems and the other four with the four processes. This series, which included eighty-five problems, was given to 1,373 pupils in Grades IV, V, and VI.

V. Later Form of the Tests

From the results of these preliminary studies was composed a series of five tests called the "Progress Tests in Arithmetic for Pupils in the First Six Grades." The first part of each test, as shown in the two tests described below, dealt with basic concepts; the other part, with problems.

Test No. 1, Number Puzzles or Problems

A. (Illustration) Put a cross (X) on the tallest boy.
B. (Illustration) Put a cross in the largest box.
C. (Illustration) Draw little rings in the long box to show how many pennies you can get for a nickel.
D. (Illustration: five clock faces) Put a ring around the clock that says 9 o'clock.
E. Put one cross (X) on the clock that says 12 o'clock.
F. Put two crosses (XX) on the clock that says 6 o'clock.
G. (Illustration) How many minutes past nine is it by the clock? Write the answer in the little box above the clock.

H. In five minutes how many minutes past nine should it be by the clock? Write the answer in the big box below the clock.
I. If you buy 3 boxes of crackers at 10 cents a box, how much change will you get back from 50 cents?
J. I first draw a line 1½ inches long and then I make the line half an inch longer. How long is it then?
K. Bananas are selling 3 for 10 cents. How much is that a dozen?
L. How much must be put with 2 quarters and 3 dimes to make a dollar?
M. A garden in the shape of a rectangle is 10 feet wide and 20 feet long. What is the distance around it?
N. To savings amounting to $2.50, I add first $1.00 and then $1.50. Out of this I spend 75 cents, 50 cents, and $2.00. How much money have I left?
O. Oranges are selling at 3 for 25 cents. Find the cost of 1½ dozen.
P. A boy works one afternoon from half past two until quarter to five. At the rate of 20 cents an hour how much money should he receive?

Test No. 2. Counting and Adding

A. (Illustration: two squares each containing three lines; one square containing four lines and an empty square beside it). The first two boxes shown here are just alike. Make the other two boxes alike.
B. (Illustration: a rectangle) In this big box show how many lines 2 lines and 3 more make.
C. (Illustration) In the first row of little boxes two boxes are left empty. Leave five boxes empty in the next row of boxes.
D. (Illustration) Leave 13 boxes empty in this long row of little boxes.
E. (A single column of three figures to be added.)
F. (A single column of five figures to be added.)
G. (One-figure number to be added to a two-figure number.)
H. (Two two-figure numbers to be added. No reduction—one figure a zero.)
I. (Two two-figure numbers to be added. Reduction required.)
J. (Three two-figure numbers to be added. Reduction required. Combinations difficult.)
K. (Three three-figure numbers. Dollars and cents involved.)
L. (Single column composed of nine figures. Difficult combinations.)
M. (Mixed numbers. Sum of fractions less than a unit.)
N. (Decimals.)
O. (Mixed numbers. Sum of fractions equal to a unit.)
P. (Fourteen two-figure numbers in a column. Number to be carried, 10.)
Q. (Mixed numbers. Unlike fractions. Common denominator not given. Sum of fractions more than a unit. Reduction required.)

Tests No. 3, No. 4, and No. 5

Tests No. 3, No. 4, and No. 5 dealt respectively with division, subtraction, and multiplication. The first four problems in each test were concrete problems based on fundamental concepts; the other problems were leading types involving abstract numbers. The test in division, rather than the one in subtraction, was placed after the one in addition, because the form of each problem was such that pupils could not mistake the process required and continue to add, as they are inclined to do when a test in subtraction follows one in addition.

VI. Checking Results

In testing the arrangement and form of the tests, records were used from two school systems differing widely geographically. The data derived were tabulated and the distance of each problem from the zero point in a scale was determined. There were, of course, some slight gaps in the various scales when any one of them for a grade was considered by itself; but when the five series were combined, the gap was not such as to interfere materially with the efficiency of the test. There were problems of many degrees of difficulty in a given grade. The fifty-two problems that were solved by pupils in the third grade, for example, could be solved correctly by the following percents of the number of pupils attempting them: 99.5, 98.0, 95.3, 94.3, 93.0, 92.3, 91.6, 90.4, 88.4, 87.0, 85.9, 84.2, 83.1, 81.1, 80.2, 79.2, 78.8, 77.1, 76.9, 74.5, 72.7, 71.2, 69.4, 68.9, 66.8, 64.2, 61.2, 60.7, 59.1, 56.2, 55.3, 53.3, 44.7, 43.3, 42.6, 39.9, 35.7, 32.0, 30.0, 28.9, 26.4, 23.0, 18.9, 17.5, 11.3, 10.6, 9.6, 5.4, 4.4, 2.7, 1.3, and 0.5.

In the upper grades there were duplicates of some of the percents that indicate that the problems were easily solved. This was a necessity, for the tests in order to be useful in the earlier grades required a number of problems that were simple for pupils in the later grades.

VII. Standards[1]

The standards for the tests were based on the examination of over 17,800 pupils in twelve cities and towns. One city was in the state of Washington, one was in Pennsylvania; the others were in Massachusetts. The school systems in the following table are referred to by numbers. The numbers of cases are given by school systems, ages, and grades in Table I.

[1] The writers express their indebtedness to Dr. Psyche Cattell, Research Associate of the Psycho-Educational Clinic of Harvard University, for assistance in the compilation and interpretation of the statistical results of this study.

TABLE I.—NUMBER OF CASES TESTED DISTRIBUTED BY SCHOOL, SYSTEMS, AGES, AND GRADES

School System No.	1	2	3	4	5	6	7	8	9	10	11	12	All
No. Cases	1779	1050	5857	642	3386	1711	1912	199	638	559	144	171	18,048
Chron. Age			5	6	7	8	9	10	11	12	13	14	All
No. Cases			242	2277	2993	3233	3190	2859	1925	911	326	92	18,048
School Grade						I	II	III	IV	V	VI	VII	All
No. Cases						3895	3272	3354	3139	2571	1452	160	17,843

It did not appear to be best to combine the data from all the towns and construct age or grade norms by calculating averages or medians from the data thus massed, since school systems vary in size and in standards of achievement. In some towns formal arithmetic is introduced early in the first grade, in others not until the second. If the results of the individual pupils of a large system, where arithmetic was not taught until unusually late, were averaged in with a number of smaller systems where it was introduced at about the average time, the data from the large towns would tend to lower the norm, at least for the lower grades, beyond that which is typical for the country as a whole. This difficulty was avoided by calculating preliminary medians for each town separately (not presented in this article) and basing the final norms on the medians of the two medians.

The twelve towns were not all examined at the same time of the year. The first was examined in October, the second in April and May, the third in November, etc. In order to render the preliminary town-grade medians comparable, from the second grade they were reduced to the middle of the school year (February 1). Two norms were calculated for the first grade: one for December 15 for the five towns which were examined during the first part of the school year and one for May first for the seven towns that were examined during March, April, May, and June.

For practical purposes it is frequently convenient to know the norm for the particular time of the year that the test happens to be given in a certain school. In order to make this information available a norm for each month of the school year was calculated from the December and the May scores of the first grade and from the February scores of the upper grades. It has been shown by Dearborn,

Lincoln, and Shaw [2] that while some pupils gained and others lost in arithmetic test scores between the end of one school year and the beginning of the next, on the average there was practically no change. To quote: "In the Courtis Tests 21 out of 48 (class) medians (44%) are exactly the same in the May and October results. Besides this, 17 October medians (37%) are only one example more or less than the median of the same classes in May." On the basis of this evidence, when interpolating to obtain norms for each of the ten school months, the year was divided into nine parts and the same norms assigned to September as to the June of the preceding year (see Table II).

TABLE II.—GRADE MONTH NORMS FOR THE PEET-DEARBORN PROGRESS TESTS IN ARITHMETIC

For Problems

	I	II	III	IV	V	VI
September	....	4.7	7.0	9.2	10.9	12.5
October	....	5.0	7.2	9.4	11.1	12.7
November	....	5.2	7.5	9.6	11.2	12.9
December	3.0	5.5	7.7	9.8	11.4	13.1
January	3.3	5.7	8.0	10.1	11.6	13.3
February	3.6	6.0	8.2	10.3	11.8	13.5
March	3.8	6.2	8.5	10.4	12.0	13.7
April	4.2	6.5	8.7	10.6	12.1	13.9
May	4.4	6.7	8.9	10.8	12.3	14.1
June	4.7	7.0	9.2	10.9	12.5	14.2

For Processes

	I	II	III	IV	V	VI
September	....	3.2	6.2	9.0	11.2	12.9
October	....	3.5	6.6	9.3	11.4	13.1
November	....	3.8	7.0	9.6	11.6	13.3
December	2.0	4.2	7.3	9.8	11.8	13.4
January	2.2	4.5	7.6	10.1	12.0	13.6
February	2.3	4.8	8.0	10.4	12.2	13.8
March	2.4	5.2	8.2	10.6	12.4	13.9
April	2.6	5.6	8.5	10.8	12.6	14.1
May	2.8	5.9	8.8	11.0	12.8	14.3
June	3.2	6.2	9.0	11.2	12.9	14.4

Age norms by months are given in Table III. They are less reliable than the grade norms and are, perhaps, finer divisions than the data justify. The greatest difficulty met with in their calculation lay in the fact that all children do not reach a given age at the same time

[2] *Standard Educational Tests in the Elementary Training Schools of Missouri.* (Harvard Monographs in Education, 1922, Series 1, No. 3.)

TABLE III.—AGE NORMS BY MONTHS FOR THE PEET-DEARBORN PROGRESS TESTS IN ARITHMETIC

For Problems

Month	5	6	7	8	9	10	11	12
0	....	3.5	4.8	6.6	8.6	10.4	12.0	12.6
1	....	3.6	5.0	6.8	8.8	10.6	12.0	12.6
2	....	3.7	5.1	7.0	8.9	10.8	12.1	12.6
3	....	3.8	5.2	7.2	9.0	10.9	12.2	12.7
4	....	3.9	5.4	7.3	9.2	11.1	12.3	12.7
5	....	4.0	5.5	7.5	9.4	11.2	12.4	12.7
6	2.9	4.1	5.6	7.7	9.5	11.4	12.5	12.7
7	3.0	4.2	5.8	7.8	9.7	11.5	12.5	12.8
8	3.1	4.4	6.0	8.0	9.8	11.6	12.5	12.8
9	3.2	4.5	6.1	8.2	10.0	11.7	12.6	12.9
10	3.3	4.6	6.3	8.3	10.1	11.8	12.6	13.0
11	3.4	4.7	6.5	8.4	10.3	11.9	12.6	13.0

For Processes

Month	5	6	7	8	9	10	11	12
0	....	1.7	3.6	5.8	8.2	10.4	11.8	12.6
1	....	1.9	3.8	5.9	8.5	10.5	11.9	12.7
2	....	2.0	3.9	6.1	8.7	10.6	12.0	12.7
3	....	2.2	4.1	6.3	8.9	10.7	12.2	12.7
4	....	2.3	4.3	6.5	9.2	10.8	12.3	12.7
5	....	2.5	4.4	6.7	9.4	11.0	12.4	12.8
6	1.0	2.6	4.6	6.9	9.6	11.1	12.5	12.8
7	1.1	2.8	4.8	7.1	9.7	11.2	12.5	12.9
8	1.2	2.9	5.0	7.4	9.8	11.3	12.6	13.0
9	1.3	3.1	5.2	7.6	10.0	11.5	12.6	13.0
10	1.4	3.3	5.4	7.8	10.1	11.6	12.6	13.1
11	1.6	3.4	5.6	8.0	10.2	11.7	12.6	13.2

of the year. Other things being equal, a first-grade child reaching the age of six years and six months at the end of the school year could not be expected to make as high a score as another child reaching the same age at the beginning of the school year. This difference is probably greater in the first than in the later grades, but it would appear that in any grade children reaching a given age easily in the year would be likely to make better scores than children reaching the same age late in the year. The average time of the year in which the tests were given was March; hence the age norms should be interpreted as being for children who reach the given age in March and allowances should be made for children tested earlier or later in the school year.

VIII. Reliability of the Tests

Three different means were used to measure the reliability of the tests. The tests were given to a group of 330 pupils in the first six grades in March and again to the same pupils three months later and the correlation between scores was made. The coefficient for the processes was + 0.95; that for the problems + 0.86. These indicate a high degree of reliability in the tests.

Beginning in the first and second grades 445 pupils in one town were given the tests at the same time of the year for five consecutive years. The scores for each year were correlated with the scores obtained in each of the other four years. The coefficients are given in Table IV. It will be noted that for the correlations of the scores made one year apart the coefficients are + 0.77 for the processes and + 0.60 for the problems; for two years apart, they are + 0.69 and + 0.53, respectively; for three years apart, + 0.61 and + 0.49; and for four years apart, +0 .50 and + 0.43. The correlation for the coefficient between the averages of the processes and the problems of the same year is + 0.66. With the element of growth and other varying conditions taken into consideration, these coefficients also indicate a high degree of reliability in the tests.

TABLE IV.—Correlation Between Arithmetic Test Scores Made in Different Years (N = 445)

Year	Problems	Processes
1st and 2nd	0.600	0.784
2nd and 3rd	0.602	0.786
3rd and 4th	0.573	0.767
4th and 5th	0.629	0.726
1st and 3rd	0.512	0.689
2nd and 4th	0.542	0.659
3rd and 5th	0.536	0.713
1st and 4th	0.456	0.552
2nd and 5th	0.515	0.667
1st and 5th	0.434	0.503

The tests were also given to 423 pupils in the upper grades to whom the advanced form of the test had been given. The coefficients of correlation were as follows:

Grade	Problems	Processes	N
V	.568	.703	144
VI	.551	.793	120
VII	.540	.627	159

IX. The Correlation Between Arithmetical Ability and Intelligence

That the scores made in the Peet-Dearborn Progress Tests in Arithmetic are influenced by, but not determined by, the level of intelligence of the pupil is indicated by the coefficients of correlation (r) given in Table V. The correlation between the Arithmetic Test scores and mental ages as determined by the Dearborn Group Intelligence Tests is markedly higher in the first than in the later grades. In general the correlations tend to be higher between mental age and ability to solve problems than between mental age and skill in the fundamental processes.

Table V.—Coefficients of Correlation (r) Between Mental Age and Arithmetical Ability, with Means and Standard Deviations

No. of Cases	Grade	Dearborn Mental Age (A or C)		Arithmetic Test					
				Problems			Processes		
		Mean	S. D.	Mean	S. D.	r	Mean	S. D.	r
461	I	7– 6	1– 0.3	4.4	2.0	0.62	3.9	1.6	0.66
300	II	9– 0	1– 5.3	7.6	2.2	0.53	7.2	1.6	0.36
271	III	9–10	1– 7.6	9.2	2.5	0.42	9.5	1.7	0.37
236	IV	10– 7	1– 6.2	11.3	2.5	0.40	11.5	1.8	0.38
1626	I–VI	9– 9	2– 4.4	8.6	3.8	0.79	8.4	3.9	0.79

CHAPTER XI

THE EFFECT OF AWARENESS OF SUCCESS OR FAILURE [1]

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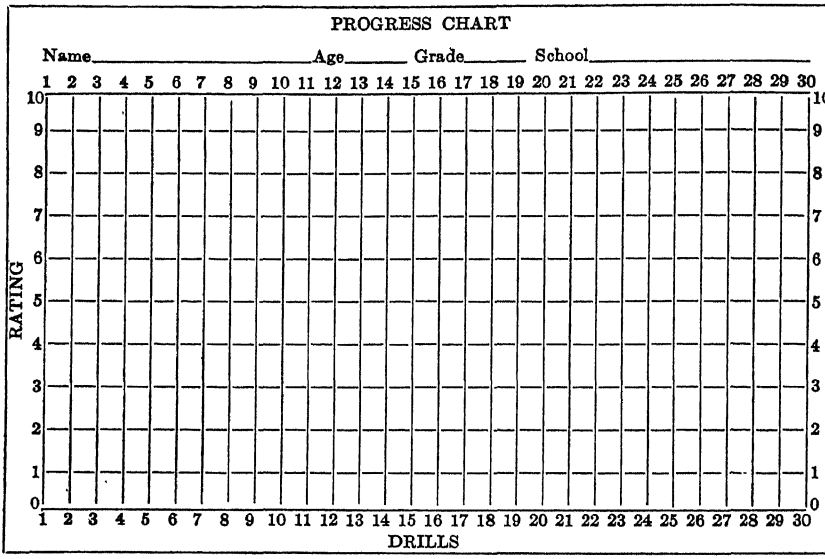
I. THE PROBLEM AND MATERIALS

The main idea of the experiment was to determine what would be the difference in gain made by two groups of fourth-grade arithmetic pupils one of which was made aware of success or failure. The one group used for the purpose of maintaining or reviewing previously learned skills in arithmetic a series of mixed fundamental drill units which contained no reference to standards of work, and hence no provision for a child or class to keep an objective record of growth (or lack of it) in power over arithmetic. The other group used exactly the same material and under comparable conditions, with this exception: on each drill unit there appeared a set of standards similar to the sample shown here.

Number Correct	0-2	3-4	5	6	7	8	9	10-11	12-13	14-19	20
Rating.........	0	1	2	3	4	5	6	7	8	9	10

By the use of these standards each pupil could determine his own rating. By averaging the pupil ratings, each class could determine its class rating. The provision of individual progress charts (as illustrated here) for each pupil and of a class progress chart for each class made feasible the systematic recording of a visual graph of progress for both the individual and the class.

[1] The experiment herewith reported was carried on by Dr. Panlasigui at the State University of Iowa during 1927-1928.



1. Determination of Standards for Ratings

The number of correct examples corresponding to each rating had been determined on the basis of the actual performance of an adequate sampling of comparable pupils. Thus, the ratings closely correspond to expected success in performance, and not to the completion of some arbitrarily assigned number of examples.

2. Nature of the Drills

Each drill contained fifteen examples, and was of the 'mixed' type.[2]

The distribution of practices on all the combinations used in examples for the various processes of whole numbers was in accordance with the experimental data reported in Chapter VI.

II. The General Experimental Set-Up

All contact with the classrooms used in the investigation was by correspondence and was similar in nature for both groups. The location and number of classes and pupils (all fourth-grade) used

[2] For the defense of mixed versus isolated drill material, see the experimental data reported in Chapter VI.

in the investigation will be clear from Table I. We shall refer to the group which used material without standards as the Control Group, and to the one which used material with standards as the Experimental Group.

TABLE I.—CLASSES USED IN THE INVESTIGATION

Experimental Centers	Exp.	Cont.	Totals	No. of Classes
1. Davenport, Iowa	74	76	150	8
2. Denver, Colorado	89	77	166	4
3. Grand Rapids, Michigan	103	96	199	6
4. Springfield, Illinois	83	78	161	5
5. Toledo, Ohio	78	68	146	4
6. St. Paul, Minnesota	109	102	211	6
7. Kansas City, Missouri	122	123	245	6
8. Pittsburg, Kansas	97	88	185	7
9. Neodesha, Kansas	53	47	100	4
10. Los Angeles, California	91	96	187	6
Total at Beginning	899	851	1,750	56
Total at End	557	439	988	—
Number Pupils Equated on Initial Test Scores	358	358	716	—

Owing to a variety of causes, such as mid-year promotions, reductions caused by pairing, and absence from school, the number of pupils used shrank considerably during the life of the experiment. The main facts relative to the final sources of data are also presented in Table I.

The usual pretests and final tests were administered by the teachers themselves or their supervisors. The experiments continued for one day a week for twenty weeks. Serious attempts were made to minimize all unusual factors and to approximate normal conditions. The most obvious exception here was probably the request that teachers send to the experimenter frequent samplings of pupils' work and of ratings, since it was desirable to determine where in the course of the experiment especial gains or losses took place, in addition to knowing what the final gain or loss was.

III. THE RESULTS

The distribution of the two groups with respect to scores made in the initial and in the final tests is reported in Table II.

TABLE II.—DISTRIBUTION OF THE PUPILS OF THE EXPERIMENTAL AND CONTROL GROUPS ACCORDING TO THEIR SCORES IN THE INITIAL AND THE FINAL TESTS

Initial Test			
Scores	Exp.	Cont.	Both
113-117	1	1	2
108-112	15	15	30
103-107	30	30	60
98-102	39	39	78
93- 97	53	53	106
88- 92	45	45	90
83- 87	39	39	78
78- 82	40	40	80
73- 77	34	34	68
68- 72	21	21	42
63- 67	25	25	50
58- 62	7	7	14
53- 57	5	5	10
48- 52	1	1	2
43- 47	2	2	4
38- 42	1	1	2
Totals	358	358	716
Mean	86.41	86.41	86.41
	±.50	±.50	±.36
Q_3	97.08	97.08	97.08
Q_1	76.55	76.55	76.55
S.D.	14.10	14.10	14.10

Final Tests			
Scores	Exp.	Cont.	Both
400-419	4	2	6
380-399	12	5	17
360-379	19	18	37
340-359	26	24	50
320-339	30	28	58
300-319	64	41	105
280-299	48	51	99
260-279	41	47	88
240-259	41	43	84
220-239	24	42	66
200-219	20	23	43
180-199	16	15	31
160-179	9	9	18
140-159	1	6	7
120-139	3	1	4
100-119	0	2	2
80- 99	0	0	0
60- 79	0	1	1
Totals	358	358	716
Mean	285.14	273.80	279.46
	±2.01	±2.05	±1.44
Q_3	320.50	313.42	317.42
Q_1	247.56	234.98	241.18
S.D.	56.48	57.68	57.14
Difference in Mean			11.34 ± 2.88

1. Status at End of Experiment

The comparability of the two groups on the basis of scores made in the final tests is shown in Table II. It will be seen that a significant difference exists between the average performances of the two groups, since the probable error of the difference of the means is only about one-fourth of the difference itself.

Lack of space prevents the insertion of data showing where in the course of learning the significant positive effect of the use of standards and records of progress apparently took place. A just interpretation of the omitted data warrants the statement that the gains were a bit slow in appearing, but that with increasing sureness the Experimental Group responded more successfully as time went on. In other words, the novelty of the progress chart idea did not stimulate a spurt of effort which then tended to die out, but, rather, an opposite effect appeared.

2. Analysis of Levels of Ability

It is of real interest to study the effect of ratings and records on different levels of ability. It is not feasible to report the status of the four quarters of the pupils on the pretests; but Table II shows the facts for the initial status of the total group, and from the figures it will be evident that the quartile scores of both groups indicated statistically identical abilities at the beginning of the experiment.

TABLE III.—DISTRIBUTION OF THE PUPILS IN THE TOP AND BOTTOM QUARTERS OF ALL GROUPS (IN THE INITIAL TEST) ACCORDING TO THEIR SCORES IN THE FINAL TEST

	Top Quarter			Bottom Quarter		
	Exp.	Cont.	Both	Exp.	Cont.	Both
Mean	337.73 ±2.85	311.69 ±3.39	324.71 ±2.32	236.00 ±3.38	236.90 ±3.57	238.02 ±2.56
Q_3	368.76	352.82	362.25	271.18	277.42	273.96
Q_1	311.46	277.04	296.85	200.00	205.14	202.28
S.D.	41.10	49.05	47.45	47.28	49.94	50.78
Difference in Mean 26.04 ± 4.43				Difference in Mean 0.90 ± 4.93		

In Table III the final status of the highest and lowest quarters based on exact pairing of scores on the initial tests will be found. Space forbids the insertion of similar data for the two middle quarters. In general they behaved in a manner easily inferred from a study of Table III and of the comments on it.

It is evident that the top quarter of the Experimental Group gained very significantly over the upper quarter of the Control Group. It is also evident that there was no significant difference between the lowest quarters of both groups, either at the beginning or at the end of the experiment. The third quarter (next to the highest) showed considerable gain in favor of the Experimental Group, but not so much as the highest quarter.

The beneficial effect of awareness of success, then, was substantially in direct proportion to the amount of success available for motivation. Those who did succeed most in the Experimental Group succeeded more than those who succeeded most in the Control Group. Those who succeeded least in the Experimental Group did just as well as (but no better than) those who were in the lowest quarter of the Control Group. We could hardly expect very modest amounts of success to exert much motivation. The fact, however, that the lowest

quarter of the Experimental Group did not suffer by knowing their progress by means of the standards and the progress chart is of much importance, because there is a notion current that awareness of failure actually reduces performance. Objective demonstration of *relative* failure (in the sense of only very modest success) had no effect; it neither helped the lowest quarter of the Experimental Group to exceed the performance of the lowest quarter of the Control Group nor did it keep the lowest quarter of the Experimental Group from doing just as well as the same quarter of the Control Group. In short, the use of objective records of success helps where the records reveal substantial progress, but has no effect where they reveal slight gains or no gain at all.

3. Shifting Within the Total Group

If the Experimental Group and the Control Group are combined, each quarter of the Total Group thus formed will contain exactly equivalent populations of the experimental and the control pupils, as paired on the initial tests. It is illuminating to study the shifts in populations of such quarters on the final tests. Remembering that both the Experimental Group and the Control Group contributed equal numbers to each quarter on the initial tests it is interesting to note that in the top quarter of the Total Group nine members of the Experimental Group who were not in the top quarter initially crowded out members from the Control Group on the final tests, and that the performance of the Experimental Group in the total upper quarter was somewhat higher than that of the Control Group (mean 362.97 vs. 355.50). Our data show similar shifting in the other quarters.

4. Sex Differences

It is quite possible that the kind of motivation used reacts more effectively on one sex than on the other. The 194 pupils in the Experimental Group provide 97 boys whose scores exactly match those of 97 girls on the initial tests. In the final test the mean for the boys was 290.57; for the girls, 286.23. This difference in the means is smaller than its probable error, whence we may conclude that our method of securing motivation was as effective with one sex as with the other.

5. Effect on Variability Within Groups

We are not sure whether a device which produces an increase or decrease in variability is to be desired. Certainly not much alteration in it would be expected. The variabilities of the two groups were identical at the beginning of the experiment. At the end, by use of Pearson's coefficient of variation, it was found that the coefficient for the Experimental Group was 20.15; that for the Control Group, 21.07.

It is obvious that the technique of motivation used had no appreciable effect on the variability of the group as a whole. Space forbids citing figures on the variability of quarters as well as for total groups. It is possible that the effect of the technique used might cause great variability in the highest but not in the lowest quarter or the reverse. A fair interpretation of the data on this point is that various levels of ability did not behave differently than the total group as far as increase or decrease in variability is concerned. Our data also show no sex differences in variability.

6. Other Comparisons

It is possible to make comparisons between two groups and between sections of two groups, in many ways; the following will be of interest:

1. If the two groups were equal on performance on the final test, 50 percent of either group would exceed the median of the other. As a matter of fact, but 40.13 percent of the Control Group exceeded the median of the Experimental Group.

2. We should expect the mean of the top quarter of both Experimental Group and Control Group to be the same standard deviation distance from the mean of the Total Group if both these groups behaved the same on the final tests. The mean of the top quarter of the Experimental Group was 1.019 S.D. above the mean of the Total Group, that of the top quarter of the Control Group was .563 S.D. above the same mean.

While it is thus clear that the top quarter of the Experimental Group is well ahead of the same quarter of the Control Group, no such disparity appears between the means of the lowest quarters of the two groups. The mean of the lowest quarter of the Experimental Group was .700 S.D., and the similar mean for the Control Group was .744 S.D., below the mean of the Total Group.

3. One further comparison seems of interest. The next table requires a word of explanation. The pupils of Experimental Group and those of the Control Group were each arranged from highest to lowest according to their scores in the initial test. Then each group was divided into twenty equal subgroups, Group 20 being the highest and Group 1 the lowest in rank. Groups 20 to 3, inclusive, contained 18 pupils each; Groups 2 and 1 contained 17 pupils each.

The average scores in the initial test and in the final test of each subgroup were computed, and converted into standard deviations on the basis of the scores of the entire group. In other words, the average scores of each of the twenty subgroups of experimental and control groups in the initial test were converted into S.D. distances in terms of the distribution of the whole group in the initial test; and the average scores of the twenty subgroups of each group in the final test were similarly converted into S.D. distances in terms of the distribution of the entire group. This process of computation gives Table IV, which makes it possible to compare the experimental and the control pupils, classified into groups each of five percent of all

TABLE IV.—DIFFERENCES IN THE FINAL TEST OF THE MEANS OF TWENTY SUBGROUPS OF THE EXPERIMENTAL AND THE CONTROL GROUPS IN TERMS OF S.D. DISTANCES FROM THE MEAN OF TOTAL GROUP

Group	S.D. Exp.	S.D. Cont.	Advantage of Exper. Group (in terms of S.D.)
20	1.577	1.285	.292
19	1.305	.495	.046
18	.689	.351	.338
17	.655	.336	.219
16	.743	.467	.276
15	.682	—.071	.753
14	.170	—.052	.222
13	.384	.085	.299
12	.002	.057	—.055
11	.048	—.127	.175
10	.035	—.157	.192
9	.087	—.218	.305
8	.092	—.236	.328
7	—.032	—.491	.459
6	—.489	—.755	.266
5	—.263	—.319	.156
4	—.714	—.058	—.656
3	—1.036	—.964	—.072
2	—.716	—.923	.207
1	—1.167	—1.195	.028

the pupils in the group. Comparisons on the initial test do not appear in Table IV, since in the nature of the case the groups must be identical on account of the pairing between them. The facts shown relative to the performance of these groups on the final test, however, are noteworthy. The table is based on the computation of the S.D. distances from the mean of the total group of the 20 subgroups for both the Experimental Group and the Control Group on the final test scores.

We find in comparing, subgroup by subgroup, that the total S.D. advantage (algebraic sum of subgroup S.D.) is 4.081 for the experimental group and .783 for the control group. Few faiths in education are based on more convincing data than those presented in Table IV.

IV. Summary

1. Our data do not pretend to throw light upon the advantages or disadvantages of any method of motivation which uses hypothetical or arbitrary standards such that the ratings are mixtures of measures of progress and inequalities of difficulty in drill material.

2. Our data apply to a method of motivation which uses a type of material wherein the ratings furnish valid measures of pupil performance. We find that for the fourth grade a clear advantage results from the use of individual and class progress charts constructed to supply such valid measures of performance.

3. This advantage accrues mostly to the pupils who have the most success, as would be expected, but no significant disadvantage is evident in the case of the slower pupils.

4. The method seems equally good for both boys and girls.

5. The method has no effect on the variability within the group.

CHAPTER XII

A STUDY OF ERRORS IN PERCENTAGE [1]

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This chapter presents a summary of an 'error study' in percentage. It does not contain any defense for the social utility of percentage nor does it contain, except by implication, any suggestions relative to the improvement of the teaching of percentage. The defense for this report is twofold. First, but little is known about the teaching of percentage unless we accept uncritically vague generalizations found in the literature which have practically no objective data to support their contentions. Hence some information, even of a limited nature, may prove of interest. Secondly, while an error study without attempts at interpretation yields facts of limited use (because their meaning may not be clear), still an error study which does attempt interpretation may point the way for learning studies through the use of which we may with confidence expect to increase our control over learning.

The main contributions of this study are: (1) data on difficulty in percentage which are of direct use in teaching, and (2) data on the nature of error which will add to our knowledge of what error really is psychologically.

I. The Pupils Studied

The data for this study were taken from the performance of 215 pupils in Grade VII, on the Compass Diagnostic Test Number XIV. These pupils were in typical classrooms in eight mid-western towns and had completed their study of percentage. Both better and poorer conditions could doubtless have been found. The pupils in this group constituted a fair sampling, however, as their normal scores on the test showed.

II. The Test Used

The test itself is a series of 113 items, all dealing either with skills in percentage or with examples in percentage. Since 215 pupils were

[1] This study was made at the College of Education, State University of Iowa, in 1928, under the direction of F. B. Knight.

tested, there were thus afforded for study 24,295 responses (including the omissions). The test items vary greatly in complexity; they call for responses to a wide and presumably satisfactory sampling of skills in percentage. The range of difficulty may be appreciated if we list here the four easiest and the three hardest items.

a. Easiest Items.—The four following items were the most thoroughly mastered by the pupils in this study. (For space considerations, the items are stripped of directions and other setting which appear in the test itself, but the work asked for will be clear to the reader.)

Items	*Wrong Answers*	*No. Among 215 Pupils Making Wrong Answers*	*PercentMaking Wrong Answers*
¼ of N = ——% of N	20	4	3
	70	3	
	12½	1	
⅛ of N = ——% of N	No response	1	4
	75	3	
	66⅔	2	
	16⅔	2	
	16⅓	1	
.55 of N = ——% of N	No response	3	2
	5.5	1	
	42	1	
.24 of N = ——% of N	No response	5	5
	.24	5	
	¼	1	

b. Hardest Items.—The following items are at the upper extreme of difficulty when difficulty is measured by percent of pupils failing to make the correct response.

Items	*Wrong Answers*	*No. Among 215 Pupils Making Wrong Answers*	*Percent Making Wrong Answers*
If 6½% of N = 70, what is 100%?	No response	102	98
	Random errors	110	
37.7 is what percent of 1908?	No response	90	94
	Random errors	112	
If 25 is 62½% of N, 100% = ——.	No response	104	84
	Random errors	76	

It may be argued that the very hard items here listed are beyond the province of arithmetic in the seventh and eighth grades. With

this judgment we have no quarrel, but a study of texts and courses of study makes it perfectly clear that we nevertheless purport to teach percentage to a level of difficulty suggested by these examples.

c. The Complete Test.—The 113 examples of the test used are presented next, along with figures showing the percents of two groups of pupils answering wrongly. One group was that quarter of all pupils making the highest total scores; the other that quarter making the lowest total scores on the test. Thus Item I was missed by no pupils in the highest quarter but by 19 percent of the pupils in the lowest quarter.

No.	*Item*	*Lowest and Highest Percents of Pupils Missing Item*
	Change the fractions below to percents:	
1	$\frac{1}{4}$ of N = ——% of N	0-19
2	$\frac{1}{3}$ of N = ——% of N	0-19
3	$\frac{3}{10}$ of N = ——% of N	0-19
4	$\frac{2}{5}$ of N = ——% of N	0-19
5	$\frac{1}{8}$ of N = ——% of N	0-19
6	$\frac{9}{100}$ of N = ——% of N	20-39
7	$\frac{7}{10}$ of N = ——% of N	0-19
8	$\frac{4}{5}$ of N = ——% of N	0-19
9	$\frac{3}{2}$ of N = ——% of N	20-39
10	$\frac{9}{9}$ of N = ——% of N	20-39
11	$\frac{17}{100}$ of N = ——% of N	20-39
12	$\frac{5}{4}$ of N = ——% of N	20-39
13	$\frac{5}{6}$ of N = ——% of N	40-59
14	$1\frac{3}{4}$ of N = ——% of N	20-39
15	$\frac{7}{8}$ of N = ——% of N	20-39
	Change the percents below to fractions:	
16	$66\frac{2}{3}$% of N = —— of N	0-19
17	150% of N = —— of N	0-19
18	$62\frac{1}{2}$% of N = —— of N	40-59
19	80% of N = —— of N	0-19
20	6% of N = —— of N	40-59
21	26% of N = —— of N	40-59
22	75% of N = —— of N	0-19
23	20% of N = —— of N	0-19
24	$16\frac{2}{3}$% of N = —— of N	20-39
25	$37\frac{1}{2}$% of N = —— of N	20-39
26	11% of N = —— of N	40-59
27	30% of N = —— of N	20-39

28 70% of N = —— of N 20-39
29 175% of N = —— of N 0-19
30 100% of N = —— of N 20-39

Write the percents below as decimals:

31 25% of N = —— of N 0-19
32 4% of N = —— of N 0-19
33 80% of N = —— of N 0-19
34 7½% of N = —— of N 20-39
35 106% of N = —— of N 20-39
36 3% of N = —— of N 0-19
37 150% of N = —— of N 20-39
38 5% of N = —— of N 0-19
39 33⅓% of N = —— of N 20-39
40 6¾% of N = —— of N 40-59
41 75% of N = —— of N 0-19
42 5¼% of N = —— of N 40-59
43 135% of N = —— of N 20-39
44 12½% of N = —— of N 20-39
45 83⅓% of N = —— of N 20-39

Write the decimals below as percents:

46 .55 of N = ——% of N 0-19
47 .4 of N = ——% of N 60-79
48 .06 of N = ——% of N 0-19
49 1.13 of N = ——% of N 0-19
50 .7 of N = ——% of N 60-79
51 .01 of N = ——% of N 0-19
52 1.63 of N = ——% of N 0-19
53 .0325 of N = ——% of N 40-59
54 .0475 of N = ——% of N 40-59
55 1.00 of N = ——% of N 20-39
56 .24 of N = ——% of N 0-19
57 .065 of N = ——% of N 40-59

Write on the line below each figure the percent of the area that is shaded.

(Note: Cuts used in the test are omitted here, but a description of each is given in parenthesis.)

58 (Cut of rectangle with one-fourth shaded) 20-39
59 (Cut of circle with sector of one-eighth area shaded and dotted lines to show the size of quarters) 40-59
60 (Cut of square entirely shaded) 0-19
61 (Cut of rectangle with second and fourth quarters shaded) 0-19
62 (Cut of rectangle with the half above the diagonal shaded) 0-19
63 (Cut of rectangle five-sixths shaded, with division mark at each sixth) 60-79

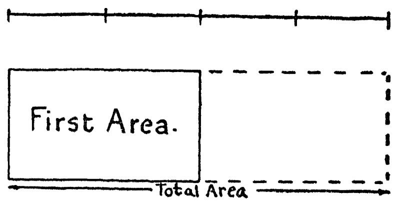
64 (Cut of rectangle three-eighths shaded, with marks at each eighth) 20-39

65 (Cut of circle two-thirds shaded) 40-59

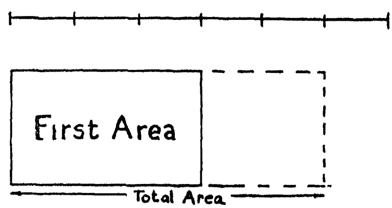
66 (Cut of square divided into 100 smaller squares with six shaded) 40-59

67 (Cut of rectangle with five division marks in right side and with top one-fifth and bottom two-fifths shaded) 40-59

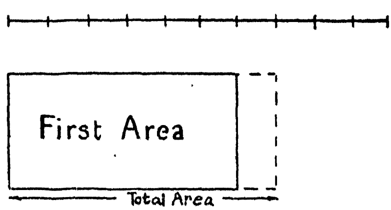
If the figures below were increased as shown by the dotted lines, what percent of the first area would the total area be? The lines above the figures will help you estimate the areas.



68 Total area = ——% of first area..................... 20-39



69 Total area = ——% of first area..................... 60-79



70 Total area = ——% of first area..................... 80-100

At the right of each example there are five proposed solutions. Do not work the examples, but simply draw a line under the way you should solve each.

71 Find 8% of 241. $241/\overline{8}$ $.08/\overline{241}$ $\begin{array}{r}241\\ \times .08\\ \hline\end{array}$

$\frac{1}{8}$ of 241 $\begin{array}{r}241\\ \times 8\\ \hline\end{array}$ 20-39

72	21 is what percent of 93?	93 — 21; .93/21; 21/93; 93 ×.21; 93/21	40-59
73	If 82 is 3% of N, 100% equals what?	.03/82; 82 ×.03; 82/3; 82 × 100; .03/100	60-79
74	When 94% of N is 3052, 100% is what?	100/94.00; 3052 ×100; 3052/94.00; .94/3052; 3052 ×.94	60-79
75	73 is what percent of 9274?	9274 ×.73; 9274/73; 73/9274; 9274/.73; 1/73 of 9274	20-39
76	Find 125% of 9672.	9672 —1250; 1.25/9672; 9672 ×1.25; 9672/125.00; 9672 ×.125	20-39
77	366 is what % of 9340?	9340/366; 9340 ×.366; 9340 ×366; .9340/366.00; 366/9340	20-39
78	Find 64% of 8321.	1/64 × 8321; 8321 ×64; .64/8321; 8321/64.00; 8321 ×.64	40-59
79	100% of N is what, if 924 is 16% of N?	1/16 × 924; .16/924; 924 ×.16; 16/.924; 924 ×100	60-79
80	When 34 = 3½% of N, 100% equals ——?	34 ×100; 34 ×.35; 35/34; .035/34; 34 ×.035	60-79
81	Find 6¼% of 9367.	9367 ×.0625; .0625/9367; 9367 ×.625; .625/9367; 9367 ×6¼	60-79
82	629 is what percent of 216?	629 ×216; 629 ×.216; 216/629	

			629/216	629 ×2.16	40-59
83	When 217% of N is 936, 100% of N is ——?	936 ×.217	217/936	936/217	
		936 ×2.17		2.17/936	80-100
84	Find 9% of 2163.	21.63 ×.09	2163 ×.09	2163 ×.9	
		.09/2163		2163/9.000	20-39
85	.7 is what percent of 1.63?	1.63/7.00	1.63 ×.7	1.63/.7	
			1.63 ×.07	.7/1.63	40-59

Write on each line the number which will make the statement true.

86 4 = ——% of 8 40-59
87 25% of 12 = —— 20-39
88 If 15 is 75% of N, 100% is —— 60-79
89 If 21 is 300% of N, 100% = —— 60-79
90 3 = ——% of 9 40-59
91 If 18 is 6% of N, 100% = —— 80-100
92 27 = ——% of 2700 60-79
93 If 9 is 33⅓% of N, 100% = —— 60-79
94 60 = ——% of 30 60-79
95 66⅔% of 15 = —— 20-39
96 2% of 400 = —— 40-59
97 5 = ——% of 40 60-79
98 37½% of 24 = —— 40-59
99 If 25 is 62½% of N, 100% = —— 80-100
100 500% of 6 is —— 40-59

Write the answers on the lines marked "Ans."

101 Find 6% of 744 .... Ans.—— 20-39
102 625 is what percent of 5000? .... Ans.—— 40-59
103 If 35 is 125% of N, what is 100%? .... Ans.—— 80-100
104 7905 is what percent of 6375? .... Ans.—— 60-79
105 Find 17% of 691 .... Ans.—— 40-59
106 If 150.64 is 28% of N, what is 100% .... Ans.—— 80-100
107 37.7 is what percent of 1908? .... Ans.—— 80-100
108 Find 83% of 305 .... Ans.—— 40-59
109 If 6½% of N equals 70, what is 100%? .... Ans.—— 80-100
110 1220 is what percent of 7625? .... Ans.—— 60-79
111 Find 4½% of 258 .... Ans.—— 40-59
112 2125 is what percent of 3125? .... Ans.—— 60-79
113 Find 118% of 465 .... Ans.—— 40-59

III. Classification of Difficulties in Percentage

Handling percentage is not a single thing to be learned; rather it involves various skills, aspects, processes, or phases, or a hierarchy of processes. In any one example not all these processes appear. The correct working of any given example demands of the pupil not only that he use the correct responses, but also that he do not use other responses which in other examples would be correct. Percentage offers to the pupil a splendid situation in which to indulge his powers of selective thinking. It will be demonstrable that much error in percentage is not doing of the thing wrongly, but doing the wrong thing. Thus, percentage is psychologically quite a different thing from, say, long division, in which doing the thing wrongly accounts for the bulk of error. It is possible to examine a child on examples, or items, representing more or less unitary phases. By such examinations the phases on which breakdowns exist may be discovered. The test used in this study was organized so that the various phases of percentage as well as the process *in toto* could be tested. This characteristic is the main distinction between diagnostic and survey tests. Table I presents the general difficulty of various aspects of percentage. Difficulty here is measured by percent of the 215 pupils failing to respond correctly.

If we take, as a criterion of utter failure to teach, a percent of difficulty score of 50 or more (assuming, of course the propriety of the test item used), it is evident that in certain aspects of percentage we are failing at the present time. These are: (a) recognition of quantities over 100 percent of a given quantity, (b) recognizing the correct solution of examples using the third case of percentage, (c) solving by inspection easy examples using the second case, (d) solving by inspection easy examples using the third case, (e) written work using the second case, (f) written work using the third case.

IV. Analyses of Specific Difficulties

Of the 24,295 responses which were made on the items, 14,402 were correct, 2,387 were omissions (a form of 'wrong' response which does not admit of analysis) and 7,506 were recorded as errors.

Of the 7,506 recorded errors, 1,955 were adjudged random (each made by few pupils), for which analysis did not seem warranted. Many of them, though distinct from others as quantitative errors, were really alike in their nature. For example, it was evident from

TABLE I.—RELATIVE DIFFICULTY OF ASPECTS OF PERCENTAGE

Aspect	For Sample, see Test Item Numbers	No. Items	Average Difficulty (in percent)[1]	Range of Difficulty			
				Percent	Item Number	Percent	Item Number
Changing fractions to percents	1–15	15	21	3	1	44	13
Changing percents to fractions	16–30	15	27	6	16	53	20
Changing percents to decimals	31–45	15	23	9	22	43	40 42
Changing decimals to percents	46–57	12	28	2	56	60	47 50
Recognizing percent of an area which is shaded when percent is less than 100	58–59, 61-67	10	36	13	61	60	63
Stating what percent a total area is of an original part when total is over 100 percent of original	68–70	3	66	39	68	81	70
Choosing the correct solution from five suggested solutions with but one solution correct[2]	71–85	15	55	33	71	82	83
Percentage examples soluble by inspection[3]	86–100	15	52	33	95	84	91
Percentage examples requiring written work for solution[4]	101–113	13	66	33	101	98	109

[1]Difficulty is wrongs plus omissions, divided by the number of pupils.

[2]The average difficulty of the 15 items of choosing correct solution may be divided on a basis of the case in percentage called for. Thus, the average difficulty of the five items calling for the first case of percentage is 42%; for the five items calling for the second case is 44%; and for the five items calling for the third case is 73%.

[3]The five items using case one had an average difficulty of 43%; those using case two, of 63%; those using case three, of 74%.

[4]The five items calling for the first case of percentage had an average difficulty of 48%; the five items calling for the second case, of 73%; the three calling for the third case, of 89%.

inspection that many seemingly different errors were really all of them errors due to inability to divide. Division was at times the right and at times the wrong thing to do. An analysis of many such errors would be properly transferred to a study of division; the errors were made in percentage, but they may not properly be regarded as an indication of difficulties in percentage.

Ruling out, then, errors of computation which we assign to analyses of the processes involved, we can at least describe 5,551 which are purely or largely percentage errors. Further, let us say that we are impressed with two types of errors: (a) errors which appear so often that a definite wrong connection is suggested, and (b) errors which vary among themselves so much that no such connection is suggested. It is possible that remedial and initial teaching influenced by one type of error would differ significantly from such teaching influenced by the other type of error. In the following description of specific difficulties, attention is given mainly to those situations (in 35 of the 113 items) to which many pupils responded wrongly in the same way. Certain generalizations will be attempted at the end of this description.

I. Difficulties in Changing Fractions to Percents

(1) $\frac{9}{100}$ of N = —— %. There were 79 recorded errors. Thirty-six of these were 90 and 17 were 900; thus these two errors, 90 and 900, account for well over half of all the errors. The remaining 26 were scattered among ten other errors.

It is evident that the pupil knows that there must be a 9 in the percent which is equivalent to the fraction $\frac{9}{100}$. Most teachers would probably insist that they stress the fact that hundredths means percent and that 9 hundredths means 9 percent. It may be that the unconscious impression of zeros in the 'picture' of the example impels the pupil to get some zeros into his answer, and that all that would be needed to correct this error would be to point this out and breed contempt for it through familiarity.

Still a child who understands percentage must know that $\frac{9}{100}$ is a small amount and that 90 percent is a large one, and hence it should appear that 90 or 900 is an unreasonable answer. Here, therefore, is an error which is unreasonable if the essential idea of percentage is understood. The pupils may be too dependent on the rule of thumb or quite unconscious of the 'thought' of the example. They may be so absorbed with computation that the feel for reasonableness is crowded out. At all events, the absurdity did not disturb them.

(2) $\frac{3}{2}$ of N = ——% of N. There were 62 recorded errors. Of these, 24 were the error $1\frac{1}{2}$. The others were scattered among 20 responses.

It seems evident that the error $1\frac{1}{2}$ is a clear case of responding to a situation more familiar than the one present. Previous and more familiar

fraction work would make 1½ the correct response. If we grant, for the average pupil, that whole numbers are more familiar than fractions, and that fractions are more familiar than percents, we can account for a significant number of errors on the theory that, given a situation which possesses the two characteristics: (1) some uncertainty or difficulty, and (2) looks more or less like some previous situation not so difficult, the pupil responds with the more familiar solution. *That is, he transfers the situation back to a better known one somewhat like it.* Should this notion be established as dynamic of error making, we would have good ground for insisting on an aspect of technology now not only slighted but even looked at with suspicion in teaching; namely, teaching what a thing is not, as well as what it is. Skillful teaching here would mean exposing the pupil to errors which he is likely to make and teaching him thereby not to make the error. This suggestion need not be argued until more data are available.

Relative to the 24 errors of 1½ on this example, it is plain that, if the example were in fractions, these 24 errors would have been correct responses. It is further plain that 1½ percent is absurd, since $\frac{3}{2}$ is more than a whole and 1½ percent is far less than a whole. This absurdity did not bother the children. The main ideas of percentage were either unknown to them or not applied to the example.

(3) $\frac{9}{9}$ of N = ——% of N. There were 63 recorded errors. Of these, 20 were the answer 1; 19 were some arrangement of 99 and a decimal point; the remaining wrong responses were scattered among 16 other errors.

If the example had been one in fractions (a more familiar situation), 1 would have been correct. The 19 errors of 99 may have been a case of too much '9-ness' in the visual picture. That $\frac{9}{9}$ of N is a whole of N and that 100 percent represents a whole was an idea either not effective at the time or one not really mastered by the error-makers.

(4) $\frac{5}{4}$ of N = ——% of N. There were recorded 78 errors, of which 23 were the response 1¼. The others were scattered among 23 different scores.

It is noteworthy that 1¼ would have been correct had the example been one in fractions rather than percentage. The absurdity of 1¼ percent did not offend the pupil. The error 1¼ could be explained by the principle of transforming a difficult stiuation into a familiar and easy one suggested by the general configuration of the difficult problem.

(5) $\frac{5}{6}$ of N = ——% of N. There were 96 recorded errors. No single error occurred with significant frequency, but there were 36 different ones. An inspection of them showed that pupils simply did not know what percent to write. Typical responses were 62½, 86⅓, 37½, 116⅔. Many of these were the correct percent equivalent for another fraction; others were 'mongrels,' partly a fragment of one common percent and the remainder a part of another common percent.

2. Difficulties in Changing Percents to Fractions

(6) 62½% of N = ——(fraction) of N. There were 109 errors recorded. Of these 11 were $\frac{5}{6}$, and 11 were $\frac{3}{8}$. The others were scattered

among 30 different responses. In the case of the 22 errors noted we seem to have a clear case of attaching the wrong response to a situation. This seems related to the well-known 'table errors' in the multiplication of whole numbers. Theoretically this error is: If S_1 should be tied to R_1 and S_2 tied to R_2, and if these two sets are taught in close proximity, we may expect that S_1 will be found often with R_2 and S_2 with R_1. A large number of errors in changing a value from one notation to another are evidently of this type. The pedagogy of fighting this 'slipping' of connections is rather neglected in much arithmetic material at present.

(7) 6% of N = —— (fraction) of N. There were 114 errors recorded: 16 were 16⅔; 8 were .6, and 11 were ⅙. The others were scattered among 37 different errors. Here it is possible that "6% of" in a setting of changing percents to fractions gets transferred to fractions in some form of a sixth. One-sixth is 16⅔ percent, and the 10 responses of ⅙ (unless utter absurdities) seem related to a subtle interference of fractions.

The eight errors of .6 and other decimal forms show that the child accustomed to changing percents to decimals does this even when a fraction form is specifically requested. This may be a form of reversion to a more familiar situation.

(8) 26% of N = ——(fraction) of N. There were 112 errors recorded, of which 12 were 26. Here attempts at writing a decimal may be present. The other errors were scattered among 33 types, some of which showed attempts to divide 100 by 26.

(9) 11% of N = —— (fraction) of N. There were 112 errors recorded. Of these, 10 were 9$\frac{1}{11}$, a clear case of dividing 100 by 11. Others were scattered among 25 different wrong responses.

3. Difficulties in Changing Percents to Decimals

Here but three of the 15 items caused serious trouble, and these three involved a fraction.

(10) 7½% of N = —— (decimal) of N. There were 81 errors recorded. Of these, 30 were 7.5, and 19 were .75. The others were scattered among 20 different wrong responses. In the 30 errors of 7.5, we see a neglect of the idea of percentage, since a decimal of 7.5 is much over 100 percent. It is probable that the previous experience of "½ is .5" dominates the reaction to the exclusion of much regard for the problem as a whole—the use of percentage. Here is a plain example of shifting a difficult situation to a familiar and less difficult one. The 19 errors of .75 show some appreciation of percentage, but the decision where to place the decimal point was not much affected by the sheer reasonableness of the relationships involved.

(11) 6¾% of N = —— (decimal) of N. Of the 93 errors recorded, 24 were 6.75, and 13 were .675. The others were scattered among 24 different responses. Most of the errors seem to arise from the same cause as in the case of the previous example.

(12) 5¼% of N = —— (decimal) of N. Of the 93 errors recorded, 35 were 5.25. The others were scattered among 25 different responses.

In the last three examples mentioned we have a total of 89 errors, all surely alike. When a pupil sees a fraction in a percent, as $7\frac{1}{2}$, $6\frac{3}{4}$, and $5\frac{1}{4}$, the fraction dominates the situation and he hastens to write the fraction rather than the whole percent as a decimal. Decimals are taught before percentage. Changing fractions to decimals is a familiar experience, and we have here, it would seem, a reverting to the more familiar. The absurdity in terms of the relatively new idea, percentage, is not effective. It is possible that we are neglecting the teaching of distinctions and also missing a good opportunity of teaching the avoidance of error by exposing a child to interferences (error) and thus preparing a guard in advance against this type of error.

4. Difficulties in Changing Decimals to Percents

Here five of the fifteen items claim a moment's attention.

(13) .4 of N = ——% of N. There were 131 errors recorded, of which 111 were the response 4. Here is a clear case of carry-over from a previous more familiar situation. In teaching decimals we confine ourselves to such facts as .4 means four-tenths, but .04 means four-hundredths. We refrain from directing attention to the fact that .4 means four-tenths, or forty-hundredths. A single-digit decimal gets associated with but one possibility; namely, the same digit as a numerator with a denominator of 10: .2 means two-tenths, .9 means nine-tenths. Whatever the decimal is, it appears unchanged when translated into fractions. Thus, when the child sees .4, even in percentage situations, the same type of translation persists; namely, 2% = .2, 4% = .4, 9% = .9, etc. Singleness of the digit in .4 gets attached to singleness of digit in equivalent fractions, as $\frac{4}{10}$. Thus, singleness carries over (in spite of absurdity) into percent equivalents of single-digit decimals. Knowing the likelihood of this error, we should enrich our first teaching of decimals to include the fact that .4, .40, .400 are equivalent. Further, we should, in the teaching of percentage itself, warn the child against the operation of this wrong inference from previous experience. As usual, the absurdity of the answer (the lack of appreciation of what the numbers mean) did not disturb the error-makers.

(14) .7 of N = ——% of N. There were 113 errors. Of these, 103 were 7. The discussion of (13) above is sufficient for this example as well.

(15) .0475 of N = ——% of N. There were 113 errors. Of these, 41 were 0475, and 43 were 475. This seems, to the writer, exactly the same thing as the errors in (13) and (14). The decimal is translated as in fractions (the more familiar situation). There were 41 pupils who translated naïvely; 43 dropped the 0 at the left since that "does not count" and hence is not written except in decimals.

(16) .0325 of N = ——% of N. Of the 110 errors, 38 were 325; 46 were 0325.

(17) .065 of N = ——% of N. Of the 116 errors 52 were 65; 41 were 065. In (16) and (17) we have the same forces at work as in the three preceding examples.

In (13) to (17) 475 of the 701 errors seemed to be definitely caused by carrying over from the previous more familiar decimal fraction to relationships (wrong cues) in spite of the meaningful ideas of percentage. Many of the other errors seem to be explainable by the notion that the pupil will 'distort' a difficult situation to one more familiar and easy which is suggested by the visual configuration of the difficult situation.

5. Difficulties in Expressing Areas in Percents

This section deals with difficulties of expressing a shaded part of a total area as a percent of that total area. Here four items warrant study.

(18) In solving Item 59, 92 errors were recorded. Of these, 20 gave 20% as the answer and 18 gave 1/8 as the answer. The others were scattered among 26 types of errors. The error 20% is evidently due to mere counting of divisions rather than judging the amount of area shaded with the aid of indicated divisions. The error 1/8 is writing the correct answer for a similar example in fractions and forgetting that answers in percent are called for.

(19) In the example Item 65, 103 errors were recorded. Of these, 44 gave 33⅓ percent, and 15 gave ⅓. The additional cause of error here is responding to the smaller of two contrasting areas rather than to the one asked for. The smaller may have greater attention value, but the impulse behind this error may also be traced to a very subtle habit certainly aided and abetted quite unconsciously by previous experience; for in most of our fraction and percentage work we deal with the smaller or possible contrasting quantities. We teach 1/3, 1/4, and 1/5 certainly first, and probably more effectively than 2/3, 3/4, or 4/5. In percentage the bulk of our work deals with relatively small percents. The general set built up is probably in favor of the smaller. The past experience forms a configuration in the mind of the pupil which causes him to see ⅓ of the figure against a background of a more general nature. This seeing is so strong that it leads(in this case) to error.

(20) In the example Item 66, there were 127 errors, but no favorite mistake; the errors embraced 38 different wrong responses.

(21) The 130 errors in Item 63 were rather impartially distributed among 36 different answers.

6. Difficulties in Expressing Areas of Over 100 Percent

(22) In solving Item 69, there were 169 errors, of which 33 were 150% and 37 were 175%. The others were scattered among 33 other wrong answers. A study of the example itself may impress the reader with the possibility that the two most frequent errors are examples of wrong reasoning. Percents over 100 are notoriously difficult. This difficulty may be partly due to the little attention given to them in contrast to the amount of practice ordinarily given to percents less than 100.

(23) With Item 70 there were 175 errors recorded. Of these 33 were 135, while the others were spread over 65 different responses.

7. Difficulties in Choosing Correct Solutions

The type of exercise involved is illustrated by four examples. This type of work was unfamiliar to the pupils, since their text and drill material did not provide this kind of practice in selective thinking. However, the unfamiliarity did not render this work as difficult as some other types of items which are practiced with universality in the teaching of arithmetic.

It will be noted that the pupil's response was limited to one of five possibilities. Several of the errors are worthy of attention.

(24) With Item 79 there were 159 errors, about equally divided among the four wrong solutions printed. In fact, the response of the 215 pupils on all five responses, one correct and four wrong, could easily be accounted for by chance. The effect of unfamiliarity with the type of work would hardly account for this. A sample exercise is worked correctly in the test, and on similar exercises the wrong responses were far less evenly distributed. It will be noted that the correct solution is a use of the third case of percentage. In this test the third case of percentage was always very difficult, irrespective of the form of the item.

(25) Example 74 caused 161 errors. The solution $3052 \times .94$ accounts for 17 of these. The remainder were about equally divided among the three other wrong solutions. Some selective thinking was present because a solution of a first case of percentage received but 17 of the 161 errors. Note again that the example used calls for the third case of percentage.

(26) For example 80 there were 164 errors recorded. Mainly the errors avoided the two examples which might be solutions to examples using the first case of percentage and concentrated on 34×100. At least, in working case three examples, pupils know that the first case type of solution is not the one to select.

(27) Example 83 induced 177 errors. Very few were the selection of the solutions which suggest the first case of percentage.

Relative to errors in selecting the correct solution from a list of suggested solutions, other types of errors of significance were: (a) failure to distinguish between $546 \times .56$ and 546×56 (the presence or absence of a decimal point was too subtle an element to bother a large number of pupils), and (b) the number of omissions (no response) on this type of work.

8. Difficulties in Examples Solved by Inspection

It will be remembered that 52 percent of the work on the 15 items of this type was wrong (or omitted).

(28) $4 = $ ——% of 8. Here there were 89 recorded errors, of which 55 gave 2 as the answer. Our theory is that when a pupil does not know what to do, he does a familiar thing as determined by the similarity of the example with some past better known example. The example must, in some way, be reminiscent of $4 \times ?$ is 8, or at least it seems plausible that 2 could be obtained in this way.

(29) 3 = ——% of 9. There were 106 errors, of which 64 gave 3 (seemingly related to 3 × ? is 9).

(30) 27 = ——% of 2700. Of 171 errors, 82 gave 100. Here again the distortion, 27 × ? is 2700, seems possible.

(31) 60 = ——% of 30. There were 168 errors recorded. Of these, 26 were 2 and 40 were 50. In the latter case a familiar situation in percentage was used as the distortion. We ask a child in beginning work in percentage what percent 30 is of 60 far more often than the reverse. When a pupil has finished percentage, if we ask him what percent 60 is of 30, we get frequent distortion to the somewhat similar and more familiar example.

(32) 5 = ——% of 40. There were 150 errors recorded, 52 of 80, and 50 of 8. 'Distortion,' as noted above, seems a plausible explanation.

(33) 25% of 12 = ——. There were 68 errors recorded. Of these, 17 were 300 (25 × 12 = 300).

(34) 2% of 8 = ——. Of the 105 errors, 35 were 200, and 17 were 800. Both of these errors could be explained as transferring a difficult situation back to a familiar one, similar to it in appearance.

(35) 500% of 6 = ——. Of the 116 errors, the most frequent was 3000 (500 × 6 = 3000).

Examples involving the third case of percentage to be worked by inspection involved many errors, but they were largely scattered.

9. Difficulties in Examples with Written Solutions

In the 13 items here, many errors were made. Inaccuracies in computation caused such an astounding variety of minor errors that an interpretation of them seems unwise without many more data.

V. Relation of Ability in Percentage to Other Abilities

At the time of giving the tests, certain other data were also obtained. These make possible the measurement of several relationships.

1. The Pearson correlation coefficient between scores on the Terman Group Intelligence Test and scores on the percentage tests is .522 ± .034.

2. The *r* between scores on the Stanford Achievement Reading Test and scores on the percentage test is .365 ± .041.

3. The *r* between arithmetic ability in general, as measured by scores on the advanced form of the Compass Survey Tests, and ability in percentage, as measured by scores on the percentage test, is .699 ± .021.

4. It is useful to study the consistency of difficulty of parts of the percentage test in respect to performance by pupils from various school systems. Eight school systems were used in this study. The

correlations between relative difficulty of parts of the test, city by city, average .938. Since different arithmetic texts were used in these school systems, it follows that the relative difficulties of various aspects of percentage, reported in this study, are not due to the use of a given textbook, but are general.

5. It is possible to divide the pupils used in this study into groups to see if able pupils make not only fewer errors, but also different kinds of errors, in comparison with less able pupils. When the total group is divided into four groups: the HH group, which are those pupils high on the Terman Group Test and high on the percentage test; the HL group, those pupils high on Terman but low on percentage; the LH group, those low on Terman but high on percentage; and the LL group, those pupils low on both Terman and percentage, we find the average percents of error for each group reported in Table II.

TABLE II.—PERCENTS OF ERRORS FOR DIFFERENT ASPECTS OF PERCENTAGE MADE BY GROUPS OF DIFFERENT ABILITIES

Parts	Nature of	H.H.	H. L.	L. H.	L. L.	All Groups
I	Equivalent Fraction, Decimal and Percent Relations	11.64	11.19	37.46	40.14	25.11
II	Expressing Areas in Percents	34.92	37.73	59.46	71.31	50.85
III	Choosing Correct Solutions in Percentage	41.49	37.25	65.10	70.26	53.53
IV	Easy Work in Percentage	41.70	38.73	79.17	81.52	60.28
V	Harder Work in Percentage	56.74	57.72	80.28	85.28	70.00
Average All Parts		37.30	36.52	64.29	69.70	51.95

Further, when we correlate the relative difficulty of examples for the low group with similar relative difficulties for the high, we get an r of $.89 \pm .03$. A low correlation here would warrant investigation, but this high one indicates that disability here is largely lack of general ability rather than a more mysterious affair. This is further substantiated by a correlation of $.99 \pm .003$ between the high group and all the others.

6. If the scores made by our low group were almost equal to those made by our high group, we might argue that on the whole most pupils gained about the same degree of mastery in percentage. However, upon a study of the average difference on the low and high groups, approximately quarters, on sections of the test as well as on the whole test, we find that on every part of the test the chances are

87 in 100 that a true difference in performance between these two groups does exist. It is important thus to study differences on parts of the test as well as on the whole test, for, if on some sections of percentage significant differences exist, on those parts of percentage provision for individual differences should be stressed. If we divide percentage into 13 specific aspects, we find that unquestioned differences do appear between the upper and the lower quarters of the pupils in every aspect. Here we have convincing evidence that there are such wide differences of learning in typical seventh-grade groups that there is a clear call for varied treatment in instruction, yet there are in texts or drill material hardly any provisions to meet such demonstrated differences, that are more than superficial in their nature.

VI. Conclusions

From this careful inspection of the errors made in percentage several conclusions seem defensible:

1. Our present methodology of teaching percentage does not build up a sufficient 'sense of the problem' to protect pupils from making altogether too many absurd errors. Answers are given which are impossible and senseless to any one who has a clear appreciation of the ideas of percentage and of the reasonable relation of numerical quantities.

2. Many errors seem traceable to an action pattern which is, one might say, a running away from a difficult situation and a making believe that a familiar and easy one similar to it will be satisfactory. The transmuting of a percentage example into a fraction or whole number example, and then solving that, is a tendency, an interference habit, which the writer at least feels should receive definite attention in teaching.

3. Many of the errors are instances of correct connections wrongly attached. Errors of this sort should be reasonably susceptible to control.

4. Broad interpretation of the data seems to fit into a feasible view of what a school subject is in terms of learning connections between specific situations and responses which are formed into groups, or hierarchies, sometimes correctly, but too often wrongly. Wrong responses too often become closely related to right ones, so that the real distinction escapes the pupil. Our technology of percentage has

probably seriously underestimated the intrinsic difficulty of the subject and would be improved by far more attention to what *not* to do and think, as well as what to do and think.

5. Error-making in percentage may be tentatively classified somewhat as follows:

(a) Mere arithmetical stuttering, errors due to sensory-motor slips. These are probably due to an imperfect nervous system and have but little pedagogical significance.

(b) Definite wrong attachments of connections of a rather simple nature, as $3/8$ of $N = 62\frac{1}{2}\%$ of N. This type of error pervades all sections of arithmetic. Practicing sets of connections in mixed rather than table order seems to be a rather adequate therapy for it.

(c) Utter confusion, as in the case of a total of 100 errors being scattered among 65 different ones—many errors each made by a few pupils instead of a few errors made by many pupils. We are not sure that this scattering is real. If it is, we have a general situation in which no connections are formed and no hierarchies organized. However, such apparently widely scattered and diverse errors may be but variations of a single error underneath the surface.

(d) A consistent and frequently repeated single error. Of this type, many illustrations have been noted. We suggest that the dynamics at work here are: When facing a difficult situation, shift it to one suggested by the visual configuration which is more familiar and easy. Reacting by analogy or mislearning by analogy may not entirely explain this action pattern. In all forms of problem-solving in the broader situations of life, we find this distortion to the easier present. Errors in percentage may be but an example of this general type of behavior, namely, regression to the easier and more familiar.

The same point of view is suggested by insistence on the presence of not only helpful transfer between whole numbers, fractions, and percentage, but also of a large amount of vicious, harmful transfer. Negative transfer from more familiar areas of arithmetic to percentage has been quite neglected in the teaching and drill material ordinarily used in percentage.

In much of the thought and practice of arithmetic in the field of error studies, remedial work, and revision of initial teaching in the light of probable errors, we find a naïve and inadequate psychology at the base. Errors have been thought of as just errors; that was all there was to them, or at least that was enough to be said. Such

a view of error is misleading, for just as we cannot say, when a man is sick, that sickness is sickness and that that is all there is to it; so we cannot be content with superficial accounts of arithmetical errors. There are different kinds of error, and each kind many require a distinct therapy of its own. It is hoped that the foregoing analysis of errors in percentage will suggest the possibility of truer understanding of errors and their significance, and will carry obvious implications of a more adequate psychology as a support for remedial techniques and for better initial teaching.

Our present methodology largely neglects instruction on how to handle error. In responding to situations in percentage, a pupil makes errors, or tends to do so. If we know what errors are probable, it would seem common sense, at least, to meet these potential errors somehow in advance.

6. The data used seem to possess internal consistency. Correlations of difficulty of items with different sets of pupils are high. Bright pupils do not make one list of errors and stupid pupils other lists. Rather, bright and dull make the same errors in about the same relative proportion, only the dull make them more frequently. However, real differences do exist in power over percentage, and such demonstrated differences should be met by genuinely validated differences in teaching techniques and material.

In general, the accomplishments in percentage are disappointing. That in this field there are genuine opportunities for research calculated to improve our control over learning is sun clear.

CHAPTER XIII

THE GRADE PLACEMENT OF ARITHMETIC TOPICS: A "COMMITTEE OF SEVEN" INVESTIGATION

CARLETON W. WASHBURNE
Superintendent of Schools, Winnetka, Illinois

I. THE PURPOSE

This report attempts to answer the following questions in terms of the results of an extensive, coöperative, statistical investigation:

1. At what stage of a child's mental growth, as measured by intelligence tests, can he most effectively learn the following phases of arithmetic: addition facts, subtraction facts, subtraction process, multiplication facts, simple long division, meaning of fractions, graphs, and Case-I percentage?

2. What degree of mastery of more elementary facts and skills is necessary for the effective learning of each of the above topics?

The capacity of a child to learn a given process in arithmetic is a function of his mental age and of the degree of his mastery of more elementary facts or processes. This fundamental fact has been but hazily recognized in organizing school curricula. As a consequence of ignoring it, schools cause very large numbers of children to fail.

How can we know when a child is mentally ready for any given group of arithmetic facts or any arithmetic process? How can we organize our curriculum in arithmetic so that the various phases of the subject succeed each other in the same order as do the mental levels at which they can be grasped by the child?

To the answering of these questions, and others related to them, over one hundred cities are applying controlled experimental research. The work is being directed by the Committee of Seven of the Northern Illinois Conference on Supervision.[1]

[1] Formerly the Superintendents' and Principals' Association of Northern Illinois.

II. The Plan of Research

The plan of research may be briefly stated as follows:

The customary grade level at which a given topic in arithmetic is taught is determined through questionnaires. A group of coöperating school systems is asked to teach this topic in the customary grade; another group is asked to teach it one grade below this customary level; and a third group to postpone it to one grade above the customary level.[2]

All three groups are then furnished with a battery of tests and detailed directions as to teaching procedure. The tests are of three kinds: (1) intelligence tests, to determine each child's mental level; (2) foundations tests, to determine each child's mastery of the more elementary facts or processes which will be used in the new topic to be taught; and (3) tests in the topic itself, consisting of from four to six equivalent forms, to determine each child's degree of mastery of the topic under consideration. The different forms of this third type of test are: (a) the pretest used before teaching, (b) the teaching tests used during teaching, (c) the final test used at the close of teaching, and (d) the retention test used six weeks later without any review during the interim.

The directions as to teaching procedure include the number of minutes per day, the number of days during which the topic is to be taught, the handling of cases of absence, the elimination of home work, and the exact order of presentation and method of teaching to be employed, including often the actual text material to be used by the children.

One word about the Committee's *modus operandi:* the arithmetic topics were divided among the committee members. Each member was responsible for the preparation of the tests and teaching procedure for his topics. He did all the corresponding with the coöperating systems, and in many cases, particularly at first, tabulated much of the incoming data. The final work has been increasingly centralized in the Research Department of the Winnetka Public Schools, under the supervision of Mabel Vogel Morphett, Vivian Weedon, and Zella Soltow, who, with clerical help, have prepared

[2] For some topics two levels suffice; for other topics four levels are needed to get the optimal placement. There are several minor exceptions to the generalizations in this and the immediately succeeding paragraphs.

all tables and graphs and, in the more recent experiments, have checked the correcting and done the tabulating of all the tests.

III. The Results

The consistency of the results indicates that they are not invalidated by any uncontrolled variables. The results show, in general, (1) that there is a definite mental level below which the attempt to teach any given process is usually futile; (2) that there is a degree of mastery of the prerequisite elements of a process without which that process cannot be efficiently learned; and (3) that if a child has reached a specified mental level and a specified skill in prerequisite topics, he is reasonably sure to learn the new topic adequately and to retain it well. These mental levels and prerequisite skill levels have now been determined for the topics herein reported and are being determined for more.

It is entirely possible that the degree of effective mastery could be raised and that the mental level at which it can be attained could be lowered if the teaching conditions as a whole were improved. Much interesting experimentation can and should be done along these lines, once the norms under the present procedure are established.

1. Addition Facts

Experiment Directed by O. E. Peterson
Northern Illinois State Teachers College
DeKalb, Illinois

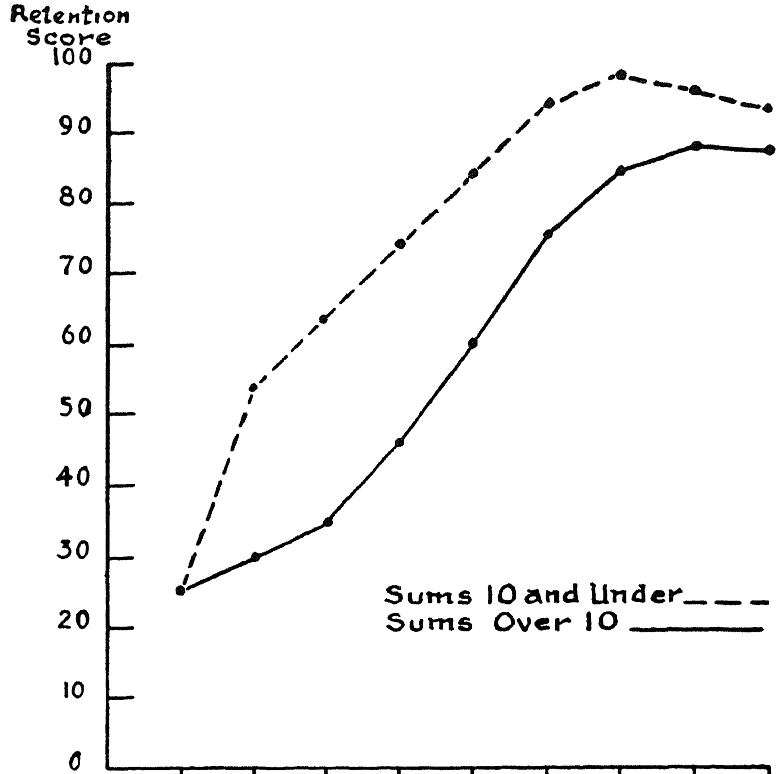
The data on addition facts have been subdivided into two groups: (a) those on the facts with sums 10 and under, and (b) those with sums over 10, although in the teaching procedure both groups of facts were included for all children.[3] An analysis of results showed that there was nearly a year of difference in the mental age of children who achieved a given degree of mastery of each group.

Graph I[4] shows the percentage of children who learn and

[3] The data were also treated on the basis of the easier and harder 50 facts (Winnetka Investigation, *Journal of Educational Research*, April, 1928). There was no significant difference between the scores treated on this basis and on the basis of sums 10 and under and sums over 10.

[4] When reading the graphs it must be remembered throughout that only smoothed curves are shown, whereas the figures given below the graphs are the data on which the true graphs were based. Since all intervals of both mental-age and foundations test scores, in all graphs, are in terms of half sigmas, and all graphs are on the same scale, graphs may be compared directly as to steepness of curves, etc.

GRAPH I.—ADDITION FACTS: PERCENT OF CHILDREN AT EACH MENTAL LEVEL MAKING 80 PERCENT OR BETTER ON RETENTION TEST



Mental* Level			5-11 to 6-4	6-5 to 6-8	6-9 to 7-3	7-4 to 7-10	7-11 to 8-4	8-5 to 9-0	9-1 to 9-7
No. Children	20	52	97	96	85	49	30	34	10
Percent making 81 percent retention on sums 10 and under	25	58	67	71	86	94	100	100	90
Percent making 78 percent retention on sums over 10	25	35	31	43	66	76	87	94	80

*Whenever mental ages are omitted, it is because age norms were not available.

retain 80 percent of the easy and hard addition facts respectively, at each mental age.[5]

[5] Eighty percent retention is used throughout the article and in the graph headings to represent percents varying from 78 to 83 percent. However, the precise percent is used to introduce the true figures (below the graphs) on which the graphs are based. When the precise percent was 75 percent, it is called 75 percent.

If one wants at least three-fourths of the children to make a retention test score of 80 percent—*i.e.*, if one wants children to have a three to one chance of learning the facts well enough to remember 80% of them—the easy addition facts should, according to this graph, be taught at a mental level not lower than 6 years, 5 months.[6]

There is no break in the curve at this point, however—children still continue to gain rapidly in the ability to learn and retain the easy addition facts until they reach a mental level of 7 years, 11 months. Then the curve flattens abruptly, practically all children retaining at least 80 percent of these facts—actually they retain much more than 80 percent, the mean retention at this level being 98 percent. There is no gain in further postponement.

Let us now turn to the harder addition facts on Graph I. If one wants three-fourths of the children to achieve 80 percent of the harder addition facts and retain them, it apparently is advisable to wait until the children have reached a mental age of 7 years, 4 months. The next interval below this, 6 years, 9 months, assures one of only three-fifths of the children attaining 80 percent retention. The peak is not reached till 8 years, 5 months.

Since at a mental level as high as 6 years, 9 months, the chances are only three to two that a child will retain 80 percent of the harder addition facts, and since the chances are three to one that he will retain them at a mental level of 7 years, 4 months, the latter would seem to be the earliest mental level at which the harder addition facts should be taught.

2. Subtraction Facts

EXPERIMENT DIRECTED BY J. R. HARPER
Superintendent of Schools, Wilmette, Illinois

The same discrepancy was found between the hard and the easy subtraction facts as between the hard and easy addition facts. The two groups of subtraction facts are therefore also handled separately. The 'hard' facts mean the harder half of the total, as determined by the Winnetka investigation.[7]

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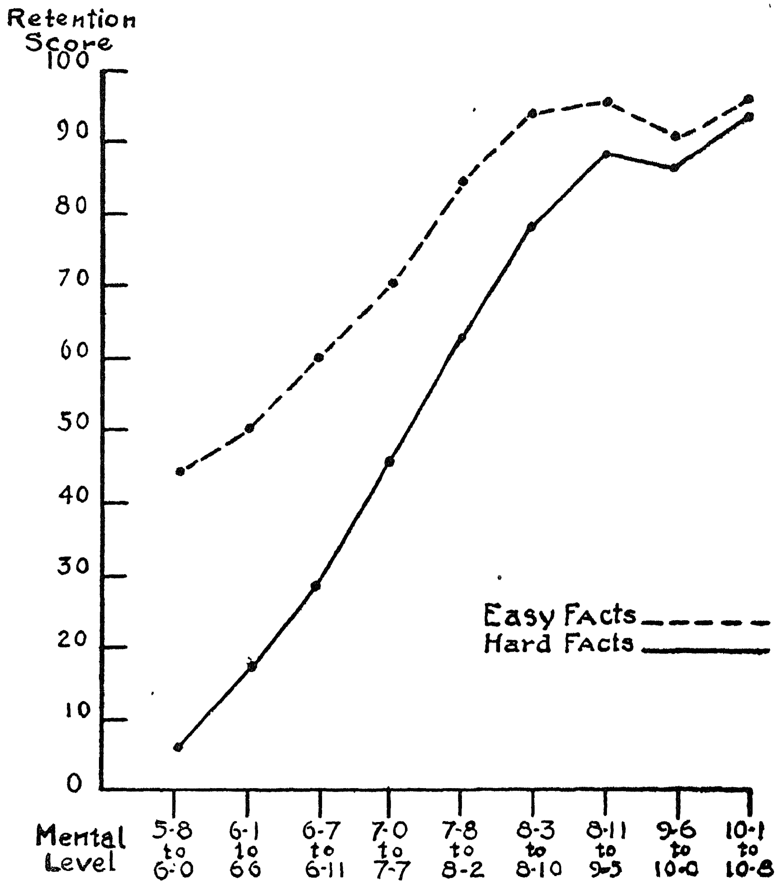
[6] For the sake of simplicity we have used the lower extreme of each interval to describe the entire interval throughout this article. Thus "a mental level of 6 years, 5 months" really means "a mental level of 6 years, 5 months to 6 years, 8 months." The complete interval is shown on all graphs.

[7] Washburne, Carleton, and Vogel, Mabel. "Are any number combinations inherently difficult?" *Journal of Educational Research*, April, 1928.

GRAPH II.—SUBTRACTION FACTS: PERCENT OF CHILDREN AT EACH MENTAL LEVEL MAKING 80 PERCENT OR BETTER ON RETENTION TEST



No. Children	16	49	60	83	75	73	34	18	12
Percent making 80 percent retention on easy facts	44	53	52	80	81	96	100	83	100
Percent making 80 percent retention on hard facts	6	22	15	53	60	78	100	78	92

Graph II shows the mental age that children have to reach in order that any given proportion of them may attain a retention test score of 80 percent in the easy and hard subtraction facts, respectively. If one is desirous of having three-fourths of the children make such a score on the easy subtraction facts, one must wait until they have

reached a mental level of 7 years (according to the unsmoothed curve—a little over 7 years on the smoothed curve). They continue to gain rapidly in ability to learn and retain these facts until they reach a mental level of 8 years, 3 months.

The earliest mental level at which three-fourths of the children attain 80 percent mastery of the hard subtraction facts is 8 years, 3 months. There are decided advantages in waiting until children reach a mental age of 8 years, 11 months.

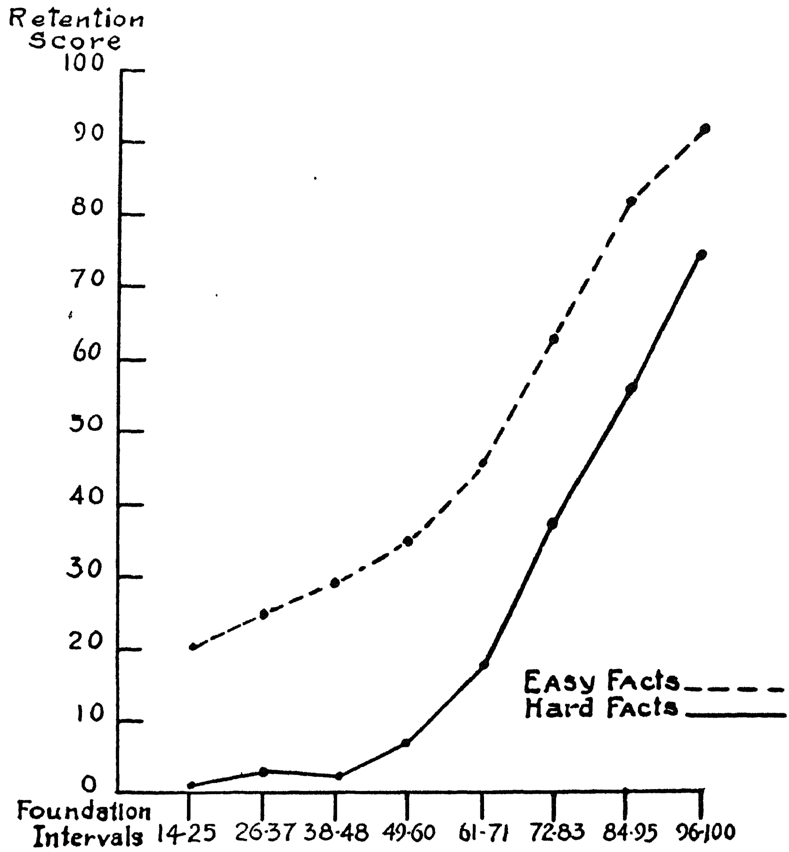
Graph III indicates that if one wishes children to have at least a three to one chance of reaching 80 percent in a subtraction facts retention test, they should have made a score in addition facts of from 84 to 95 percent for the easy subtraction facts and from 96 to 100 percent for the hard subtraction facts.

Graph IV is a curve showing the ratio of average final test scores to average retention test scores;[8] *i.e.*, showing what percent of the facts learned are retained at various mental-age levels. The preceding graphs have all shown raw retention, and therefore have been conditioned by the amount of time allotted to the experiments. Thus, if children of lower mental levels had been given a longer time in which to learn the facts, they would probably have increased their retention scores. Graph IV depicts the ratio of the facts learned to the facts retained. If, for example, a child of 7-year mentality learned only 50 percent of the facts in a specified time, but retained 40 percent; and a child at a higher mental level learned 75 percent of the facts and retained 60 percent; each child would be retaining the same proportion of what he had learned. Graph IV shows, however, that this is not likely to be the case. Not only do children at the lower levels learn both hard and easy subtraction facts less rapidly, but they remember a smaller proportion of the few that they have learned than do children of a higher mental level.

All things considered, it would appear that the easy subtraction facts should not be taught until children have reached a mental level of at least 7 years, and that there is marked gain if they are postponed to a mental level of 8 years, 3 months. Similarly, the lowest mental level at which one can reasonably afford to teach the hard subtraction facts is 8 years, 3 months, and there is a marked gain if they are

[8] In order to determine whether to use the average of the ratios or the ratio of the averages, a sampling was worked by both methods. The results varied so little that it was decided to use the ratio of the averages, as this involved less work.

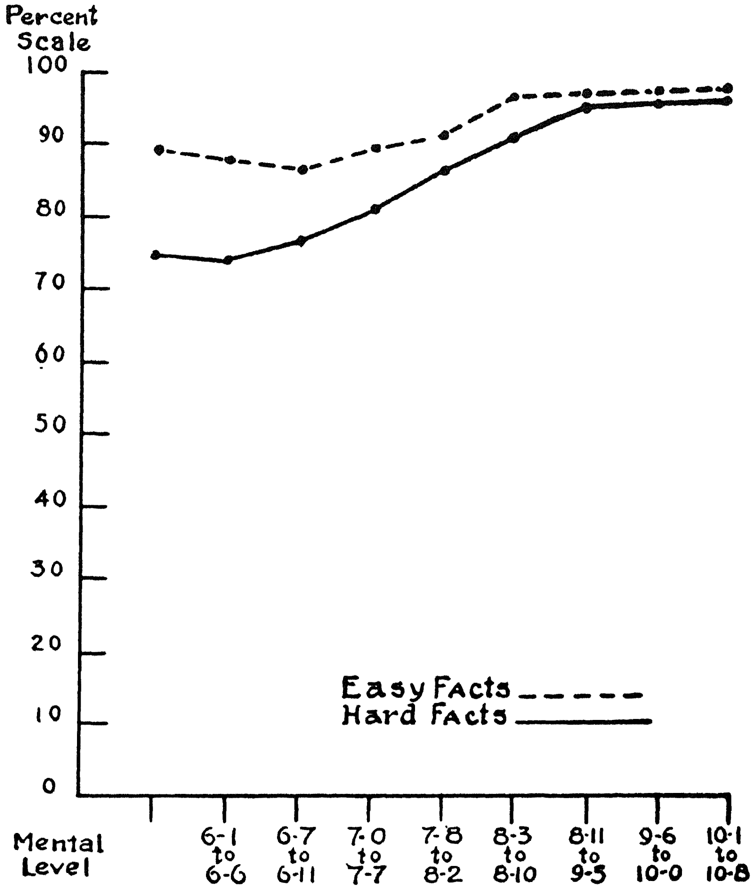
GRAPH III.—SUBTRACTION FACTS: PERCENT OF CHILDREN AT EACH INTERVAL OF ADDITION FACTS FOUNDATIONS MAKING 80 PERCENT OR BETTER ON RETENTION TEST



No. Children	14	15	18	29	24	28	65	233
Percent making 80 percent retention on easy facts	21	27	28	41	33	75	85	73
Percent making 80 percent retention on hard facts	0	7	0	7	13	39	57	76

postponed to 8 years, 11 months. Children should not begin a study of easy subtraction facts until they have an arithmetic ability indicated by a score of 84 to 95 percent in the addition facts, while a score of 96 to 100 percent in the addition facts is necessary for adequate retention of the hard subtraction facts.

GRAPH IV.—SUBTRACTION FACTS: RATIO OF AVERAGE RETENTION TO AVERAGE FINAL TEST SCORES



N › Children	16	49	60	83	75	73	34	18	12
Ratio of retention to final test for easy facts (Percent)	90	88	85	93	92	98	99	97	100
Ratio of retention to final test for hard facts (Percent)	76	76	75	84	85	93	98	96	99

If a child has reached a mental age of 7 years *and* has made a score on an addition facts test of from 84 to 95 percent, his chances of making at least an 80 percent retention test score on the easy subtraction facts are nearly six to one.

Similarly, if he has reached a mental age of 8 years, 3 months *and* has made a score of 96 to 100 percent on addition facts, his chances of

learning and retaining at least 80 percent of the hard subtraction facts are almost five to one.

3. Subtraction Process [9]

Experiment Directed by J. R. Harper
Superintendent of Schools, Wilmette, Illinois

What is meant by subtraction process can be best understood by reference to the test that was used, in various forms, as pretest, teaching test, final test, and retention test. One such form follows:

Subtraction Process Test

Form 1

271 —88	127 —89	540 —16	786 —125
601 —303	602 —297	49 —46	508 —199

Graph V shows a rapid increase in ability to solve correctly six out of eight of the examples six weeks after the close of teaching. Up to a mental level of 8 years, 9 months, the chance of making even a 75 percent score on a retention test is decidedly poor; at that level it is only a little better than two to one.

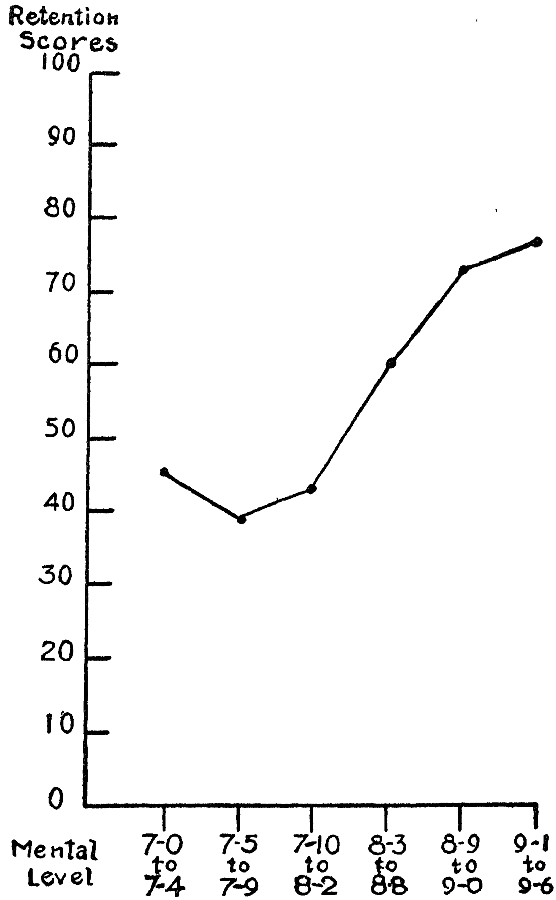
In terms of arithmetic ability as measured by a foundations test in subtraction facts, one must make a score of from 57 to 70 percent in subtraction facts in order to have a two to one chance of making a 75 percent retention score in subtraction process. This is shown in Graph VI. If, however, a child has reached a mental level of 8 years, 9 months, *and* has made a score in a foundations test of 57 to 70 percent, his chances of making a score of at least 75 percent in a retention test are six to one. The average score of such children on retention tests will be 87 percent.

Graph VII shows a steadily mounting ratio of retention test scores to final test scores as one goes higher and higher up the scale of mental level. At the mental level of 8 years, 9 months, the retention test scores are 78 percent of the final test scores.

In general, it may be said that the subtraction process should not be taught to children who have not at least attained a mental age of 8 years, 9 months, and a score on a subtraction facts test of from 57 to 70 percent.

[9] Subtraction Process is also called Subtraction with Carrying or Subtraction with Borrowing.

GRAPH V.—SUBTRACTION PROCESS: PERCENT OF CHILDREN AT EACH MENTAL LEVEL MAKING 75 PERCENT OR BETTER ON RETENTION TEST



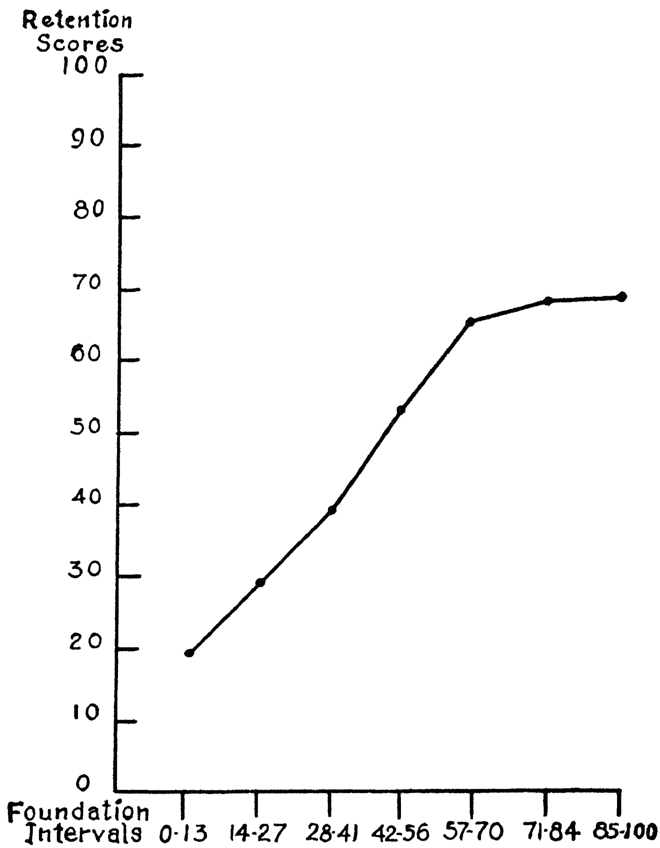
No. Children	15	22	31	36	21	11
Percent making 75 percent on retention test	47	32	42	58	71	82

4. Multiplication Facts

EXPERIMENT DIRECTED BY O. E. PETERSON
Northern Illinois State Teachers College
DeKalb, Illinois

Multiplication facts were not divided into hard and easy facts because most of the hard facts in multiplication are the zero facts.

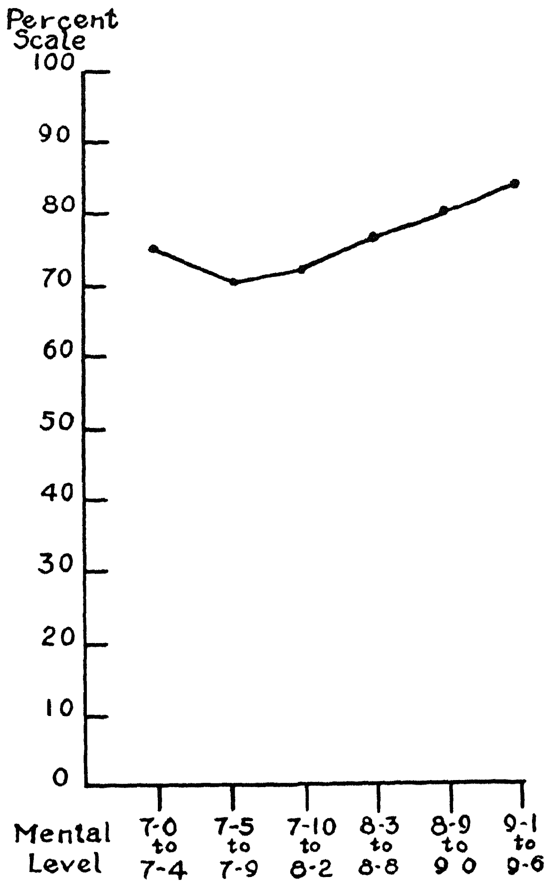
GRAPH VI.—SUBTRACTION PROCESS: PERCENT OF CHILDREN AT EACH INTERVAL OF SUBTRACTION FACTS FOUNDATIONS MAKING 75 PERCENT OR BETTER ON RETENTION TEST



No. Children	10	20	24	32	18	15	38
Percent making 75 percent on retention test	20	35	29	63	67	67	68

These, being practically meaningless when not a part of a simple multiplication example, like 3 × 402, were not included in the testing or teaching materials. They should probably not be taught as separate facts; rather the principle involved should be taught. We have no data, therefore, as to the mental age at which a child should learn the zero facts or whether he should learn them at all before he undertakes the process of simple multiplication involving a two- or three-digit multiplicand.

GRAPH VII.—SUBTRACTION PROCESS: RATIO OF AVERAGE RETENTION TO AVERAGE FINAL TEST SCORES



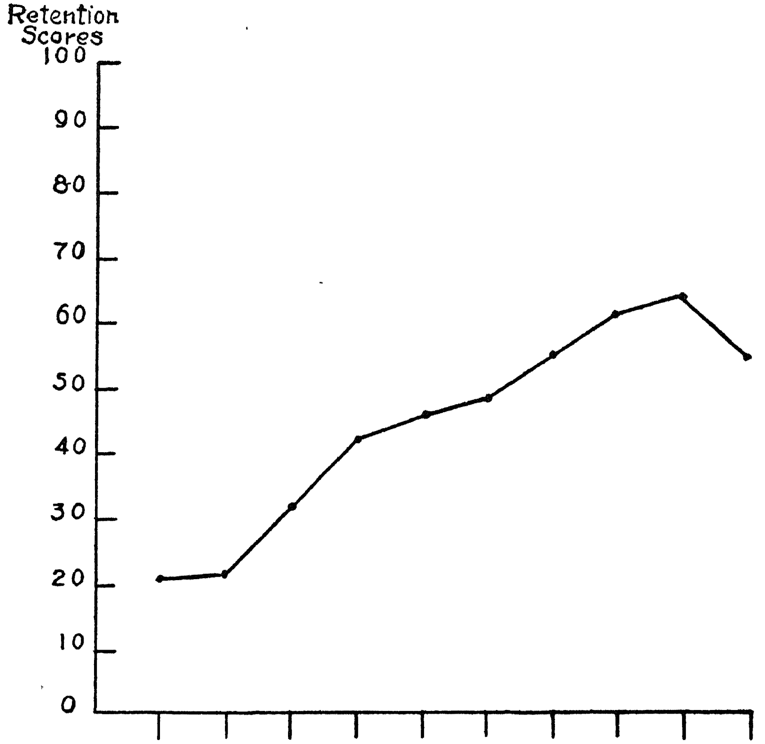
No. Children	15	22	31	36	21	11
Ratio of retention to final test (Percent)	74	69	72	78	78	84

The foundations test used in this experiment consisted of the 100 addition facts.[10]

Graph VIII shows the mental level at which various proportions of children can attain 80 percent mastery of the multiplication facts. The number of children that make this score is relatively low, owing

10 The graph showing the relation of foundations to retention is omitted to economize space.

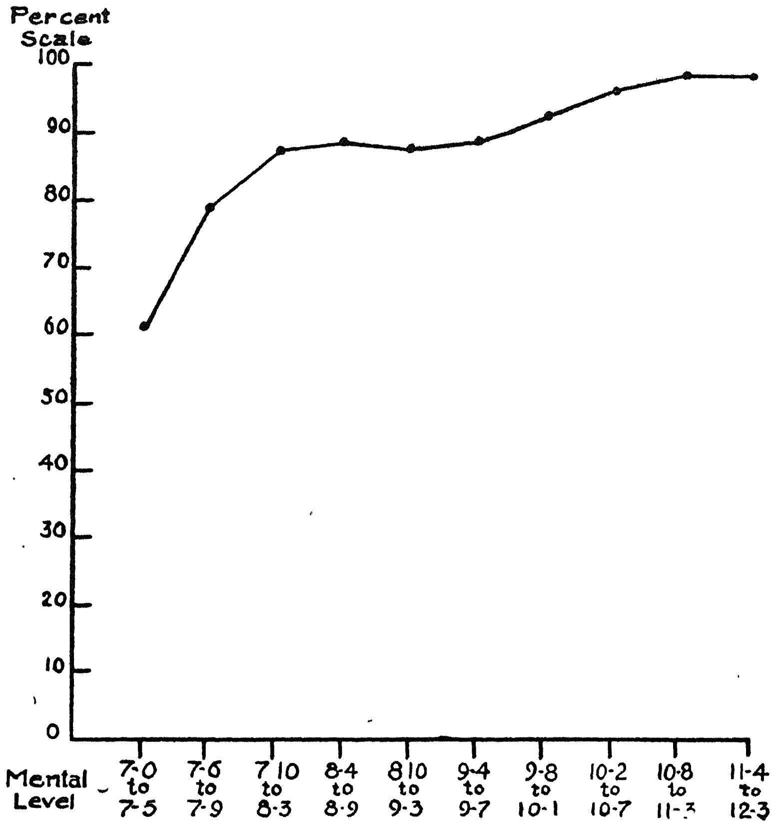
GRAPH VIII.—MULTIPLICATION FACTS: PERCENT OF CHILDREN AT EACH MENTAL LEVEL MAKING 80 PERCENT OR BETTER ON RETENTION TEST



Mental Level	7-0 to 7-5	7-6 to 7-9	7-10 to 8-3	8-4 to 8-9	8-10 to 9-3	9-4 to 9-7	9-8 to 10-1	10-2 to 10-7	10-8 to 11-3	11-4 to 12-3
No. Children	10	25	64	111	149	144	101	77	35	11
Percent making 80 percent on retention test	20	16	34	44	46	44	55	61	69	55

perhaps to the shortness of the duration of the experiment (6 weeks, 20 minutes a day, to learn 50 of the multiplication facts—less than 2 facts a day. One would think it was enough time!) The curve shows a definite tendency to flatten out at 8 years, 4 months. There is a second point of flattening at 10 years, 2 months. It is interesting to note on other curves, not reproduced here, that the final test scores show the same flattening point (8 years, 4 months) and that the curves representing 90 percent and 100 percent retention again parallel closely the 80 percent curve. Apparently, therefore, the

GRAPH IX.—MULTIPLICATION FACTS: RATIO OF AVERAGE RETENTION TO AVERAGE FINAL TEST SCORES



No. Children	10	25	64	111	149	144	101	77	35	11
Ratio of retention to final test (Per-cent)	61	83	87	90	86	90	92	97	99	99

mental age 8 years, 4 months, represents a definite level of mental growth before which there is rapidly increasing capacity to learn and after which the capacity to learn the multiplication facts increases less rapidly.

This outcome appears again in Graph IX, which shows the ratio of retention test scores to final test scores. There the smoothed curve breaks at the mental level 7 years, 10 months, but the unsmoothed curve breaks, as in Graph VIII, at 8 years, 4 months. The second flattening takes place, as before, at the mental level 10 years, 2 months.

Although nearly 90 percent of what is actually learned is retained by the time a child reaches a mental level of 8 years, 4 months, it is hardly safe to recommend that multiplication facts be taught as early as this level, even to children who can make a score of 92 to 100 percent on an addition facts test, because such children have only an even chance of reaching a retention test score of 80 percent, even in six weeks of teaching. They will learn on the average only 42 facts in this period, but they will retain 92 percent of the number learned.

If, on the other hand, one could manage to postpone teaching multiplication facts until children have reached a mental level of about 10 years, 2 months, and have (as such children are apt to) a score in an addition facts test of from 96 to 100 percent, then their chance of making 80 percent in a retention test score would be 70 percent, or a little better than two to one, while their mean retention score would be 84 percent.

Since, however, almost all subsequent arithmetic work requires a knowledge of multiplication facts, the Committee hesitates to recommend a postponement to 10 years, 2 months. Therefore, final recommendation is withheld until data for the subsequent and dependent processes are available.

This much, at least, can be said with certainty: Multiplication facts should not be taught below a mental age level of 8 years, 4 months, nor to children who have not attained virtual mastery of the addition facts.

5. Simple Long Division

EXPERIMENT DIRECTED BY RAYMOND OSBORNE
Francis W. Parker School
Chicago, Illinois

Simple long division is defined as long division involving a two-place divisor and a one- or two-place quotient, and involving all characteristic difficulties possible under these circumstances.

The foundations test used to discover the children's prerequisite knowledge consisted of a test in short division, simple multiplication, and subtraction, as follows:

FOUNDATIONS TEST

$$\begin{array}{r} 8769 \\ \times 8 \\ \hline \end{array} \qquad \begin{array}{r} 561 \\ -209 \\ \hline \end{array} \qquad \begin{array}{l} 3\overline{)393} \\ \\ 6\overline{)660} \end{array}$$

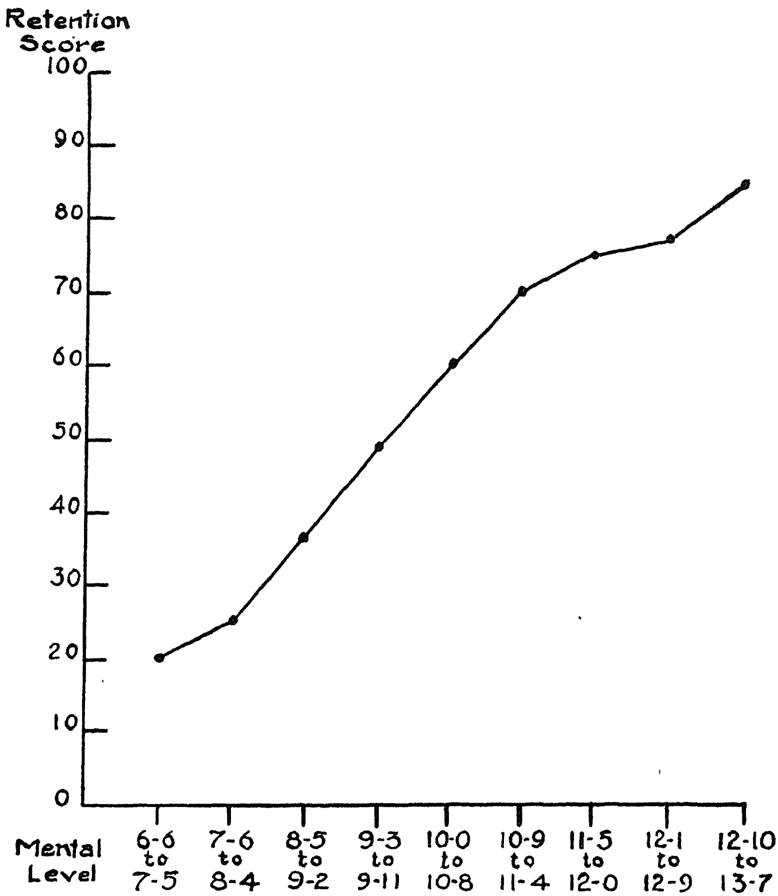
4579 ×7	403 —198	3)714
		4)408
7927 ×9	826 —77	3)7000
		6)949
6937 ×6	807 —309	7)147
		7)425

Graph X shows a steady and steep rise in ability to learn and retain simple long division up to a mental level of 10 years, 9 months. By the time this mental level is reached, children have a seven to three chance of making a 75 percent score on a retention test. The next higher mental level, 11 years, 5 months, gives the children a three to one chance of making such a score. The next lower mental level, 10 years, gives the children only a three to two chance, or very little better than an even chance, of making a satisfactory retention test score.

Graph XI shows a very steep rise in ability to do long division, as determined by the foundations test scores. A child should evidently be required to attain a score of at least 81 percent in a foundations test if he is to retain ability to do even simple problems in long division at all adequately.

Graph XII (the ratio of retention test scores to final test scores) shows a very rapid but diminishing rate of increase in ability to retain what is learned in simple long division, as the child increases in mental age. Thus, at the lowest mental level, over half that is learned is forgotten within six weeks, but by the time a mental level of 10 years is reached, children retain 85 percent of what they learn. Thereafter the curve rises much less steeply, indicating that, while learning may be so slow and laborious at a mental age of 10 years as to make for very low final and retention test scores at the end of the given teaching period, what little is learned is reasonably well retained. If one accepts 10 years, 9 months, as the lowest desirable mental level for teaching long division, one can count on children retaining, on the average, 84 percent of what they have learned.

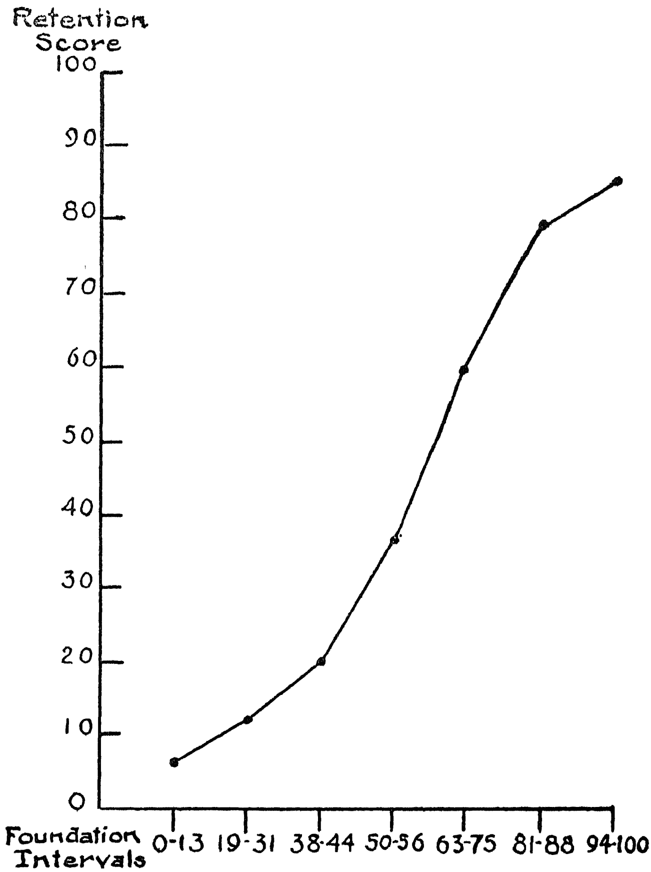
GRAPH X.—Simple Long Division: Percent of Children at Each Mental Level Making 75 Percent or Better on Retention Test



No. Children	20	45	114	120	95	62	37	24	19
Percent making 75 percent on retention test	20	20	38	50	58	74	73	79	79

It is, however, by seeing that a child who has reached an appropriate mental level has an equally appropriate proficiency in foundations that one can be assured of satisfactory work in long division. Children who have a mental age of 10 years, 9 months, and who at the same time can make a score on the foundations test of 81 percent, have a seven to one chance of making a 75 percent retention

GRAPH XI.—SIMPLE LONG DIVISION: PERCENT OF CHILDREN AT EACH INTERVAL OF FOUNDATIONS SCORES MAKING 75 PERCENT OR BETTER ON RETENTION TEST

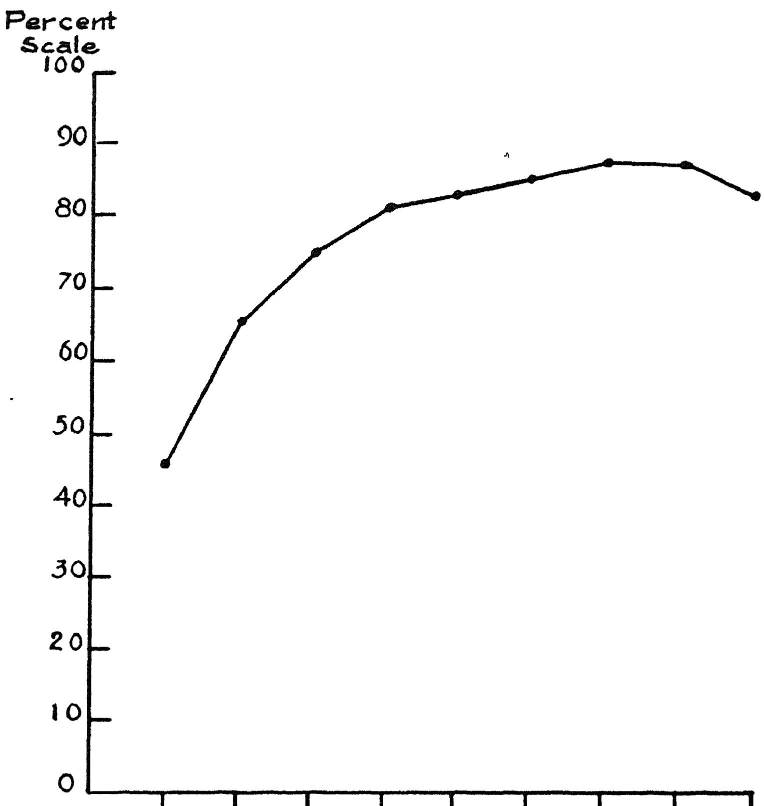


No. Children	45	76	58	61	84	100	124
Percent making 75 percent on retention test	7	11	16	36	60	86	85

test score or better. The average of the retention test scores of such children will be 86 percent.

Experiments with more advanced forms of long division, involving three- and sometimes four-place quotients, are now under way. It seems probable that when the long-division test includes the more difficult forms, the mental level necessary for a thorough understanding will be even higher.

GRAPH XII.—SIMPLE LONG DIVISION: RATIO OF AVERAGE RETENTION TO AVERAGE FINAL TEST SCORES



Mental Level			8-5 to 9-2	9-3 to 9-11	10-0 to 10-8	10-9 to 11-4	11-5 to 12-0	12-1 to 12-9	12-10 to 13-7
No. Children	20	45	114	120	95	62	37	24	19
Ratio of retention to final test (Per-cent)	53	60	80	80	85	84	89	88	86

6. Meaning of Fractions

EXPERIMENT DIRECTED BY HARRY O. GILLET, Principal,
and LOUIS E. RATHS, Research Assistant,
University of Chicago Elementary School,
and ORVILLE T. BRIGHT, JR., Superintendent of Schools,
Dolton, Illinois

The experiment in meaning of fractions was the outgrowth of the attempt to prepare a foundations test for addition and subtraction

of fractions. A tentative form of such a foundations test, involving fraction concepts, was given to children in Grades IV, V, and VI, and showed such a lack of basic understanding that it was decided to determine the grade placement of the meaning of fractions before attempting to determine that of the addition and subtraction of fractions.

In order to understand the experiment, one must be familiar with the meaning of fractions test. Ten of the sixteen items are accordingly reproduced herewith:

MEANING OF FRACTIONS TEST

Form 1

1. Mary has broken this stick of candy into two equal pieces. What part of the stick is in each piece? (Picture of a stick of candy broken in halves.) [11]

2. Color ½ of this glass to show that it is ½ full of milk. (Picture of an empty glass.)

3. Helen has divided her pennies into three equal piles. What part of her pennies is in each pile? (Picture of three piles of five pennies each.)

4. These are John's pennies. Draw a ring around ⅕ of his pennies. (Picture of five piles of two pennies each.)

5. It takes two pints of milk to make a quart of milk. A pint is what part of a quart? (Picture of a pint and a quart bottle of milk.)

6. Draw a line ¼ as long as this line. (Picture of a line divided into fourths.)

7. Here are three eggs. (Picture of three eggs.) Here are nine eggs. (Picture of three groups of three eggs each.) Are three eggs ½ or ⅓ or ¼ as many as nine eggs?

8. Here are eight oranges. (Picture of four groups of two oranges each.) Draw ¼ as many oranges.

9-10. Omitted.

11. George has divided his pennies into five equal parts. He has a ring around part of them. What part of his pennies has he drawn a ring around? (Picture of five piles of three pennies each with a ring around two piles.)

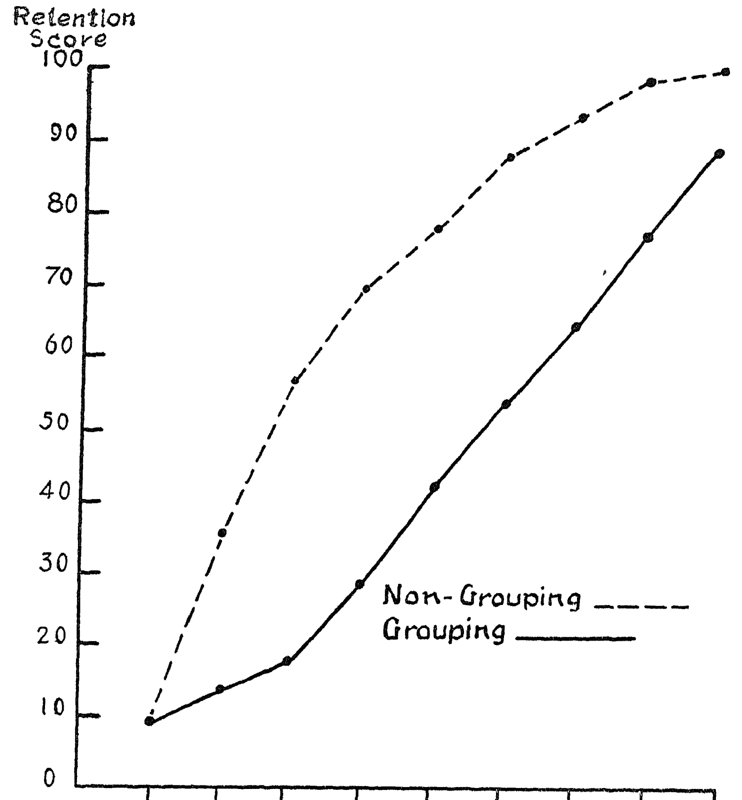
12. Here are some oranges on three plates. Draw a ring around ⅔ of all the oranges. (Picture of three plates, each holding three oranges.)

13-16. Omitted.

In this test, Examples 3, 4, 7, 8, 11, and 12 involve grouping—*i.e.*, recognizing a group of objects as a single unit. The retention test scores for grouping, particularly Examples 11 and 12, involving

[11] Because of limited space the pictures themselves cannot be given, and a description is given instead; also only ten of the sixteen exercises are reproduced.

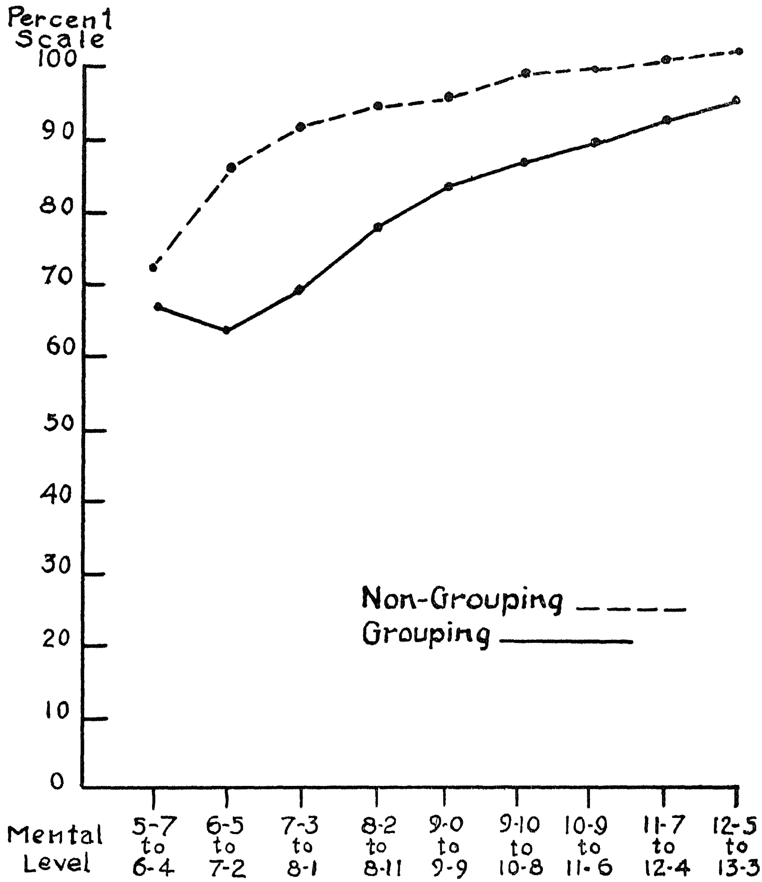
GRAPH XIII.—MEANING OF FRACTIONS: PERCENT OF CHILDREN AT EACH MENTAL LEVEL MAKING 80 PERCENT OR BETTER ON RETENTION TEST



Mental Level	5-7 to 6-4	6-5 to 7-2	7-3 to 8-1	8-2 to 8-11	9-0 to 9-9	9-10 to 10-8	10-9 to 11-6	11-7 to 12-4	12-5 to 13-3
No. Children	12	34	71	111	109	117	131	54	41
Percent making 80 percent retention on simple fractions	8	41	54	75	73	91	90	100	98
Percent making 83 percent retention on grouping in fractions	8	18	13	30	39	58	61	78	88

groupings where the numerator is over 1, were so much lower than those for the other parts of the test that for our present purposes the test scores have been divided into two parts. The first part involves simple meaning of fractions (all grouping problems excluded), and the second part includes only the grouping problems.

GRAPH XIV.—MEANING OF FRACTIONS: RATIO OF AVERAGE RETENTION TO AVERAGE FINAL TEST



No. Children	12	34	71	111	109	117	131	54	41
Ratio of retention to final test for simple fractions (Percent)	73	89	90	97	96	99	99	102	101
Ratio of retention to final test for grouping(Percent)	66	62	66	79	82	87	90	91	97

Graph XIII shows the percentage of children making a retention test score of 80 percent at each mental-age level for simple meaning of fractions (non-grouping) and for grouping.

The graph for simple meaning of fractions shows steadily decreasing steepness as it rises from one mental age level to the next.

By the time the mental level 9 years has been reached, a child's chances are three to one in favor of making 80 percent or better on his retention test. At the next mental age level, 9 years, 10 months, the chances are at least seven to one that a child will make such a score.

For grouping problems it is necessary to wait until a mental level of 11 years, 7 months, has been reached before children have a three to one chance of making a retention test score of 80 percent (5 problems out of 6).

Graph XIV shows the ratio of average retention test scores to average final test scores at each mental-age level for simple meaning of fractions and for meaning of fractions involving grouping. The ratio is consistently lower for meaning of fractions involving grouping, since at even the higher mental levels this is a more difficult process. However, by the time the mental age, 10 years, 9 months, is reached, 90 percent of what has been learned is retained, even in grouping problems, while for simple meaning of fractions 90 percent retention *of what has been learned* is reached as low as the mental level 7 years, 3 months. From this it appears that, although learning of this simpler aspect of meaning of fractions is somewhat slow and laborious at the lower mental-age levels, what there is of it is comparatively permanent.

Summing up, meaning of fractions not involving grouping can be safely taught to children with mental ages of 9 years or over with reasonable certainty of success, from the standpoint both of learning and of retention. Meaning of fractions involving grouping should, however, be postponed until a mental level of at least 10 years, 9 months, and preferably 11 years, 7 months, has been reached.

7. Simple Bar Graphs

EXPERIMENT DIRECTED BY HOWARD C. STORM
Superintendent of Schools, Batavia, Illinois

The experiment on the teaching of graphs was one of the first undertaken, and the data were handled in a manner different from that used later. The results, however, were definite and unequivocal. The test in graphs is reproduced herewith:

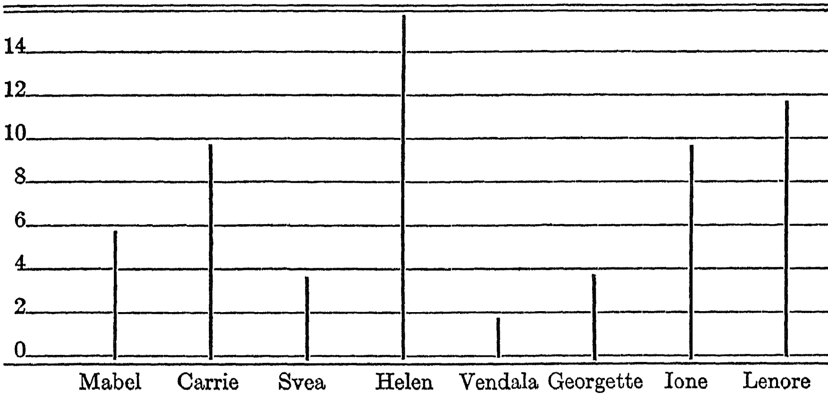
TEST ON GRAPHS

FORM II

GRAPH FROM BATAVIA TRACK MEETS
GIRLS CHINNING THEMSELVES

Seventh- and Eighth-Grade Girls

(One space represents two chinnings)



Which girl seems to be strongest in her arms?
Which is the next strongest?
Which is the weakest?
Name two that can chin themselves an equal number of times.
Name two more that are equal.

The foundations test covered the essential elements of multiplication facts, simple multiplication, multiplication of fractions, division facts, short division, long division, and ability to read numbers. The intelligence test was National, Scale A, Form 2.

The results of the experiment are exhibited in the accompanying table:

Median Percentile Rank on National Intelligence Test	Median Chronological Age	Median Score on Foundation Test	Median Score on Pretest	Median Number Days to Finish	Median Score on Retention Test	No. of Classes	No. of Children
Grade V 62......	10 yr.-5 mo.	100	25	10	95	9	220
Grade VI 63......	11 yr.-4 mo.	100	30	10	97	6	150
Grade VII 61......	12 yr.-4 mo.	100	30	12	94	8	180

The percentile ranks on the intelligence test indicate the percent of children of the same grade who would make a lower score than the children in question. The three groups are almost equally bright. The median for all groups in the final test was 100 percent and in the retention test had dropped only slightly.

It therefore appears to be quite certain that fifth-grade pupils can easily learn to read the type of bar graphs which we taught. We feel that graphs should be introduced in the fifth grade, because they are so convenient for use in various projects in the school and because they are being used so widely in geographies and histories. We find that graphs appeal to the child, and we think they will help him in his everyday life and his school work.

Practical considerations make it impossible, with the present organization of the arithmetic curriculum, to teach graphs earlier than Grade V, because some knowledge of fractions is necessary. This experiment clearly indicates that there is no gain, in terms of learning ability, through postponement of simple graphs beyond Grade V.

8. Case-I Percentage

Experiment Directed by Carleton Washburne, Supt. of Schools
and
Vivian Weedon, Research Assistant
Winnetka, Illinois

One form of the percentage test used is reproduced herewith:

Case-I Percentage Test
Form 1

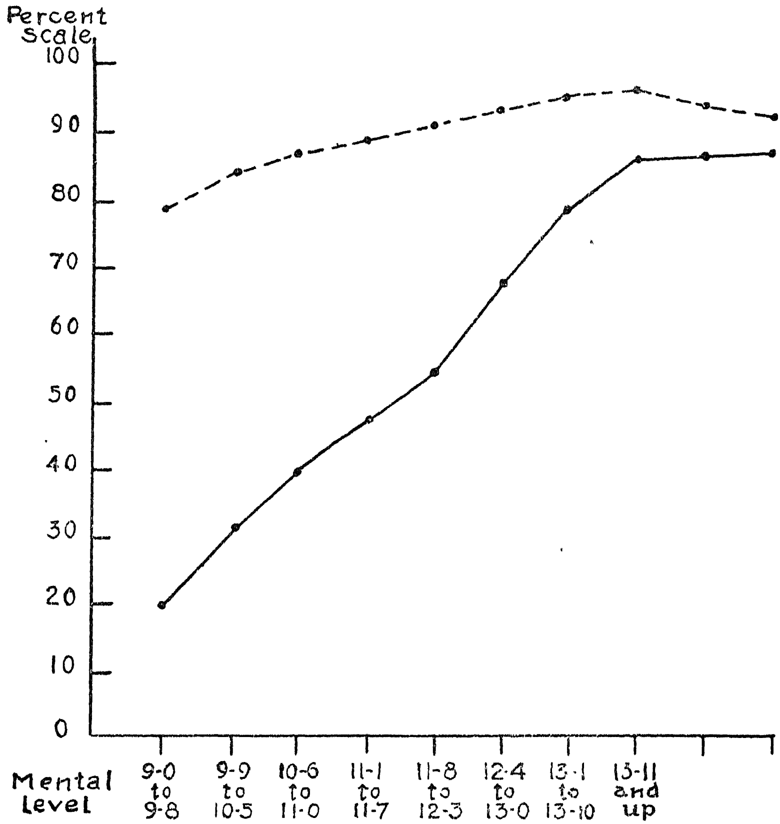
1. Write the fraction that is equal to the following percents:

50% =	66⅔% =	16⅔% =	30% =	12½% =
20% =	33⅓% =	87½% =	90% =	40% =
62½% =	60% =	75% =	37½% =	10% =
25% =	70% =	83⅓% =		80% =

2. Find 12½% of 24.
3. Find 80% of $15.
4. Find 83⅓% of 371.
5. Find 36% of 49.
6. Find 3% of 73.
7. Find 100% of $2.50
8. Find 175% of 4000.
9. Which is the largest, 8%, ⅛, or .8?

The lower curve in Graph XV shows a remarkably rapid increase in ability to learn and retain Case-I percentage right up to a mental level of 13 years, 11 months. By the time 13 years, 1 month, is reached,

GRAPH XV.—CASE-I PERCENTAGE: PERCENT OF CHILDREN AT EACH MENTAL LEVEL MAKING 80 PERCENT OR BETTER ON RETENTION TEST AND RATIO OF AVERAGE RETENTION TO AVERAGE FINAL TEST SCORES

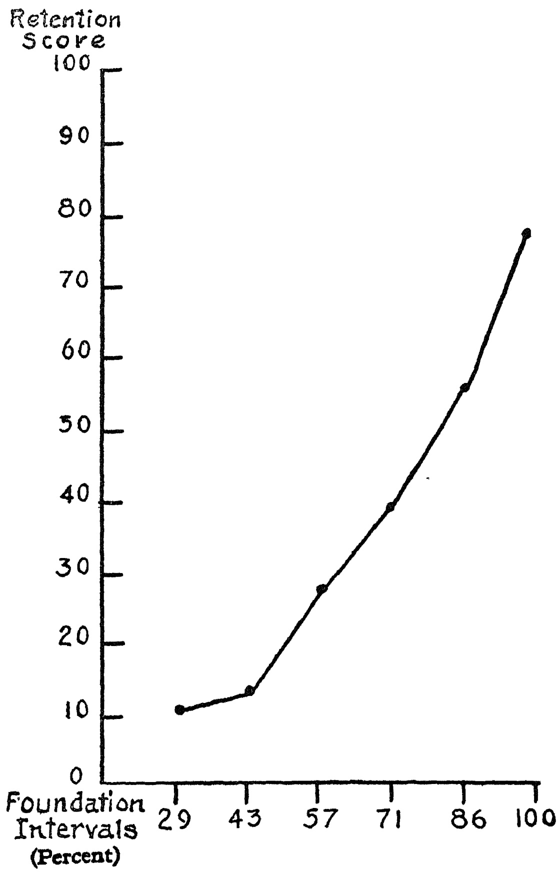


No. Children	11	39	81	147	179	140	148	75	51	13
Ratio of retention to final test, in percent (broken line)	78	86	86	91	91	92	96	96	96	92
Percent making 81 percent on retention test (solid line)	18	28	43	47	52	67	79	87	92	77

however, children have better than a three to one chance of making a retention test score of 80 percent. At 12 years, 4 months, they have about a two to one chance.

The upper curve in Graph XV shows a steady rise in the ratio of retention test scores to final test scores up to a mental level of at least 13 years, 11 months, or, according to the unsmoothed curve, to 13

GRAPH XVI.—CASE-I PERCENTAGE: PERCENT OF CHILDREN AT EACH INTERVAL OF FOUNDATIONS SCORES MAKING 80 PERCENT OR BETTER ON RETENTION TEST



No. Children	12	30	50	66	125	596
Percent making 81 percent on retention test	8	7	30	36	52	74

years, 1 month. There is a slight flattening of the curve at the level 9 years, 9 months, and again at the level 10 years, 6 months. Retention test scores do not reach 90 percent of final test scores until a mental level of 11 years, 1 month, is reached.

Graph XVI, showing the ratio of retention to foundations test scores, indicates that only children who make a score in a foundations test of 100 percent have a three to one chance of making an 80 percent

retention test score. The foundations test was solely multiplication of decimals.

If, on the other hand, children of a mental level of 12 years, 4 months, have also reached a foundations test score in multiplication of decimals of 100 percent, their chances of making a score of 80 percent or better on a percentage test six weeks after the termination of six weeks of teaching are better than three to one.

Case-I percentage is apparently a subject which continues to profit by postponement to 13 years, 11 months. Children whose foundations scores are high in a test on multiplication of decimals, and who have reached a mental level of 12 years, 4 months, however, can be effectively taught Case-I percentage.

IV. Summary

The experiments of the Committee of Seven indicate very clearly the need for taking into serious account the mental level and the arithmetical foundations of each child before attempting to teach him any given arithmetic topic. Not to do so is to doom many children to failure; to do so adequately is to insure reasonable success to the large majority of the children.

Recommendations are given in terms of mental age and arithmetic ability rather than in terms of school grade because there is such a wide range of mental age within each grade, and so much difference between grades having the same designation in different school systems, that grade levels have comparatively little meaning. The data on graphs, however, were the first obtained, and were treated by grades instead of by mental ages; results for this one topic, therefore, are given by school grades.

To use the Committee's results satisfactorily, superintendents and teachers will want to give group intelligence tests and arithmetic foundations tests to determine the approximate level of each child as to mental age and ability in arithmetic. The following summary table indicates the mental-age levels and arithmetic foundations scores children should first attain if at least three out of four of them are to make the very modest mastery represented by a retention test score of 80 percent in the various topics herein reported:

Topic	Minimal Mental-Age Level	Minimal Arith. Foundations Test Score (in Percent)
Addition Facts		
Sums 10 and under	6 yr. 5 mo.	—
Sums over 10	7 yr. 4 mo.	—
Subtraction Facts		
Easier 50	7 yr. 0 mo.	84
Harder 50	8 yr. 3 mo.	96
Subtraction with Carrying	8 yr. 9 mo.	57
Multiplication Facts	8 yr. 4 mo.	
	or 10 yr. 2 mo.	96
Simple Long Division		
1- and 2-place Quotient	10 yr. 9 mo.	81
Meaning of Fractions		
Non-Grouping	9 yr. 0 mo.	—
Grouping	11 yr. 7 mo.	—
Graphs, Simple Bar	10 yr. 5 mo.	—
Percentage, Case I	12 yr. 4 mos.	—
	or 13 yr. 4 mo.	100

CHAPTER XIV

A REVIEW OF EXPERIMENTS ON SUBTRACTION

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and
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Four principal methods of subtraction have been in use throughout the world. Each will be described briefly as a background for the reviews of the literature dealing with experimentation on subtraction.

I. Principal Subtraction Methods

Method 1. The Complementary Method. This method, already regarded as old by 1,150 among the Hindus, first appeared in printed arithmetics in 1478, was later widely used in England, and was the method of the famous Pike arithmetics of 1788 in America.[1]

The method follows:

13	becomes	13	
— 8		2	(10—8)
		5	

"Instead of taking 8 from 13, we may add 10—8 to 13 and then drop 10. This depends upon the relation $13 - 8 = 13 + (10-8) - 10$, and since 10—8 is called the complement of 8 (to the number 10), this is known as the complementary method."[2]

Method 2. The Borrow and Repay Method (known also by the names *Equal Additions, Carrying,* and *Second Italian* methods.) This method, known also to the Hindus, antedates printing, and was the only method in use in America in 1820.[3]

The procedure is:

82	5 from 12 equals 7
—45	4 becomes 5
37	5 from 8 is 3

The 1 which is borrowed in the minuend is added or repaid in the subtrahend, hence the name "borrow and repay."

[1] Smith, D. E. *History of Mathematics,* Vol. II, pp. 95-99.
[2] Smith, D. E. *The Teaching of Arithmetic,* p. 76.
[3] Smith, D. E. *The Teaching of Arithmetic,* p. 76.
Stone, J. C. *How We Subtract,* p. 98.

Method 3. Take-Away Method (known also as *Simple Borrowing, Decomposition* and *First Italian* methods). This plan came to Europe from India and is perhaps the most commonly used method in America.

An example is:

82	5 from 12 is 7	
—45	The 8 becomes 7	
37	4 from 7 is 3	

In this method, as in the preceding, the subtractions may be thought of in two ways: "5 from 12" or "12 less 5."

Method 4. Additive Method (known also as the *Austrian* and *Making Change* methods, although the latter is something of a misnomer). This method appeared in Italy in the sixteenth century, being suggested by Butteo (1559) and probably other early writers.[4] It came to the attention of German writers in 1874 from Austrian arithmetics of 1848, hence its somewhat inappropriate name. It claims the advantage of eliminating the need of learning the basic subtraction facts, employing instead the already mastered addition facts.

The method follows:

82
—45
37

Two solutions are used:

A. Borrow and Repay:
5 and <u>7</u> are 12
4 becomes 5
5 and <u>3</u> are 8

B. Simple Borrowing:
5 and <u>7</u> are 12
8 becomes 7
4 and <u>3</u> are 7

The remainder of this article summarizes studies based on the foregoing methods, which may be referred to in explanation of terms.

II. Experimental Studies

1. England

Study 1. Ballard, P. B. "Norms of performance in the fundamental processes of arithmetic." *Jour. of Exper. Ped.*, 2; 1914, 396-405 and 3; 1915, 9-20.

Thirty-five boys' schools (9176 boys) and thirty-six girls' schools (9502 girls) were given an arithmetic test (apparently an adaptation of

[4] Smith, D. E. *History of Mathematics*, Vol. II, p. 101.

the Courtis Series B) to establish norms. The test was timed and the scoring was entirely objective. Later testings confirmed the results.

Ballard states: "Suffice it to say that for every age the E.A. [equal additions] children of both sexes were found to work subtraction more expeditiously than the D. [decomposition] children. And on the whole the number of errors is less."

Study 2. McClelland, W. W. "An experimental study of the different methods of subtraction." *Jour. of Exper. Ped.,* 4; 1918, 293-299.

The pupils tested were 143 thirteen-year-olds, of whom 63 used the method of equal additions (presumably borrow and repay) and 80 used the decomposition method (presumably simple borrowing). The former group subtracted 161 columns in 10 minutes with 6.7 percent errors; the latter subtracted 134 columns with 9.7 percent of errors in the same time.

Both groups were then given additional practice over a considerable period of time.

McClelland concluded: "The method of equal additions appears superior in speed, accuracy, and adaptability to new conditions, while the method of decomposition is superior in speed after long practice."

Study 3. Winch, W. H. "Equal additions versus decomposition in teaching subtraction." *Jour. of Exper. Ped.,* 5; 1920, 207-220 and 261-270.

Thirty-eight girls, averaging about twelve years old, previously taught by the decomposition method, were divided into two equal-ability groups. For the following two months one group continued by the decomposition method; the other was introduced to, and practiced in, the equal-additions method for the same length of time. All teaching and testing was done by the same head mistress.

Winch concluded: (1) "The method of equal additions in subtraction taught to children late in school life, who have hitherto worked by decomposition, produces results in a few weeks equal on the whole, and superior in the weaker children, to those produced by the method of decomposition." (2) "The amount of the gain involved does not justify a change at this late period of a child's school career."

In a second study made during the same year, using 46 younger pupils (girls averaging 8½ years), more marked results were obtained. Winch states: "The method of equal additions shows to decided advantage with young children in accuracy and rapidity; and this is true both in the case of the superior children, who had already learned something of both methods, and also in the case of the inferior children, who, prior to the experiment, really knew nothing of either method."

2. United States

Study 4. Mead, C. D. and Sears, I. "Additive subtraction and multiplicative division tested." *Jour. of Educ. Psych.*, 7; 1916, 261-270.

Two second-grade classes of approximately equal ability were taught subtraction for a period of four months, one class by the additive and one by the take-away method. No borrowing was introduced at any time during the experiment; *i. e.*, the minuend was invariably larger than the subtrahend. An initial test was given on the addition combinations. Six tests were given during the experiment and conditions were kept closely similar for both groups. The median scores on the six tests are given below:

Method	*Test 1 (Addition)*	*Test 2*	*Test 3*	*Test 4*		*Test 5*	*Test 6*
Take-away	6.7	6.5	13.4	13.4	10 minutes daily addition practice	16.2	15.4
Additive	7.7	8.5	13.6	14.7	introduced	12.7	15.7
Difference	1.0	2.0	0.2	1.3	here	−3.5	0.3

Two things are to be noted in these results: (1) the additive group showed an initial superiority (on addition) of 1.0, and (2) the additive group lost on Test 5. This may be explained upon the basis that, during the month preceding this test, 10 minutes of daily practice on the *addition* facts was introduced (both additive and take-away groups). The differences are invariably small. The probable errors of the medians were not computed, but they can be estimated as falling within the limits 0.5 and 0.8, making the probable errors of differences about 1.0 or slightly less.

With the exception of Test 5 the results appear to favor the additive group, remembering however its slightly greater initial ability. Perhaps the most interesting result is the evidence of interference, or negative transfer, arising from the mixing of addition and subtraction practice in the case of the additive group and the absence of such an effect under the take-away method. If allowance is made for the slight initial superiority of the additive group (by subtracting 1.0 from each difference), the tests favor the take-away group three times out of five; if Test 5 is also ignored, the situation is about 50:50. Attention should be called to the fact that Test 6 introduced three-place numbers to be subtracted (no borrowing) while the earlier tests had employed only one-place numbers. The transfer effects from simple examples to more difficult ones were therefore about equal for the two methods.

Study 5. Taylor, J. S. "Subtraction by the addition process," *Elem. School Jour.*, 20; 1919, 203-207.

Taylor gave a simple test to 11,368 pupils in two districts of New York City in grades four to six. In these districts the Austrian (additive) method had been compulsory for six years. Analysis of the subtraction

methods followed by these pupils showed that 21.8 percent used the decomposition method, 40.6 percent used the method of equal additions, and that 37.6 percent used the Austrian (additive) method. Moreover, the percent of pupils actually using the Austrian method dropped from 52.7 in the fourth grade to 21.2 in the sixth grade. Taylor's conclusion was that the Austrian method was a failure in the schools of the two districts studied. He says: "In other words, by the time the children reach the sixth grade, 88 out of every 100 subtract by a method which is officially excluded from the schools."

Study 6. Beatty, W. W. "The additive versus the borrowing method of subtraction." *Elem. School Jour.*, 21; 1920, 198-200.

Using a method similar to that of Taylor, Beatty studied groups of pupils from two different schools where the Austrian method had been taught. Of 175 upper-grade pupils, 115 actually used the borrowing method. The Austrian group showed a median accuracy of 81.7 percent, in comparison with 79.3 percent for the borrowing group. The median rate, however, was 9.2 for the borrowing group and 8.2 for the Austrian group. The borrowing group was therefore about twelve percent more rapid and about three percent less accurate.

Study 7. Johnson, J. P. "The merits of different methods of subtraction." *Jour. of Educ. Research,* 10; 1924, 279-290.

Johnson gave Lesson Card No. 33 of the Courtis Standard Practice Tests to 277 normal-school students, timing the test, to ascertain how many methods of subtraction were used. The methods found and their frequency will be made clear by the following tabulation:[5]

Method	*Number Employing*	*Average Accuracy*
I. Decomposition (take-away—borrowing)	220	15.7[6] ± 0.9
II. Equal Additions (take-away—carrying)	23	16.2 ± 0.7
III. Addition-Borrowing; sometimes called Austrian, but not by Johnson	8	15.7 ± 0.8
IV. Addition-Carrying; called Austrian by Johnson	13	16.4 ± 0.6
V. Mixed and 'freak' methods	13	15.5 ± 1.1

Study 8. Buckingham, B. R. "The additive versus the take-away method of teaching the subtraction facts." *Educ. Research Bull., Ohio State Univ.*, 6; 1927, 265-269.

Pupils in seven centers were carefully paired by an intelligence test into parallel groups in each center. Instruction was kept constant for the

[5] P.E.'s computed by G. M. Ruch. See Ruch, G. M., Knight, F. B., and Lutes, O. E. "On the relative merits of subtraction methods; another view," *Jour. of Educ. Research,* 11; 1925, 154-155, for further criticism of Johnson's statistical treatment.

[6] Erroneously reported as 15.3 by Johnson.

parallel groups, but one group was taught by the take-away and the other by the additive method. The results follow:

Center	No. of Pairs of Pupils	Average in Final Test: Additive	Average in Final Test: Take-Away	Differences in Favor of: Additive	Differences in Favor of: Take-Away	P.E.diff
1	15	59.9	73.4	....	13.5	6.4
2	11	59.4	72.2	....	12.8	9.8
3	17	42.8	51.0	....	8.2	5.3
4	24	73.3	65.0	8.3	....	4.0
5	29	63.9	71.1	....	7.2	3.8
6	5	56.0	64.4	....	8.4	5.4
7	9	68.1	78.1	....	10.0	7.4

Six of the seven comparisons favor the take-away group, although the probable errors are individually large. Had it been possible to pool the results from the seven centers, it would have been readily apparent that the take-away group was superior, and the probable error would beyond doubt be small in comparison with the average difference in the composite.

Study 9. Osburn, W. J. "How shall we subtract?" *Jour. of Educ. Research,* 16; 1927, 237-246.

Reviews the earlier studies and adds new data based upon 1,414 pupils in the fourth to sixth grades of 23 Wisconsin school systems. A test was given which included all of the 45 subtraction facts in which borrowing or carrying is needed. Osburn's essential data follow:

Method	N	Mean Errors
I. Take-Away Decomposition	917	2.7 ± 0.13
II. Take-Away Equal Additions	238	1.8 ± 0.18
III. Additive	202	2.3 ± 0.25

It seems clear that the equal-additions method is slightly superior, with the additive second, and the decomposition method last. It is to be noted that extensive data have hitherto been lacking on the relative merits of the two take-away methods. The difference between Methods I and II is statistically significant; the other differences are not. However, the reader should bear in mind that, from a broad point of view, the difference between 1.8 percent and 2.7 percent of errors is after all not very great.

III. Critical Comment

The following statements give the present writers' reactions to the nine papers summarized above.

1. Results in four of the studies, McClelland, Winch, Mead and Sears, and Johnson, are rendered somewhat uncertain by the small numbers of cases treated. When differences range as small as those reported, hundreds or thousands of cases are needed for reliable results. Careful control of experimental conditions offsets this criticism, however, to some degree in certain of the studies (Studies 1, 2, and 3).

2. The three English investigators, Ballard, McClelland, and Winch, agree closely that the equal-additions method is superior to the decomposition method. This is also supported by the finding of Osburn (Study 9). These English writers have not raised the question of the relative merits of additive versus subtractive (take-away) methods.

3. Mead and Sears present the first experimental evidence on the relative merits of additive and subtractive methods. Taking the scope of their evidence into account, and bearing in mind the limited sampling, equality of the methods is the only possible conclusion.

4. The greatest interest in the Mead and Sears results rests on the fact that the introduction of addition practice during the teaching of additive subtraction set up interference, or negative transfer. "The papers of the additive class showed over and over again a confusion of the two processes (addition and subtraction), a slipping back into the habit of adding the lower figure to the top one."[7] This confusion was not found for the take-away group.

The major argument for the additive method has always rested upon the assumption that this method is built upon the addition skills already mastered and hence eliminates the necessity for learning a whole new set of subtraction combinations. The results of Mead and Sears suggest a fallacy in this position.

5. Taylor's results (Study 5) showed that when pupils were taught by the Austrian method in the lower grades, almost ninety percent had abandoned it by the sixth grade. It need not be assumed, however, that *all* the pupils had been taught additive subtraction, since certain teachers doubtless failed to follow the official prescription. Beatty (Study 6) tends to confirm this by his finding that two-thirds of pupils taught additive subtraction switched over to take-away methods.

6. The results obtained by Taylor and Beatty may tend to support the contention of Ruch, Knight, and Lutes[8] that the method used by Johnson (Study 7) and others may be vitiated by selective factors. If pupils 'switch' methods during the middle and upper grades, this phenomenon might occur most often among those who had a low mastery of subtraction and who suffered from a persistence of errors, tending in the long run to cause them to change methods.

[7] Mead and Sears, *loc. cit.*, p. 269.
[8] See footnote 5.

If this is true, in any adult group, those subtracting additively might be a superior-to-average sampling, and hence not a true criterion of the intrinsic worth of the method.

7. Buckingham's material (Study 8) appears to be worthy of considerable confidence. The results are internally consistent. Had his data been susceptible to pooling, the statistical significance would unquestionably have been increased. Of seven comparisons, six favor the take-away method over the additive.

8. Osburn's results (Study 9) carry us farther in the direction of analysis of probable truth, since he differentiates between the two types of take-away methods, *viz.*, decomposition and equal additions. The latter is superior, the former inferior, to the additive method, according to his results. The superiority of equal additions over decomposition agrees with the work of the three English investigators. Osburn suggests that the complementary method might well be investigated, although it is little used in this country.

9. None of these studies exhausts the argument. To settle the question of the relative merits of the four principal methods of subtraction, the ultimate investigations must follow pupils year-by-year through the elementary school and into the high school and higher branches of mathematical study. The initially-best method may ultimately be undesirable, owing to such factors as negative transfer, entrance of new difficulties in the subtraction of fractions, mixed numbers, and compound denominate numbers, low transfer value to problem-solving and life situations, circumscribed thinking in algebraic uses of the fundamental operations, etc.

10. The writers cannot refrain from suggesting that the differences among the rival methods of subtraction must be small; otherwise centuries of observation and a dozen empirical studies would long since have laid down the broad outlines of truth. If one method is one or two percent better than another, the present confusion is understandable; but if one method is really five, ten, or twenty percent better than another, a few crucial investigations on a large scale would soon give us the truth. Such investigations should include pupil performance in subtraction of fractions, mixed numbers, and compound denominate numbers, and all types of numbers in problem situations, as well as abstract whole numbers to which experimentation has so far been limited. It is total superiority, not superiority in but one phase, that needs to be known.

APPENDIX

A CRITIQUE OF THE YEARBOOK

NOTE TO THE APPENDIX

The members of the Society's Committee on Arithmetic have been desirous from the inception of their undertaking to secure extended and unbiased discussion of their work. They recognized that there were numerous other members of the Society who were interested in the various problems attaching to the teaching of arithmetic and who were entirely competent to contribute expert assistance in the production of the Yearbook. Experience has shown, however, the wisdom of operating as a small committee, at least under the conditions that have prevailed. Consequently the Committee conceived the plan, not hitherto tried by our yearbook committees, of submitting its product to a subcommittee that should serve as a Reviewing Committee charged with examining the text before publication and with preparing a critique of the Yearbook to be bound therewith, so that our readers might have the benefit of the reactions of nearly twenty experts as a stimulus to their own careful consideration of the work of the Yearbook Committee. The active members of the Society who coöperated in this way are listed in the front matter of this volume.

This plan has been carried out, despite certain unavoidable handicaps arising from lack of adequate time between the appearance of galley prooof and the call for the report of the reviewers. The Appendix which follows is a summary by Professor Brueckner of the several critiques forwarded to him by these reviewers.

The editor trusts that members of the Society will find this innovation sufficiently valuable to warrant its trial in other yearbooks to come.

G. M. W.

APPENDIX

A CRITIQUE OF THE YEARBOOK

Prepared by a Special Reviewing Committee
LEO J. BRUECKNER, *Chairman*

I. THE PLAN OF THE REVIEWERS

The plan for preparing the report of the Reviewing Committee was as follows: Each chapter in Part I was given to either three or four members of the committee for review. Some members were sent more than one chapter. These reviews were sent to the chairman of the committee to be used as might seem best in the preparation of this final report. Because of unavoidable delays in receipt of the galleys it was not possible to secure reviews on the complete book from individual members of the committee nor on the chapter on teacher training by other members of the committee than the chairman. The studies in Part II were reviewed by Dr. Osburn and the chairman. The chairman is the only reviewer who has read the Yearbook in its entirety.

In preparing this report every effort has been made by the chairman to present adequately the gist of the more important points in the several reviews. Some of the reviews submitted were quite lengthy; others were very brief. In some places in the following report reviewers are quoted directly. Because of lack of time it was not possible to submit this final report to the several reviewers; hence what follows should be regarded largely as the chairman's reactions to the reports on the various chapters, supplemented by his own reactions to the whole volume.

Because the time between receipt of proof of some portions of the Yearbook and its printing has been so brief this critique will undoubtedly be 'short' in many respects. It is also quite likely that the point of view of the Yearbook Committee has not been correctly interpreted in several places. Questions and comments by some of the reviewers, Dr. Ruch and Dr. Osburn, for example, have led to the reformulation by the original contributors of certain statements originally made in the galley proof.

One conference was held between the chairmen of the Yearbook and of the Reviewing Committee, Professors Knight and Brueckner. Otherwise the work of the Reviewing Committee has been done wholly by correspondence.

II. The Content of the Yearbook

The Twenty-Ninth Yearbook contains carefully considered discussions of important principles that should underlie instruction in arithmetic, with sufficient elaboration and illustrative content to suggest the methods of applying these principles in the evaluation of materials and techniques of instruction. The decision by the Yearbook Committee not to proceed beyond general principles to specific and detailed suggestions for classroom procedure was made because of the difficulty of reaching "sufficiently close agreement on details" and because the Committee did not wish to seem to "propagandize any particular course of study, textbook, or type of instruction." The inadequacy and lack of scientific information on many phases of the subject are clearly presented in a critical survey of previous research which makes evident the inadvisability of dogmatic statement of doctrine at this time. Important researches in arithmetic hitherto unpublished constitute Part II of the Yearbook.

1. Outstanding Contributions

Some of the outstanding contributions of the Yearbook are: (1) the challenging of the 'reductionist' point of view and of the social utility theory of curriculum-making; (2) the evidence as to the necessity of more adequate consideration of the psychological analysis of the learning process and of instructional material; (3) the discussion of principles underlying diagnostic and remedial work in arithmetic; (4) the critical survey of previous research; and, (5) the research studies of Part II, especially those dealing with various aspects of primary number work.

2. Possible Limitations

Possible limitations that are revealed by a reading of the galleys of material are: (1) the stress placed on the computational function of arithmetic to the neglect of other important functions; (2) the extensive use of historical material to show trends, selected in such a way that the information presented may be misleading since no

evidence is given to show that the selections are typical; (3) the lack of specific recommendations and suggestions after a faulty present status is revealed, as for example, desirable specifications for organizing drill materials in fractions; (4) the apparent stress on adult needs rather than on the child's needs of number; (5) the failure to broaden the concept of problem work to include other than textbook problems that may arise in local activities outside of school and in applications of number in other school subjects; (6) the highly technical nature of some of the material, which will be of little immediate, practical value to classroom teachers, though admittedly of great value to specialists and textbook writers.

In the following pages these contributions and possible limitations will be discussed, either directly or incidentally. The discussion will follow more or less closely the chapter organization, chiefly because of the way in which the reviews were prepared. The comments will be limited to what seem to be fundamentals rather than matters of detail. The reviewers entertain mental reservations regarding some of the points raised in several chapters, which will not be presented in this report, because they do thus concern matters of detail, which it would not be appropriate to argue in this chapter of appraisal.

III. The Educational Philosophy Underlying the Yearbook

Some one has classified individuals according to their views on current social and educational problems into four groups; (1) the reactionaries, who wish to prevent change and to return to conditions as they were at some former time; (2) the conservatives, who make progress slowly but under pressure; (3) the progressives, who believe that gradual change from the present status is desirable but should ideally be made in the light of scientific evidence; (4) the radicals, who believe that the present status should be completely and abruptly overthrown and a new one substituted on a purely theoretical basis. The Yearbook Committee clearly is progressive. It does not propose radical changes, but prefers "to proceed slowly and on sure ground, to be content with sane and moderate progress." The changes that are proposed are "changes which can be made and held, changes based on a sober psychology of learning and of human nature." The treatment of arithmetic that is suggested is "as progressive as modern thinking, courses of study, textbook construction, and scientific ex-

perimentation give it a right to be, and much more progressive than much of present classroom practice.''

The spirit in which the Yearbook was written ''assumes the desirability of a liberal school in contrast to the lock-step teacher-driven school of the '90's. It recognizes the child as the center of interest. The final criterion of all values is considered to be the effect any technique of teaching or any content of instruction has upon the child.'' ''It is not enough to cast education in terms of the child's present interest and desires alone.'' The committee believes that the education of the child should be conceived ''in terms of his becoming an adult as well as in terms of his present status.'' It is held to be evident that many of the demands of the future generation in the United States ''can be predicted with reasonable certainty.'' What the basis of such a prediction should be is not pointed out. ''How to teach is fundamentally more a matter of psychology based on research and investigation than a matter of philosophy.''

It should be pointed out that educational method may be determined in part by social theory. If one accepts the point of view that this is a period of rapid changes, much of it unpredictable, ''one of the primary objectives of education is to prepare for activities and duties that cannot be foreseen at the present time.'' As Bode [1] points out, ''the most important educational problem to-day is the problem of direction.'' Significant progress consists in the creation of new standards ''so that the schools may reinforce the desirable social changes that are already going on.'' ''The danger in the enthusiasm for the scientific determination of objectives in education lies in the fact that it obscures the need of breaking from old standards and old ideals,'' unless there is vision and a program looking into the future. One of the problems of the curriculum-maker is to consider the contribution arithmetic can make to this forward-looking program and the methods by which this contribution can be made most effectively. The content of the curriculum—that is, ''what to teach''—proposed in the Yearbook, consists of ''those skills, informations, judgments, attitudes, habits, ideals, and ambitions, which the child will find adequate and satisfying to the most important part of his whole self; namely, to his future adulthood as well as his present childhood.''

It is only a guess on the part of the reviewers that the point of view of the Yearbook with respect to the basis of selecting the content

[1] Bode, B. H. *Modern Educational Theories.* Macmillan, 1927.

of the curriculum is expressed in the sentence, "What to teach should be decided by as wise adults as are available for the task, who will base their decisions as far as possible upon the available body of objective scientific data." Elsewhere in the Yearbook the paucity of such data is pointed out. Because of this lack, the following point of view presented in the *Twenty-Sixth Yearbook* of this Society should be considered as an alternative:

"That part of the curriculum should be planned in advance which includes (1) a statement of objectives, (2) a sequence of experiences shown by analysis to be reasonably uniform in value in achieving the objectives, (3) subject matter found to be reasonably uniform as the best means of engaging in the experiences, and (4) statements of immediate outcomes of achievements to be derived from the experiences. That part of the curriculum from which selection of supplementary experiences and materials are to be used as conditions locally suggest, should be planned partly in advance and should be made partly as new materials become available. That part of the curriculum which represents the daily life-situations and interests from which the immediate specific needs of students arise should be—can only be—made from day to day."

IV. Reaction Against the 'Reductionist' Point of View

In this Yearbook a vigorous attempt has been made to counteract a strong tendency on the part of the curriculum-makers to give inadequate consideration to the educational contributions of the subject of arithmetic. Data are presented which show that the amount of time devoted to arithmetic has decreased steadily during the past few decades. This tendency is probably due to the reductionist point of view, which has stressed the idea that much of what has been taught as arithmetic has been useless, of little social value, and hence should be eliminated. As is stated in the Yearbook, the present curriculum in arithmetic is the result of a process of subtraction. It is pointed out that the process should now be reversed and that attention should be given not only to the question of what should be eliminated but also to the consideration of what should be added to the curriculum. The Reviewing Committee heartily indorses this point of view.

Curriculum investigations of arithmetic have stressed the principle of 'social utility' as a basis of selecting curricular content. In this Yearbook the point is clearly made that social utility, as usually considered, is an inadequate basis for determining curricular content. Utility has usually been determined on the basis of frequency of use of number, chiefly in computations, that occur in daily adult life.

Few investigations have been made of how number functions in child life. Frequency of use may not be a valid criterion on the basis of which to select curricular content. As is pointed out in the Yearbook, the curriculum-maker should not be guided in his selection of the content solely by the frequency with which a particular process or topic is found in the activities of daily life. Pickell writes: "The comments in the second chapter under the heading 'An Opposite Interpretation' are so strikingly challenging that everyone interested in curriculum-making should read them." The curriculum-maker should consider not only what number people *do* use but also what number people *ought* to use and what its functions are, other than mere computation.

The earlier curriculum studies stressed the computational function of number and paid little attention to the other important contributions that arithmetic can make when properly taught. More recently studies have been made of the arithmetic needed to read books, papers, and magazines intelligently, the arithmetic of the consumer, the arithmetic of production and distribution, and the arithmetic used in the daily activities of children. These studies show that increased thought is being given to the selection of a body of subject matter which will function directly in the lives of the pupils and contribute to the socialization of the individuals. The addition of such subject matter to the curriculum may be the direct result of an effort to check the present tendency to teach the abstract phases of algebra, geometry, and trigonometry in the upper grades and the consequent lack of consideration given to some of the important social aspects of arithmetic that cannot well be taught until the pupils have sufficient maturity to appreciate their importance. The socialization of the subject matter and the development of more efficient drill materials will do much to insure a more adequate consideration of the functions of arithmetic other than the computational.

V. The Special Functions of Arithmetic

The special functions of arithmetic that are discussed in the Yearbook may be considered under four heads: (1) the computational, (2) the informational, (3) the sociological, and (4) the psychological. It was the consensus of the reviewers that certain of these functions should be more adequately discussed than has been done. Therefore

the following paragraphs are presented to emphasize functions other than the computational, which are now largely neglected in teaching.

1. The Computational Function

Surveys of classroom teaching in elementary schools, such as those by Steel,[2] have revealed the fact that approximately 86 percent of the time devoted to arithmetic is used for the presentation of new processes of computation in arithmetic and practice in working abstract examples in new and previously taught processes. The remainder of the time, only 14 percent, is used for problem work, applications of arithmetic, projects, diagnostic work, and other important aspects of instruction. The apparently excessive amount of time devoted to the phases of arithmetic dealing with computation shows that under present conditions the computational function of number is considered to be its major objective. Judging from the space in the Yearbook given to aspects of instruction related to computation, it would seem that in the opinion of the Yearbook Committee this phase of the subject is the most significant and important. Relatively little attention is given to the other functions of arithmetic.

The recognition by the school that computation should probably be stressed less than has been the case in the past has resulted in the elimination of much of the difficult subject matter and process work formerly taught. More effectively organized instructional materials, constructed according to principles ably presented in the Yearbook, have made it possible to reduce the amount of time devoted to mere drill work. The study of pupil difficulties and the development of effective remedial exercises have reduced the extent of pupil mortality in arithmetic which has uniformly been found to be one of the chief causes of pupil failure in the elementary grades.

It has also been recognized that mere repetition of examples in a process is not an adequate basis of drill. The modern child-psychology movement, with its emphasis on the importance of meaningful, purposeful experiences on the part of the pupil, has resulted in the elimination of many of the faulty kinds of routinized practice on processes. Stress is also being placed on the concrete application of computation in the daily activities of the children and of adults both in and out of school. Hence the introduction of units of work

[2] "Scientific method in supervision." *Second Yearbook of the National Conference of Supervisors and Directors of Instruction,* 1929, Chapter IX.

in the arithmetic curriculum which stress other functions than computation.

While, under present conditions, apparently undue stress seems to be placed on the computational function of arithmetic, it seems reasonable that much of the instruction is ineffective because of faulty instructional materials. This Yearbook makes a large contribution in the detailed consideration of principles that should underlie the development of skill in computation and the construction of instructional materials.

2. The Informational Function

The informational function is concerned with the problem of teaching various aspects of arithmetic in such a way that the whole subject will have a richness of meaning and content that is not possible when the major stress is placed on mere process work in computation. For example, a pupil may be taught how to make change as a process in computation. The real social significance of the concept of money is not made clear if the teacher does not at the same time bring out the fact that a coin is the end-product of a large number of endeavors by the human race to develop a convenient, efficient means of expressing value. As Judd points out, the coin is the result of the evolution of a long series of social institutions. A child in the elementary grades can readily appreciate the difference between barter and our modern system of values. He can sense the utility and simplicity of the system of exchange which includes such elements as checks, bills, notes, trade acceptances, bonds, stocks, etc. In a word the mere teaching of how to make change which is concerned wholly with the computational application of number, falls far short of the educational possibilities.

In the same way richness of meaning may be brought to such concepts as the measurement of length, area, weight, time, volume, and other quantitative aspects of the environment. The work in arithmetic from this point of view will consider various subjects as large units for study, much as is now done in history and geography, in which the evolution of transportation, lighting, etc., are considered at length. There is no reason why a reasonable portion of the time in arithmetic should not be devoted to the consideration of topics of social significance because of the information that the child will acquire rather than because of the practice that may be given in computation. Con-

sideration of such a problem as "How does the grocer use arithmetic?" provides more possibilities of the consideration of desirable quantitative relationships than the solution of the problem, "How many boxes of crackers at ten cents each can I buy for a dollar?" which merely gives practice in computation but in no way enriches the pupil's concepts of the functions of arithmetic.

Many situations in geography, history, art, and other school subjects present opportunities for the consideration of important quantitative aspects of the environment. It is essential that the teacher recognize the application of number in all phases of the curriculum and use the opportunities thus presented to enrich and vitalize the meaning and utility of number. Much vital information may be imparted through the consideration of such topics.

Special attention should be given to the problem of helping the pupil to acquire meaningful and reasonably accurate concepts of units of measure and rules of procedure. A pupil may learn how to compute the number of units of length in the perimeter of a field with given dimensions or the number of units of area in a room with given dimensions, yet have no real concept of the meaning of the units themselves or of how to find the perimeter or area of a room of which he must first find the dimensions. One great deficiency of much of our so-called 'problem work' is that the pupil is usually given all of the facts needed to solve the problem—a wholly unlife-like situation. Undoubtedly the pupil will acquire a much richer variety of number concepts by the actual application of units of measure in real situations where he must manipulate the units as they are intended to be used, where he must eliminate non-essentials and select and organize the essentials in arriving at the answer to a problem or the solution of some difficulty.

The consideration of the significance and structure of our number system presents much rich material which is often entirely overlooked as a means of enriching and broadening the pupil's concepts of number and of developing his appreciation of the many efforts of the human race to develop the simple number system which has played such a vital part in the social, economic, and industrial progress of the past few centuries.

Probably many such topics as longitude and latitude, building and loan associations, and certain aspects of investments should be taught largely on an informational basis rather than because of the

computations involved. Increased stress on the informational function of arithmetic will result in more discussions of the meaning of number and the significance of the application of number to all sorts of situations. Topics will not be selected merely because they give practice in computation, but in part at least because of the contribution they may make to "richness and fullness of life."

It is the judgment of the Reviewing Committee that the informational function of arithmetic should have been given more consideration in the discussions of the social value of arithmetic.

3. The Sociological Function

Arithmetic should make a vital contribution to the intelligent consideration of various aspects of business, consumption, production, government, and social relationships which lend themselves to quantitative study and analysis. The real meaning and significance of profit and loss, the responsibilities and difficulties of the retailer and the wholesaler, and the relationships of the home and business establishments are part of the field of arithmetic. Harap [3] has demonstrated the necessity of giving definite attention to the presentation of the types of quantitative information that will make the individual more intelligent in the selection of his food, clothing, and shelter; that is, a more intelligent consumer. Very little consideration has been given to this aspect of arithmetic in this Yearbook. In the teaching of taxes increased stress should be placed on the meaning, necessity, and importance of taxes as a social institution, rather than on the methods by which taxes are computed. In no other subject is there the opportunity to teach the pupils the considerations that should underlie the investing and saving of money, the methods of investing money, the ways of sending money in safety from one place to another, convenient ways of carrying money, and many other problems dealing with the quantitative aspects of social relationships.

A consideration of certain topics in arithmetic may well be used as a means of showing how complex international problems have been settled by common agreement. Even if it may seem unnecessary to teach in detail the metric system, this topic represents an interesting illustration of an attempt to solve by the common acceptance of a simple decimal system the vexing problems due to a lack of uniformity in measurement units among the peoples of the earth. Although the

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[3] Harap. *Education of the Consumer.*

metric system has not yet been accepted as the standard in the United States and Great Britain, it seems certain that the question of its acceptance is a mere matter of time and an intelligent appreciation of the simplicity of the system. The universal acceptance of the calendar and of the method of reckoning time according to standard time belts is another illustration of the way necessity has forced the human race to agree on a common method of dealing with a quantitative aspect of the environment.

In the pages of the Yearbook there are to be found from time to time references to topics which suggest that its writers had in mind the sociological function of arithmetic. However, these items are buried in the stress that has been placed on the computational function.

4. The Psychological Function

Number may be thought of as a method of thinking. The scientist thinks with precision because he has devised ways of expressing his concepts quantitatively and compactly by means of number. Order, arrangement, and precision characterize his methods of thinking. The concepts of the savage are vague and indefinite since he has not developed a basis of expressing them quantitatively or in an orderly manner. He has no way of expressing relationships clearly because he lacks the techniques that have been invented for stating these relationships simply, such a ratio, averages, medians, modes, correlations, and similar statistical devices. Such terms abound in books, newspapers, and magazines. To read intelligently the reader must have a true appreciation of their functions.

The frequency table and the correlation table have provided methods by which a large number of facts may be organized and interpreted. Graphs and diagrams help to present the facts in convenient form. The pupil should learn to read and to construct the simple forms of graphs found in his reading and to sense their limitations. The modern basis of insurance and of many other forms of investment is the statistical organization and interpretation (often with the aid of the mathematics of probability) of a vast amount of quantitative data. Arithmetic can contribute much to the development of a realization of the utility and importance of commonly used statistical and graphic devices.

The concepts of ratio and proportion are methods of thinking which can only be developed by their repeated application in concrete

situations in which they function. They cannot be developed merely as abstract notions.

A recognition of the value of accurate quantitative information as a basis for forming judgments and making decisions should result from the repeated consideration and weighing of such facts in the solution of problems of concern to the individual or group. The ability to recognize the limitations of data and to evaluate them is an important aspect of quantitative thinking in reading which has often been overlooked as a desirable outcome of arithmetic instruction.

A careful reading of the manuscript of the Yearbook does not reveal sufficient evidence to warrant the assumption that the Yearbook Committee has given this important function of arithmetic adequate consideration. The importance of number as a tool needed in all lines of human endeavor is pointed out; however, its importance as a method of thinking is not stressed. In his review Horn especially stresses the necessity of considering the contributions arithmetic can make to the development of ability to "think in reading" when quantitative concepts and relationships are being considered.

VI. The Place of Problem Work in Arithmetic

When has a child learned a thing? As judged by life values, a thing is not adequately learned if it is merely given back on the demand of another; the learner must recognize the situation that calls for the response and then make the response in an effective manner. The danger in teaching processes in isolation, without regard to their application in normal activities, is that the learner may master the ability to compute answers to textbook problems but fail to recognize the situations in life in which the same processes are needed. Under such conditions the response may come from some artificial element in the situation, such as the oral directions of the teacher or the printed statements in the textbook, and not from the normal situation. The process lacks felt pertinence to life as the learner sees it, especially if what is to be learned is on the adult level and is unrelated to his own experiences.

In much of current practice the printed problem is used as a means of showing the child the use of numbers in life situations. In these problems all of the facts needed to solve the problem, and also at times unessential facts, are given. The Yearbook points out the need of a rich variety of problem work, but apparently fails to stress

the use of real problems in the sense of problems involving numbers that arise in normal activities of daily life. This interpretation of the concept 'problem' by the reviewers may be incorrect, yet it is clearly implied in such expressions as "In arithmetic problems do not 'arise.' They are stated by one person to another, the latter being remote from the scene of action."

The real task is not "to make the descriptive problems in arithmetic go as far as possible in affording life-like opportunities for thought;" the real task is to utilize the possibilities of normal activities to give the pupil practice in thinking quantitatively in a normal way and to apply number as it is applied in life. The nature of the work with verbal problems can be much improved by not limiting them to mere statement of facts and questions about the facts, but by basing them on tabular data, charts, diagrams, maps, and similar types of material presented in such a way that the pupil must select the essential facts needed to get answers to questions about the material presented. This type of activity is similar to that used by a person who comes into contact with quantitative data in his reading and in other situations. The more nearly the activities of the school can approximate such situations, the more likely will the study of numbers be to function in life.

Harap points out that "there is an abundance and variety of child and adult arithmetical experience which, if fully utilized, would have a decidedly enriching effect on the life of the child. A freer and liberalizing outlook comes only through many concrete and meaningful experiences. Useful skills are set off by other useful skills and not by abstractions." It seems to be suggested in the discussion that it is necessary to introduce certain processes and important forms of thought, such as ratio and proportion, even though they do not occur frequently in practice, because they give a freer, wider, more liberalizing outlook and because they set off the skills which do function in life. Harap believes that exception should be taken to this point of view. Facility in the use of numbers does not depend on the pupil's attainment of a high degree of abstraction of number concepts, but rather on the pupil's practice of arithmetical operations in connection with many particular situations. Difficulty with unfamiliar units is not due to the fact that the pupil has attained an insufficient degree of abstraction, but rather to the fact that he has not gained a sufficient degree of *particularization*. The advocate of reducing arith-

metic to its social uses has no prejudice against ratio, proportion, and percentage, as such. He demands a selection of those processes which are relatively high in *value*. To teach them purely as "forms of thought" is worthless. Harap says: "It would be of distinct value to weigh the advantages of systematic, economical learning of useful arithmetic processes against those of learning the same processes through the medium of an organized arrangement of life situations in which number would function directly and concretely."

"Differences of opinion still exist as to the extent to which immediate experiences of child life may determine the practices and procedures constituting this proper child training, but all agree that its ultimate test is the effectiveness with which subsequent situations are met by the individual so educated. In the selection and validation of curriculum materials expert analysis must be made both of the activities of adults and of the activities and interests of children. The ultimate test of the value of an organization of curriculum materials is the effectiveness of child learning. The curriculum can prepare for effective participation in social life by providing a present life of experiences which increasingly identifies the aims and activities derived from analysis of social life as a whole."

VII. The Time Allotment for Arithmetic

Several of the reviewers commented on the advisability of presenting a suggested schedule of the time that should be allotted to arithmetic. The schedule actually presented is based on extensive surveys of present practice and undoubtedly may be used as a conservative guide by those who do not wish to deviate widely from present practice. The fact that variation may be desirable is suggested by the inclusion in the table of upper and lower limits for each grade. Some consideration should be given to the ways in which the time allotted to arithmetic is divided among the various aspects of instruction, such as drill, problem work, discussions, and the like. Judd points out, however, that any attempt to suggest a schedule of the amount of time to be devoted to arithmetic is "likely to be a serious handicap to the complete reorganization of the arithmetic course in the schools." It is his belief that "there should be a very great elaboration of the instruction in arithmetic. If this elaboration is to be made, all of the conventional time arrangements will have to be modified." It should therefore "be perfectly clear that no sin has

been committed in differing very radically from the average times shown in the table." If the curriculum is organized, even in part, on an activity basis, a definite amount of time cannot be set aside for arithmetic, since number will be an inherent part of the total situation in which the need arises naturally in the course of the activity. If a time allotment is adopted, it is obvious that the consideration of arithmetic can not and should not be limited entirely to the arithmetic period, since applications of number arise in all subjects and these should be used as a means of enriching the number concepts of the pupils.

It is conceivable that a complete reorganization of arithmetic as now taught especially in the first four grades may be necessary. The tendency to reduce the amount of time devoted to formal process work in arithmetic and to substitute a different kind of instruction at these levels suggests the advisability of allowing even wider ranges in time allotments than are provided by the table. This possibility is also hinted at by the data as to the time for beginning formal arithmetic, which may be given quite the opposite interpretation to that given them by Buckingham. While no one would maintain that the formal teaching of number in Grades I to IV should be eliminated, it seems reasonable to assume that the character of the work in these grades should be materially different from the abstract drill now so characteristic. Some of the outstanding facts presented by this Yearbook pertain to the large amount and variety of number concepts that pupils acquire in their school and play activities before formal number work is undertaken and the amount of arithmetic they already know when formal instruction is begun. The reports of Collings[4] and Meriam,[5] taken together with the new facts presented in this Yearbook, suggest that a considerable change in the nature of arithmetic work in all grades would not result in any marked reduction in the efficiency of computation and at the same time would make it possible to enrich greatly the work in arithmetic by emphasizing the informational, sociological, and psychological functions of the subject.

VIII. The Psychology of Learning Assumed

The Yearbook emphasizes the fact that teaching based on felt needs of pupils and interest in the subject at hand is inadequate.

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[4] Collings, E. *An Experiment with a Project Curriculum.*
[5] Meriam, J. L. *The Child and the Curriculum.*

These elements are regarded as important in any learning situation; but, as is pointed out, consideration of them alone overlooks other important matters, such as the characteristics of effectively organized practice, the analysis of the learning steps involved in mastering a process, the construction of materials of instruction, and other similar "dynamics of learning." Clearly, a pedagogy based on "adequate consideration of all of the dynamics of learning in proper proportion is preferable to one that emphasizes at one time one dynamic" and at another time a different one. The Yearbook contains ample evidence that many of the important elements in instruction have not been given adequate consideration by the schools of to-day and that others have apparently been overlooked.

The basic psychology is avowedly behavioristic. It views skills, abilities, and habits as "fabrics of connections of unit skills and elements." The application of this point of view in the analysis of unit skills in instructional materials is well illustrated in Chapter V. Its limitations as a basis of analysis are discussed in Chapter VII of Part II. The analyses included in Chapter V are based on a determination of the frequency with which unit skills occur rather than on an analysis of the "fabrics of patterns" in which they occur. The expression "fabrics of patterns" smacks strongly of the *Gestalt* psychology, the application of which to arithmetic is not discussed in the Yearbook. It would be of great interest to consider the differences in the methods of analysis of instructional materials used by *Gestalt* psychologists and those discussed in this Yearbook.

One of the reviewers comments on the statement of psychology as follows: "The offhand dismissal of *Gestalt* psychology and the avowal of adherence to behaviorism and 'fabrics of connections' leaves one's mind in a state of bewilderment. 'Fabrics of connections' seem to be contrasted with structural psychology. How can this be when 'fabrics' and 'connections' are strictly structural terms?"

IX. New Data on Primary Number Work

Much new information of significance regarding arithmetic in the primary grades is given in Part II of the Yearbook. The data on types of quantitative knowledge possessed by young children on entering school raise questions as to what the nature of the arithmetic curriculum in the primary grades should be. Apparently children early develop number concepts which have not been com-

monly recognized as existing in young children. Many number facts are learned incidentally in activities and are known before formal instruction is begun. This information must be borne in mind in devising methods of instruction in these grades and should be considered together with the experimental evidence as to the grades in which formal arithmetic should be begun. Possibly the reason why pupils who have had no formal arithmetic as such in certain of the lower grades are equal or almost equal in ability to those who have had a continuous course in arithmetic in all grades may be found in the large amount of arithmetic they have learned incidentally in the ordinary activities of the school. As Courtis has shown, growth may be a function of maturity plus native endowment rather than of the methods of instruction used in the classroom.

The study of the difficulty of multiplication facts supplies vital information on this phase of instruction similar to the data in the study by Knight and Behrens on the learning difficulty of addition and subtraction facts. Such research is fundamental and highly suggestive. One significant factor in this study to which special attention should be directed is the evidence that the percentages of children who know the basic multiplication facts vary widely and that an astonishingly large number of these facts have already been learned in incidental ways. The low correlation between the rankings of the difficulty of the combinations by Clapp and those reported in this study is similar to those found for the other processes in the study by Knight and Behrens.

The results of the Beito-Brueckner study of transfer in learning addition combinations suggest the necessity of a whole series of careful studies of the question of transfer between unit skills in all processes. The amount of transfer may be much greater than has been suspected; for example, the amount of transfer of skill in the multiplication of fractions to skill in the division of fractions is undoubtedly great. Number is an abstraction and general; it may be thought of as an intellectual habit. The learning of number is therefore decidedly different from the learning of such motor habits as walking. More investigations similar to those of Judd and Brownell are needed to discover the ways in which children develop number ideas. It may be that new data regarding the intellectual capacity to generalize as to number and its uses will greatly simplify the psychological

factors at present thought to be basic in the analysis of learning and in the preparation of instructional material.

The data on the value of drill on mixed processes as against single isolated processes show the superiority of mixed as against isolated drill, as these two kinds of drill were considered in the investigation reported. It should be pointed out that this study is concerned with the problem of devising effective means of insuring retention of acquired skills, not with the type of drill most effective when new processes are first presented or when remedial work is to be done. Mixed drills serve admirably as diagnostic exercises, but they should undoubtedly be supplemented by remedial exercises directed upon isolated topics or skills in which deficiency has been revealed.

X. Diagnostic and Remedial Work in Arithmetic

One of the most significant and important chapters of the Yearbook is the discussion of diagnostic and remedial work in arithmetic (Chapter V). The discussion summarizes the best of current thought on that phase of instruction. The principles underlying diagnostic techniques with processes are considered. The importance of analyzing the responses and thought process of pupils having difficulty, in order to determine the cause of the weakness, is pointed out. This technique is a definite professional approach to educational problems similar to that of the clinical expert in medicine.

Ruch believes that the review of this part of the Yearbook should point out the possibilities of diagnosis of difficulties by group diagnostic test techniques. By means of such tests he feels that adequate determination of general causes of deficiency can be discovered, on the basis of which proper remedial work may be undertaken. "Such group diagnosis is not complete or final, but it frequently represents the practical limit of diagnostic testing under average school conditions." The detailed analytical study of the difficulties of individual pupils is, of course, the ideal method.

In connection with the section on "Norms," Ruch thinks that it may be of interest to present certain data gathered by Miss Eunice Adams under his direction. One hundred fifty-two slightly above-average pupils were given seven well-known arithmetic tests (survey type) suitable for use in Grades VII and VIII. The scores of this group of pupils when averaged and referred to the norms supplied by the makers of the seven tests gave these results:

Test*	Average Score of 152 Pupils	Average Score in Terms of Grade Equivalents (Norms)
A	224.3	8.5
B	70.8	H 6
C	32.0	H 7
D	26.8	H 5
E	30.1	9 (Estimated; above H 8)
F	38.0	H 6
G	56.1	11

When seven standard tests rate the same pupils all the way from upper fifth-grade ability to eleventh-grade ability, surely some of the norms are wrong—and who knows which of them? The moral seems to be: "Pay your money and pick the test that will give the showing you wish to make."

At the present time there are available many so-called 'diagnostic' tests, both group and individual. The theories underlying their construction and application vary widely. Standards are given in different ways; some are based on age norms, others on percentiles, others on the amount of time needed to complete a test or practice test with 100 percent accuracy. Words like *diagnosis* have a wide variety of meanings. The Yearbook raises the question of the need of standardizing the technical meanings of commonly used words, such as *test, norm,* and *standard,* and of methods of test contruction. While such an attempt may be desirable, any effort at this time to standardize might tend to check the present rapid developments in this field.

Ruch would have welcomed even more stress on the 'anticipatory' use of tests described in the section entitled "Tests Used After Instruction." A concrete illustration may be "cited for the cases of multiplication and division of fractions. Since the addition and subtraction operations with fractions are usually taught first, a certain denominator difficulty in multiplication (and division) of fractions may be anticipated, owing to the interference of preëxisting habits—*e.g.*, the following error: $\frac{1}{8} \times \frac{3}{8} = \frac{3}{8}$. This fairly common error arises beyond doubt from the interfering habit generalized in addition and subtraction of fractions, vis., that the 'denominator (if common) is written unchanged.' The symbol of operation in the case of $+$ and $-$ has previously appeared to affect only the handling of the numerators. Many errors of this type are open to logical analysis; many others have been found through error studies. Anticipation of such learning difficulties can minimize them greatly."

*The names of these tests will be supplied to responsible persons by personal letter upon request to G. M. Ruch.

XI. Crucial Problems of Methodology

Instruction in arithmetic may be considered from the point of view of the methodology that will insure adequate consideration of its four chief functions. The present practice in most schools is to organize the work about the presentations of processes and the establishment of skill in computation either in solving verbal problems or in working examples to the neglect of the other functions of arithmetic. Much of present research work has been directed to the analysis of skills involved in the several processes and to the development of more adequate materials of instruction in computation. Comparatively little research has been done to determine the effective methods of achieving the other functions of number.

The basic methodology for development of skills in computation must differ from that by which the other functions may be achieved. Educational science has demonstrated the necessity of considering individual differences in rates of learning skills. Arithmetic differs from such subjects as history because it consists in part of a hierarchy of skills which apparently must be developed to optimal levels in a fairly logical order, since one skill is often dependent on skills of a more rudimentary type. There is some evidence presented in the Yearbook that children apparently learn a great deal of arithmetic in informal activities in which the use of number arises incidentally rather than according to some logical system set up in advance. The usual procedure in teaching has been to limit the arithmetic of the class period to processes that have been taught or are about to be taught. This has resulted in a rigidity and narrowness of content which has greatly reduced the educational possibilities of the subject.

Much of the social value of arithmetic lies in the opportunity that it affords to deal with the quantitative aspects of life in which there is present no computation, as such. As is true for any subject matter, the pupils learn to grasp the significance and meaning of number by using it in their everyday activities. The units of work in which applications of number arise need not stress computations, nor need they be undertaken chiefly because of the opportunities afforded for computation. It is conceivable that much of the instruction in arithmetic may be organized about large units for study, much as is now done in the social studies, reading, and science. This seems especially desirable in the upper grades in the consideration of such topics as insurance, banking, taxes, and similar phases of the

subject, concerning which an individual should have accurate concepts and a true appreciation. Similarly, in the lower grades such topics as the methods used now and in the past to tell time and the uses of arithmetic in various life activities may be made the basis for worthwhile lessons in arithmetic dealing with quantitative aspects of the environment without stressing computations.

These larger units of study also afford opportunity for group and individual activities, for the exploration of matters of special interest to individuals, and for adaptation of the subject to local conditions. The methods that are used to teach geography, history, and similar subjects may be used in the presentation of the units of socialized subject matter in arithmetic. In these activities will arise the need for computations in arithmetic that will make number meaningful and vitalize the quantitative aspects of the subjects. From this point of view, therefore, the subject of arithmetic may be considered to be a social study of a value equal to geography and other social studies. Less stress is placed on the solving of problems based on processes being taught. The unit of subject matter and the normal activities of everyday life, rather than formal process work, determine the major work in the arithmetic period.

Bobbitt [6] makes this contribution to the problem of the curriculum:

> In school life mathematics should be primarily not a matter of solving difficult problems, but rather a matter of continuously viewing for many years the quantitative aspects of things, and of thinking in accurate terms . . . The way to learn to think quantitatively in fields where this is desirable is mainly to think quantitatively relative to matters in those fields. The ability to do quantitative thinking is to be developed in youth under conditions as nearly like those in which it is to function as is practicable. . . . An appreciation of the worth of mathematics in human life should be developed by using mathematics in one's affairs; by seeing its uses in human affairs in general.

In order to think accurately in quantitative terms the pupils must obviously use number concepts and a specialized vocabulary. It is one of the chief functions of the arithmetic classes to see to it that the pupil understands the number system and the socially significant number concepts and vocabulary. In addition to this, these classes must make certain that the pupil learns to perform the necessary operations in the various processes with reasonable speed and with a high degree of accuracy.

[6] Bobbitt, J. F. *How to Make a Curriculum*, pp. 150 and 158.

Among the most significant facts revealed by educational science are the differences in the rates and ways in which individuals acquire skills. Some pupils make rapid progress in all phases of arithmetic computations; others require much more time to achieve the same level of ability. This wide variation in progress among individuals has been clearly shown by such studies as those of Burk, Courtis, Washburne, Thorndike, and others.

Various attempts have been made to provide for these individual differences. Burk first developed a plan whereby the pupils progressed from skill to skill in arithmetic as rapidly as they achieved a desirable standard. Burk stressed the computational aspects of arithmetic almost to the exclusion of the important social values of the subject. At almost the same time Courtis was developing his individualized practice exercises, which made it possible to adapt the practice or previously learned processes in whole numbers to the varying needs of individuals in the class in an economical, efficient way. At Winnetka, Washburne has extended the work of Burk, and in addition has made an attempt to socialize some of the work in arithmetic. This plan of instruction has been criticized because there is no definite tying up of the drill and socializing work.

The development of the practice exercises by Burk and Washburne has been based on a careful analysis of the steps in the learning process and the presentation of the various units in self-teaching and self-checking instructional units. Some of the problems that arise in connection with the construction of such units are clearly presented in several of the discussions in the Yearbook. The careful analysis of the unit skills involved in each process merits the critical study of every teacher of arithmetic and of every person who is engaged in the preparation of instructional materials. Much of the failure of pupils to make satisfactory progress in arithmetic may be due to the use of instructional materials which have been constructed without a clear recognition of the various skills involved and with an inadequate conception of the difficulties pupils encounter at each new step of the process.

Equally as important as the recognition of the unit skills in a process is an analysis of the various types of situations, or combinations, in which the several skills may appear. An analysis of the frequency with which each of the unit skills in a process may occur gives an unreliable index of the adequacy of the instructional con-

tent. For example, in Part II of the Yearbook Brueckner and Kelly show that a set of sixteen types of examples in subtraction of fractions contains all of the unit skills listed in an analysis of unit skills in that process. It is shown that thirty-seven other possible important combinations of unit skills are not included in the set of thirteen examples which contain all the skills. As is pointed out in the discussion, in the construction of instructional materials both the unit skills in a process and the various combinations in which these unit skills appear in significant types of examples must be considered. Overman comments on the possible inadequacy of present lists of skills. Careful study is needed to determine the completeness both of the lists of unit skills and of example types that are available. It should be pointed out that it is a simple matter to determine the relative difficulty of types of examples in a process, while it is probably quite impossible to isolate the difficulty of a particular skill in a complex of skills.

The curriculum-maker is faced with the problem of integrating the curriculum in arithmetic from the point of view of its social values and contributions and from the angle of the computational function of the subject. In the order of time, activities in which number is used concretely should precede the abstract consideration of number and drill work on processes. When the use of number has been discovered in concrete settings, it may be abstracted from them and studied in isolation. As has been pointed out, the study of number and its uses in processes must be individualized because of the differences in the rates and ways in which the pupils learn to deal with abstract number processes and concepts.

The difficulties of administering the curriculum with these two points of view in mind is well expressed in the following statement by Courtis in the *Twenty-Fourth Yearbook* of this Society:

> The work already done has revealed certain difficulties and defects and suggested desirable modifications. For one thing, it is apparent that an ideal course of study would consist of two parts: (1) a series of social projects in which there would be need for the use of fundamental skills in meaningful situations, and (2) a series of self-instructive, self-appraising practice exercises, so closely correlated with the project work that children could avail themselves of drill exercises as they became conscious of the need. The danger of the completely organized drill system, however perfectly individualized, is that both teachers and children will come to consider skills as ends in themselves. Under such conditions, the transfer value of the skill developed is small. While the danger is negligible for competent

teachers who neither overemphasize the drill work, nor permit the use of drill exercises by children except in response to a felt need, it is a very real danger for teachers without vision. In many ways the problem is not "How to individualize school work?" but "How to secure teachers with the right points of view?" On the other hand, the unit-task idea, democratically administered by a competent teacher, solves very many of the distressing problems caused by individual differences and greatly increases the efficiency of teaching.

XII. The Training of Teachers of Arithmetic

Although no data are presented in the Yearbook to describe current classroom practices in the teaching of arithmetic, the assumption is apparently made in the chapter on teacher training (Chapter VI) that on the whole the teaching of arithmetic is at a low level. The inference from the discussion is that the inferior type of instruction is largely due to the limitations of present courses on the teaching of arithmetic.

Data are presented showing that in typical courses a large amount of time is spent in mere review of abstract arithmetic processes, on the assumption that prospective candidates for teaching certificates need this review so that they may be able to compute at least as efficiently as the pupils whom they are to instruct. The results of tests on processes and on problem-solving that have been given to students in training classes, as reported in this Yearbook, show that on the whole these students are far superior to the pupils whom they are to teach and will likely be even more so after two or three years of teaching. It therefore seems evident that much less time need be given to mere routine review of processes than has been the case in typical courses, such as those analyzed in the chapter in question. It is pointed out that a much better practice, such as is now followed in schools offering excellent courses in the teaching of arithmetic, would be to require review work on processes of only those students whose test scores do in fact reveal marked deficiency and need for review. It is also pointed out that this review work should be individualized.

The lack of professional study of technical aspects of instruction in arithmetic is clearly the result of the large amount of time devoted to the review of arithmetic processes, mostly on the elementary-school level. This situation is apparent from an examination of current courses, outlines, examinations, and catalogs. The points against present-day courses are summarized in the following statement from the chapter:

(1) The body of professionalized subject matter justifies and requires, in order that it may receive any adequate treatment, a differentiation of courses. Half of the courses (leaving courses for rural teachers out of account) examined have no differentiation and the extent of it in the others is often insufficient.

(2) A considerable proportion of the meager time available for the course is devoted to reviewing elementary-school arithmetic. Twice as many courses give time to review as get along without review; and among those courses which teach elementary-school arithmetic the commonest practice is to devote 50 percent of the time to it.

(3) The textbooks and collateral readings do not suggest—at least in many institutions—that dignified collegiate courses in arithmetic are being offered. For the most part the textbooks themselves are of an elementary character, having been written, of course, for these very courses. Moreover, the collateral readings fail to come to the rescue. They fail to include many of the important studies and sources of material. They tend to be merely more books like the textbook.

(4) Most of the courses fail to utilize as materials for study textbooks such as the schools are using. It has already been pointed out that, owing to the important part played in American education by the textbook, the omission of these materials from the professional courses of study is serious.

(5) A study of the final examinations and outlines used in these courses confirms the general impression as to the elementary character of the courses and the large amount of attention devoted to the arithmetic of the grades.

Curiously enough, no attempt is made to describe what features constitute excellent courses on the teaching of arithmetic that may be found at the present time in teacher-training institutions or by what standards such courses may be evaluated. Instead of describing any of them, an attempt is made to outline the contents of three "differentiated" courses on the teaching of arithmetic in the primary, the intermediate, and the upper grades. The basis of the proposed outline of courses is: "the outlines of courses submitted by teacher-training institutions" (the above quotation characterizes the inadequacy of such courses); "a series of problems about the teaching of arithmetic" submitted by Professor Buswell; case studies (242 of them) on student-teacher difficulties in the teaching of arithmetic; catalogues; typical courses of city systems; selected textbooks (the inadequacy of which is brought out in the Yearbook); and "books and articles on the teaching of arithmetic." The following comment on the method by which the courses outlined were developed by the writer of the chapter may be illuminating to the reader: "Data for

this section (primary arithmetic) and for the next two sections have been gathered in many ways. In fact, some of the sources are so remote that they defy identification.'' No data are presented to show that student teachers who have taken the courses that are outlined are better equipped to teach than those who have taken any of the better ones of the courses now available, since no comparisons are made between the proposed courses and those now found in the best training institutions. It would have been very enlightening to have included, as a basis of comparison, outlines of present courses that are judged by competent authority to be excellent—even more enlightening to have included outlines of those courses that have been found by careful investigation to produce strong teachers of arithmetic.

No one will take exception to the inclusion in training courses on arithmetic of a rich body of professionalized subject matter, similar to that now found in technical courses in medicine, dentistry, engineering, and other professional courses. The amount of such material available for educational courses is very great and is constantly increasing. Judging from the data presented, in many courses inadequate consideration is given to many of the important investigations that have been made on crucial points in the teaching of arithmetic. The outlines of the three proposed courses contain many suggestions as to the kinds of materials that might well be included in current courses to place them on a higher professional level and to insure insight on the part of teachers into significant instructional problems.

Experimental work is much needed to determine the best methods of training teachers of any subject. As it stands, there is no way of knowing that the proposed set-up of courses will produce better teachers of arithmetic than are produced by the best of the courses now in operation. Certainly the basis of the selection of the course content may be criticized on the grounds of its incompleteness and the unproved merit of its proposed content. One important basis of evaluating present courses that is not mentioned in Chapter VI is the evaluation of present courses by teachers in service who have taken them. Haggerty and Peik have demonstrated the feasibility of this procedure. The use of the job analysis of the teaching of arithmetic as a technique for determining the content of training courses is hinted at in the use that was made of courses of study of city schools. A much more careful study of the professional aspects

of the teaching of arithmetic is necessary than the one protrayed by this chapter.

Undoubtedly the chapter will have served its purpose if a survey of the data convinces those concerned that there should be included in training courses for teachers a much greater body of professionalized subject matter than is found at present in typical courses. It is obvious that 50 percent of the time need not be devoted to mere routine review of number processes—the median in present courses where time is devoted to review work. Many helpful suggestions as to what the basis of the selection of such professional material should be, and what its probable nature should be, are included in the discussion.

No attempt is made to indicate the nature of the training that may be required by teachers in service. Someone has said that the typical teacher training institution can provide only approximately five percent of the training needed by the teacher and that the other ninety-five percent are acquired 'on the job.' An investigation of the typical training activities that have been found fruitful with teachers in service would undoubtedly produce significant and valuable information.

XIII. Comments on Part II

From time to time throughout the preceding discussion reference has been made to pertinent studies appearing in Part II of the Yearbook. In view of the fact that each of the studies is specific in nature, detailed discussion of them is not necessary. They suggest a whole series of programs of research on crucial problems of methodology in arithmetic. Reference has already been made to the significance of the findings related to primary number work and their possible implications as to curriculum-making. Some of the findings are quite contrary to present opinion on several points. Almost all of the studies are concerned with investigations connected with some aspect of arithmetic computation; there is very little of significance regarding other important functions of arithmetic. There is no reason to believe that a final answer has been found for any question in arithmetic. However, a good beginning has been made. The studies that are reported indicate the painstaking research that is being undertaken throughout the country to attempt to arrive at solutions of instructional problems in arithmetic. The chapters presenting a

tabulation of the problems and techniques that have been employed in research in the field of arithmetic (Chapter II) and a critical review of this research (Chapter III) should be thoroughly digested by every student of the teaching of arithmetic.

XIV. Conclusion

The value of this Yearbook is undoubtedly very great. The rejection of the reductionist point of view in curriculum-making by its authors and their consequent demand that instruction in arithmetic be enriched will have a great influence on educational theory and practice. Their proposals for improving the instructional content of arithmetic as related to computation by a careful psychological analysis of the basic elements and skills in learning a process are of vital importance to classroom procedure. Careful consideration of the implications of the procedures of analysis that are described should result in the development of instructional materials more efficient than much that is now available. The stress placed on the value of diagnostic and remedial work should emphasize the professional aspects of the teaching of arithmetic. In brief, this Yearbook has contributed much to aid in the improvement of instruction of arithmetic as far as the computational function of the subject is concerned.

This Yearbook will be of great value to supervisors because of its richness in suggestions and because of the new researches that have been made available. Many applications to local situations will suggest themselves. In view of the lack of information on certain points, it might have been helpful if the Committee had formulated tentative proposals as guides for further study and research. It is quite likely that the general nature of much of the discussion will make it less immediately helpful than the Yearbook on Reading published by this Society in 1925. However, in the long run the influence of the present Yearbook on educational practices may be greater.

If it is recalled that the Committee did not think it wise to proceed beyond general principles to specifics because of the lack of scientific information on details, it will be realized how great is the amount of research necessary to arrive at specifics for questions that are raised but not answered by this Yearbook. Research on arithmetic should be stimulated and directed by the situation thus revealed.

The belief of the Yearbook Committee that issues must be evaluated from the adult point of view is sound. However, when that has been accomplished, the details by which the decisions are to be solved in the classroom are still to be determined. When one notes the stress that has been placed on only one of the chief functions of arithmetic in this Yearbook, the question must be raised as to who will make the evaluation. It would seem that among others the philosopher, the sociologist, and the psychologist must all have their voice in the matter.

All "dynamics of the learning situation" must be considered in the development of adequate methods of instruction, and undue stress should not be placed now on one dynamic and now on another. A methodology based solely on children's immediate interests and purposes does not take into consideration other important dynamics of learning. The discussion of this point in the Yearbook should serve as a prophylactic against loose thinking and sentimental interpretations of essentially sound educational theory.

CONSTITUTION OF THE NATIONAL SOCIETY FOR THE STUDY OF EDUCATION

(As Revised at the 1924 Meeting and Amended at the 1926, 1928, and 1929 Meetings of the Society)

Article I

Name.—The name of this Society shall be "The National Society for the Study of Education."

Article II

Object.—Its purposes are to carry on the investigation of educational problems, to publish the results, and to promote their discussion.

Article III

Membership.—Section 1. There shall be three classes of members —active, associate, and honorary.

Section 2. Any person who is desirous of promoting the purposes of this Society is eligible to membership and shall become such on payment of dues as prescribed.

Section 3. Active members shall be entitled to vote, to participate in discussion, and under certain conditions, to hold office.

Section 4. Associate members shall receive the publications of the Society, and may attend its meetings, but shall not be entitled to hold office, or to vote, or to take part in the discussion.

Section 5. Honorary members shall be entitled to all the privileges of active members, with the exception of voting and holding office, and shall be exempt from the payment of dues.

A person may elected to honorary membership by vote of the Society on nomination by the Board of Directors.

Section 6. The names of the active and honorary members shall be printed in the Yearbook.

Section 7. The annual dues for active members shall be \$2.50 and for associate members, \$2.00. The election fee for active and for associate members shall be \$1.00.

Article IV

Officers.—Section 1. The Officers of the Society shall be a Board of Directors, a Council, and a Secretary-Treasurer.

Section 2. The Board of Directors shall consist of six members of the Society and the Secretary-Treasurer. Only active members who have contributed to the Yearbooks shall be eligible to serve as directors.

Section 3. The Board of Directors shall be elected by the Society to serve for three years, beginning on March first after their election. Two members of the Board shall be elected annually (and such additional members as may be necessary to fill vacancies that may have arisen).

This election shall be conducted by an annual mail ballot of all active members of the Society. A primary ballot shall be secured in October, in which the active members shall nominate from a list of members eligible to said Board. The names of the six persons receiving the highest number of votes on this primary ballot shall be submitted in November for a second ballot for the election of the two members of the Board. The two persons (or more in the case of special vacancies) then receiving the highest number of votes shall be declared elected.

Section 4. The Board of Directors shall have general charge of the work of the Society, shall appoint its own Chairman, shall appoint the Secretary-Treasurer, and the members of the Council. It shall have power to fill vacancies within its membership, until a successor shall be elected as prescribed in Section 3.

Section 5. The Council shall consist of the Board of Directors, the chairmen of the Society's Yearbook and Research Committees, and such other active members of the Society as the Board of Directors may appoint from time to time.

Section 6. The function of the Council shall be to further the objects of the Society by assisting the Board of Directors in planning and carrying forward the educational undertakings of the Society.

Article V

Publications.—The Society shall publish *The Yearbook of the National Society for the Study of Education* and such supplements as the Board of Directors may provide for.

Article VI

Meetings.—The Society shall hold its annual meetings at the time and place of the Department of Superintendence of the National Education Association. Other meetings may be held when authorized by the Society or by the Board of Directors.

Article VII

Amendments.—This constitution may be amended at any annual meeting by a vote of two-thirds of voting members present.

MINUTES OF THE CLEVELAND MEETING OF THE SOCIETY FEBRUARY 23 AND 26, 1929

I

The first session of the Society, Saturday evening, February 23, was devoted to a discussion of the *Twenty-Eighth Yearbook* of the Society, entitled *Pre-school and Parental Education* with special reference to *Part I, Organization and Development.*

The Public Auditorium, South Wing of Music Hall, where the meeting was held, was one of the most satisfactory rooms that has fallen to the lot of the Society for many years, in respect to comfort, ease of speaking, and esthetic setting.

With Dr. Koos, Chairman of the Board, presiding, the following program was presented to an audience of some eight hundred persons:

I. "The Purpose and Plan of the Yearbook"
Lois Hayden Meek, Educational Secretary, American Association of University Women, Washington, D. C., and Chairman of the Yearbook Committee
(15 minutes)

II. "The Organization of Education for Preschool Children"
Helen T. Woolley, Director, Institute for Child Welfare Research, Teachers College, Columbia University, New York City
(15 minutes)

III. "Education for Parenthood"
Edna White, Director, Merrill-Palmer School, Detroit, Michigan
(15 minutes)

IV. "The Function of the Public School in Preschool Parental Education"
Carleton W. Washburne, Superintendent of Schools, Winnetka, Illinois
(15 minutes)

V. "Critical Evaluation of Part I of the Twenty-Eighth Yearbook"
Frank N. Freeman, Professor of Educational Psychology, University of Chicago, Chicago, Illinois
(10 minutes)

VI. Discussion
Open to all active members of the Society. Time-limit, each three minutes

Mr. E. C. Lindeman, who had been scheduled to divide the time with Dr. Freeman in a critical evaluation of Part I of the Yearbook, was not present.

II

The second session of the Society was held in the same auditorium Tuesday evening, February 26. Prior to our meeting, the curtain at the back of the stage, which divides the South Wing from the larger auditorium at the other side of the stage, was raised while orchestral selections were played by pupils from the Cleveland schools both to our audience and that assembled in the other auditorium for a session of the National Education Association.

Our second session was devoted to further discussion of the Yearbook on *Preschool and Parental Education,* with special reference to *Part II, Research and Methods.*

Dr. Koos called the session to order and asked the Secretary, Dr. Whipple, to preside.

Following informal introductory remarks by Dr. Lois Meek, chairman of the Yearbook Committee, the following program was formally presented:

I. "Research in Child Development"
Arnold Gesell, Professor of Child Hygiene and Director, Yales Psycho-Clinic, New Haven, Connecticut
(15 minutes)

II. "Methods of Educating Children of Preschool Age"
Patty Smith Hill, Professor of Education and Director, Department of Kindergarten First-Grade Education, Teachers College, Columbia University, New York City
(15 minutes)

III. "Mental Hygiene Aspects of Preschool and Parental Education"
Douglas A. Thom, Director, Division of Mental Hygiene, Massachusetts State Department of Mental Diseases, Boston, Massachusetts
(15 minutes)

IV. "Critical Evaluation of Part II of the Twenty-Eighth Yearbook"
a. Edward A. Bott, Professor of Psychology and Director, Psychological Laboratory, University of Toronto, Toronto, Canada
(15 minutes)
b. William H. Kilpatrick, Professor of Education, Teachers College, Columbia University, New York City
(15 minutes)

V. Discussion
Open to all active members of the Society. Time-limit, each three minutes

The discussion, owing to the delayed opening and the impending Business Meeting, was limited to replies by the speakers to two or three questions directed to them from the audience and to a brief defense by the Yearbook editor of a footnote (page 825 of the Yearbook) that had been questioned by Drs. Thom and Kilpatrick.

III

There were three matters of moment at the annual Business Meeting which followed.

1. The Secretary read the following resolution, prepared by Dr. Ernest Horn, endorsed by the Board of Directors, and recommended by them for adoption by the Society:

In the death of Dr. Bird T. Baldwin on May 12, 1928, the National Society for the Study of Education has suffered a great loss. He was elected to active membership at the Washington meeting, February, 1908, and was continuously and helpfully active in the work of the Society for the past twenty years. He was a welcome speaker in the meetings of the Society and an important contributor to its yearbooks. He made notable contributions to the yearbooks entitled "Standards and Tests for the Measurement of Efficiency of Schools and School Systems," "The Education of Gifted Children," and "Nature and Nurture." In the months immediately preceding his last illness, he gave his inspiration and help to the preparation of the current yearbook on "Preschool and Parental Education."

The members of this Society were constantly impressed by his warm interest in the work of the Society; his unselfish encouragement of the work of others; his public address; his charm of manner in friendly and informal discussion; his unusual thoughtfulness and consideration for those with whom he worked; his generous giving of himself to assist in worthy causes; his earnest belief in the ideals and methods of science in the study of education; and his devotion to the welfare of young children.

THEREFORE, *be it resolved:* That the National Society for the Study of Education direct its Secretary to transmit to Dr. Baldwin's nearest relatives, as a token of appreciation of his services to education and of the personal esteem in which he was held by his fellow members in this organization, this memorial statement; and *be it further resolved:* that a copy of the statement and resolutions be published as a part of the minutes of the annual meeting of this Society and in the columns of "School and Society."

By a rising vote the Society expressed unanimously its approval and appreciation of this resolution.

2. The Board of Directors recommended that Article III, Section 7, of the Constitution of the Society be amended by changing "$1.50" to "$2.00," to read: "The annual dues for active members shall be $2.50 and for associate members $2.00. The election fee for active and for associate members shall be $1.00."

This recommendation was adopted unanimously.

3. The Secretary read to the members of the Society the "Plans for the Subvention of Research" substantially as printed on page 844 of the Twenty-Eighth Yearbook, and urged members to assist the Board of Directors in making a success of this new line of professional activity on the part of the Society.

On motion, the Society was adjourned.

GUY M. WHIPPLE, *Secretary.*

SYNOPSIS OF THE PROCEEDINGS OF THE BOARD OF DIRECTORS OF THE SOCIETY DURING 1929

This synopsis, indicating matters of importance only that have been considered by the Board of Directors, is presented in order that the members of the Society may be informed concerning the acts and policies of those who are directing the work of the Society.

FIRST 1929 MEETING OF THE BOARD

(Hotel Hollenden, Cleveland, Ohio, February 23, 1929)

Present: Directors Bagley, Buckingham, Horn, Judd, Koos, Rugg, and Whipple.

1. Resolutions prepared by Dr. Horn with respect to the death of Dr. Bird T. Baldwin were read and endorsed for presentation at the Business Meeting.

2. The nominations to the Textbook Committee of Dr. W. C. Bagley and Professor Raleigh Schorling were approved.

3. The nomination to the Arithmetic Committee of Dr. W. A. Brownell was approved.

4. A number of special bills against the Society were considered and approved.

5. The chairman and the secretary of the Board were appointed representatives of the Society on the Council of the A.A.A.S. at the Des Moines meeting.

6. The proper allocation of costs for the extra space used by the Committee on Preschool and Parental Education in the production of its yearbook was discussed at length and referred to the chairman and the secretary, with power, but with the stipulation of certain general principles that should control the allocation.

7. There was a brief and informal discussion of the desirability of a yearbook on "The Fine Arts."

8. It was voted to hold a more extended meeting of the Board in May at Cleveland, Chicago, or Iowa City.

SECOND 1929 MEETING OF THE BOARD

(Hotel Stevens, Chicago, Illinois, May 11, 1929)

Present: Directors Bagley, Charters, Horn, Judd, Koos, Terman, and Whipple; also, by invitation, Committee-Chairmen Brim, Edmonson, and Knight.

1. Dr. Koos was re-elected chairman, to serve until March 1, 1930.

2. An invitation to send a delegate to the International Convention for the Exchange of Youth was declined with thanks.

3. The National Committee on Calendar Simplification was notified that the Board of Directors endorsed that Committee's proposals personally, but were not in a position to do so officially.

4. It was voted that the active members of the current yearbook committee and the speakers on the Society's program discussing that yearbook might be entertained at dinner at the Society's expense during the annual meetings, and that members of the Board of Directors should be invited to attend, but at their own expense.

5. It was voted to renew the special offer to active members whereby a paper-bound yearbook could be had substantially at cost, and to set the additional charge at $1.50.

6. It was voted to permit Winnetka Materials, Inc., to purchase for resale 500 to 1000 offprints of Supt. Washburne's chapter on the Winnetka curriculum in the Twenty-Sixth Yearbook, Part II, at a price which should afford the Society a profit comparable to its profits on commercial sales of yearbooks in general.

7. Reports were received, in person, from the chairmen of the Society's yearbook committees as follows:

a. Committee on Arithmetic. Chairman Knight reviewed the work of this committee, outlined the contents of the proposed yearbook, and applied for an additional appropriation of $1200. The Board discussed the report at length, voted the appropriation requested, and made suggestions concerning the Atlantic City program.

b. Committee on The Textbook. Chairman Edmonson distributed typed copies of the minutes of the last meeting of his committee and copies of a tentative table of contents for the proposed yearbook. The Board expressed general approval of the work of this committee after adding a few constructive suggestions, and urged completion of the work for publication in 1931. The Board voted to appropriate $350 additional for the use of this committee in 1929.

c. Committee on Rural Education. Chairman Brim explained the proposal of this committee to prepare a yearbook in which Part I would deal with "Present Status and Trends," and Part II with "Principles." He also explained how the work on this material was to be divided among members of his commitee and a group of associates. The Board voted an appropriation for 1929 of $300, plus unexpended balance, with informal approval of the plan to use a portion of this sum to finance a meeting of the group of associates (in view of the fact that clerical expenses are in general being borne by other funds than those of the Society).

8. Yearbooks newly proposed or already under consideration were discussed and handled as follows:

a. Efficiency, Training, and Placement of Teachers. Director Bagley reported the outcome of a conference between Earle Rugg, F. N. Freeman, and himself, indicating the possibility of a future yearbook to deal with the curriculum of teachers colleges, the success of teachers, prognostic tests

of teaching ability, and the placement of teachers. After discussion it was voted to lay this proposal on the table until it seemed more sure that the Society could proceed to advantage in this field already being studied actively by certain other agencies, notably the Federal Bureau of Education.

b. Secondary Education. Director Koos made a similar report with respect to the proposal to produce a yearbook on "Secondary Education," and a similar disposition was made of it.

c. The Fine Arts. Director Horn reported on the feasibility of producing a yearbook on this topic. The Board agreed with his contentions that this topic was important, but that it bristled with controversial questions, and that the selection of a competent committee would present numerous difficulties. It was voted to lay the proposal on the table.

d. Science. Director Whipple called attention to several requests from members looking toward a yearbook in the teaching of science, with special reference to science in the elementary-school curriculum. The Board requested him to communicate with some half-dozen persons interested in these problems and to report at the next meeting upon the feasibility of a yearbook on this topic.

e. Special Abilities and Disabilities. Director Horn was requested to investigate the possibilities of a yearbook dealing with those aspects of mental hygiene associated with special abilities and disabilities among pupils.

f. Geography. Director Horn also called attention to the important rôle played by geography in European schools and suggested it would make an excellent topic for a yearbook. This proposal was finally referred for report to Director Bagley, who was asked also to confer with geographers themselves on the possibilities of coöperation with this Society.

g. Practical Application of New Ideas. Director Judd, having called attention to the difficulties felt by persons interested in educational research and theory in getting new ideas translated into practice, was asked to report at the next meeting upon the possibilities of a yearbook on this topic.

h. The Liberal Arts College. Director Charters was asked to confer with various persons and report at our next meeting on the feasibility of this topic as material for a yearbook.

9. In response to the request of the Board for suggestions for the subvention of educational research by the Society, Professor G. M. Ruch, of the University of California, presented in four typed pages a plan for "The Evaluation of Standard Educational and Mental Tests," with the suggestion that the Society stand sponsor for this piece of research and contribute a portion of the needed funds. As the Board was not unanimous in its reaction to this proposal, it was finally voted, after extended discussion, to refer the matter to Directors Horn, Terman, and Whipple for further study and report.

AUDIT OF ACCOUNTS OF THE TREASURER OF THE SOCIETY FOR 1928

STATEMENT OF RECEIPTS AND EXPENDITURES FOR THE YEAR JANUARY 1, 1928, TO DECEMBER 31, 1928

Balance on Hand, January 1, 1928, per prior report..................$21,286.87

RECEIPTS

From Sale of Yearbooks by the Public School Publishing Company:			
Royalties, June to December, 1927	$6,795.52		
Royalties, January to June, 1928	7,738.48	$14,534.00	
Interest on Bonds, etc.:			
Interest on Registered Liberty Bond	$ 42.50		
Interest on Other Liberty Bonds	42.50		
Interest on U. S. Treasury Bond	42.50		
Interest on Dominion of Canada Bond	55.00		
Interest on Detroit-Edison Bond	50.00		
Interest on Interstate Power Bond	25.00		
Interest on Royalties	112.93		
Interest on Savings Accounts	377.35		
Interest on Checking Account	71.24	819.02	
Securities Received (cost value):			
$1,000 Utah Power and Light Bond	1,004.50		
1,000 Alabama Power Bond	1,027.50		
1,000 Chicago Junction R. R. Bond	1,022.00	3,054.00	
Dues from Active and Associate Members		4,447.52	
Total Receipts for the Year			22,854.54
Total Receipts, Including Initial Balance			$44,141.41

EXPENDITURES

Yearbooks

Manufacturing and Distribution		
Printing and binding 27, I and II	$7,301.73	
Mailing 27, I and II	861.80	
Mats and plates for 27, I	589.60	
Mats and plates for 27, II	508.00	
Plates for 11, II	169.30	
Reprinting 1000 24, I	533.50	
Reprinting 1017 24, II	509.25	
Reprinting 5000 24, I	1,647.80	
Plating and reprinting 1017 26	728.50	
Reprinting 2032 26, II	566.50	$13,415.98

Preparation

Nature-Nurture Committee	$ 34.83	
Preschool Committee	424.85	
Textbook Committee	117.23	
Arithmetic Committee	328.74	
Rural Education Committee	200.70	1,106.35
Total Cost of Yearbooks		$14,522.33

Meetings

A. A. A. S. Council Meeting	$ 4.81	
Boston Society Meeting	387.36	
Rochester Board Meeting	284.63	676.80

Secretary's Office

Salary	$2,500.00	
Office Rent	300.00	
Clerical Assistance	354.45	
Travelling Expenses	89.64	
Postage and Express	137.48	
Stationery and Printing	294.25	
Telegraph and Telephone	20.66	
Bonding	12.50	
Auditing	40.00	
Refunded dues, collections and bad checks	32.03	
Miscellaneous	6.00	3,787.01

Investments

$1,000 Utah Power and Light 5's 1944	$1,004.50	
$1,000 Alabama Power Co. 5's 1951	1,027.50	
$1,000 Chi. Junc. R. R. 5s 1940	1,022.00	
Interest Purchased on the Above Bonds	33.90	3,087.90
Total Expenditures for 1928		$22,074.04
Balance on Hand December 31, 1928		22,067.37
Total Expenditures and Closing Balance		$44,141.41

ANALYSIS OF BALANCE ON HAND DECEMBER 31, 1928

Balance on Hand December 31, 1928:

Checking Account, Danvers National Bank, $1,344.13, Less $235.87 checks outstanding	$ 1,108.26
Savings Account, Danvers National Bank	11,598.48
Undeposited Dues	470.00
$1,000 Dominion of Canada Bond 5½ 8/1/29	979.75
1,000 Detroit-Edison Bond 5 1940	940.00
1,000 U. S. Treasury Reg. 4¼ 1938	1,000.00
2,000 U. S. Liberty and Treasury Bonds 4¼	1,926.88
1,000 Interstate Power 1st 5 1957	990.00

1,000	Utah Power and Light 5 1944	1,004.50
1,000	Alabama Power Co. 5 1951	1,027.50
1,000	Chicago Junction R. R. Refng. 5 1940	1,022.00
	Balance December 31, 1928	$22,067.37

We hereby certify that we have verified the securities, reconciled the bank balance and footed the receipts and disbursements of the National Society for the Study of Education for the year ended December 31, 1928, and that we found all income recorded as received when due and disbursements supported by vouchers.

White and Paton,
By A. H. Paton
Certified Public Accountants.

HONORARY AND ACTIVE MEMBERS OF THE NATIONAL SOCIETY FOR THE STUDY OF EDUCATION

(This list includes all active members enrolled on December 31, 1929.)

HONORARY MEMBERS

DeGarmo, Professor Charles, Coconut Grove, Fla.
Dewey, Professor John, Columbia University, New York City.
Hanus, Professor Paul H., Harvard University, Cambridge, Mass.

ACTIVE MEMBERS

Abernethy, Professor Ethel M., Queens College, Charlotte, N. C.
Adams, Jesse E., College of Education, University of Kentucky, Lexington, Ky.
Ade, Lester K., Prin. New Haven State Normal School, New Haven, Conn.
Aherne, Mrs. Vina M., 146 Grafton St., New Haven, Conn.
Aitken, C. C., State School, Walkaway, Australia.
Albright, Denton M., Supt. of Schools, Rochester, Pa.
Alder, Miss Louise M., State Teachers College, Milwaukee, Wis.
Alderman, Dean Grover H., University of Pittsburgh, Pittsburgh, Pa.
Alexander, Professor Carter, Teachers College, Columbia Univ., New York City.
Alger, John L., Pres., Rhode Island College of Education, Providence, R. I.
Alleman, S. A., Supt. of Schools, Napoleonville, La.
Allen, C. F., School Administration Bldg., Little Rock, Ark.
Allen, Professor Fiske, State Normal School, Charleston, Ill.
Allen, I. M., Supt. of Schools, Highland Park, Mich.
Allen, Miss Willette A., 33 Cain St., N. E., Atlanta, Ga.
Allison, Dr. Samuel B., Dist. Supt., Board of Education, Chicago, Ill.
Almack, John C., Stanford University, Cal.
Alter, Harvey E., Thomas Street School, Rome, N. Y.
Althaus, Carl B., Univ. of Kansas, Lawrence, Kan.
Amann, Miss Dorothy, Southern Methodist University, Dallas, Texas.
Anderson, Professor E. J., Shanghai College, U.S.P.O. No. 964, Shanghai, China.
Anderson, Ernest W., 64 Fulton St., Medford, Mass.
Anderson, Harold A., School of Education, University of Chicago, Chicago, Ill.
Anderson, Harold H., State University of Iowa, Iowa City, Iowa.
Anderson, Mrs. Helen B., 414 W. Fayette St., Pittsfield, Ill.
Anderson, John A., Washington School, Miles City, Montana.
Andrews, Professor B. R., Teachers College, Columbia University, New York City.
Andrus, Dr. Ruth, State Department of Education, Albany, N. Y.
Angell, Miss L. Gertrude, Buffalo Seminary, Bidwell Parkway, Buffalo, N. Y.
Anspaugh, G. E., Komensky School, Chicago, Ill.
Antholz, H. J., Supervising Principal, Spooner City Schools, Spooner, Wis.
Anthony, Miss Katherine M., State Teachers College, Harrisonburg, Va.
Archer, C. P., State Teachers College, Moorhead, Minn.
Arnold, E. J., Supt. of Schools, Bremen, Ohio.
Ashbaugh, Dr. E. J., Miami University, Oxford, Ohio.
Ashley, Myron L., 7113 Normal Blvd., Chicago, Ill.
Atkins, Miss Ruth E., 723 Third St., St. Cloud, Minn.
Augustin, Miss Eloise D., "The Maples," Otsego Co., Laurens, N. Y.
Avery, Geo. T., Colorado Agricultural College, Fort Collins, Colo.

Axtelle, George E., Prin., Fort St. School, 3670 Sierra Ave., Honolulu, T. H.
Ayer, Miss Adelaide M., Dir. Training, State Teachers College, Milwaukee, Wis.
Ayer, Fred C., University of Texas, Austin, Texas.
Ayer, Jean Y., Macmillan Co., 60 Fifth Ave., New York City.

Bachrodt, Walter L., Supt. of Schools, San Jose, Cal.
Badanes, Saul, Public School 173, Pennsylvania Ave., Brooklyn, N. Y.
Bader, Miss Edith M., Ann Arbor, Mich.
Bagley, Professor William C., Teachers College, Columbia Univ., New York City.
Bailey, Francis L., 509 W. 121st St., New York City.
Baird, Don O., 509 W. 121st St., New York City.
Baker, C. A., Dean of Normal School, Rio Baptist College, Rio de Janeiro, Brazil.
Baker, Francis M., Prin. Alcorn School, Philadelphia, Pa.
Baker, Frank E., President State Teachers College, Milwaukee, Wis.
Baker, Dr. Harry J., 100 E. Grand River Ave., Detroit, Mich.
Ballou, Frank W., Supt. of Schools, Washington, D. C.
Balyeat, F. A., School of Education, University of Oklahoma, Norman, Okla.
Bamberger, Miss Florence E., Johns Hopkins University, Baltimore, Md.
Bamesberger, Miss Velda, Dir. Elem. Educ., Toledo, Ohio.
Barber, Fred H., Box 247, Emory, Virginia.
Bardwell, R. W., Supt. of Schools, Madison, Wis.
Bardy, Joseph, 5231 Wayne Ave., Philadelphia, Pa.
Barfoot, Harry N., Frankford High School, Philadelphia, Pa.
Barnes, Harold, Supervisor Elem. Educ., Girard College, Philadelphia, Pa.
Barnes, Percival Simpson, Supt. of Schools, E. Hartford, Conn.
Barringer, Benton E., State Normal School, Bowling Green, Ohio.
Barton, W. A., Jr., Coker College, Hartsville, S. C.
Baumgardner, Miss Nina E., State Teachers College, Mankato, Minn.
Bawden, Herrick T., Teachers College, Temple University, Philadelphia, Pa.
Bayles, E. E., Univ. of Kansas, Lawrence, Kans.
Beall, Ross H., Quadrangle 90-B, Iowa City, Iowa.
Beattie, Alfred W., Supervising Prin., Ben Avon, Pittsburgh, Pa.
Beatty, Willard W., 15 Meadow Ave., Bronxville, N. Y.
Beck, G. Herman, Prin., 3009a Victor St., St. Louis, Mo.
Bedell, Ralph C., Southwest High School, Kansas City, Mo.
Bednar, Miss Christine, 132 West Marquette Road, Chicago, Ill.
Beeby, Daniel J., 7740 Ingleside Ave., Chicago, Ill.
Beechel, Miss Edith, 1230 Amsterdam Ave., New York City.
Beito, E. A., Columbia University, New York City.
Bell, Dr. J. Carleton, 1032A Sterling Place, Brooklyn, N. Y.
Bemiller, J. F., Supt. of Schools, Galion, Ohio.
Bender, John F., School of Education, University of Oklahoma, Norman, Okla.
Benedict, Ezra W., Arlington, Vt.
Benson, Dr. C. E., New York University, Washington Sq., New York City.
Benson, J. R., 6131 Magnolia Ave., St. Louis, Mo.
Benton, G. W., 88 Lexington Ave., New York City.
Benz, H. E., College of Educ., Ohio University, Athens, Ohio.
Bergman, W. G., School of Education, Univ. of Michigan, Ann Arbor, Mich.
Berkson, Dr. I. B., Palestine Zionist Executive, Jerusalem, Palestine.
Berman, Samuel, 2127 N. Natrona St., Philadelphia, Pa.
Berry, Professor Charles S., University of Michigan, Ann Arbor, Mich.
Berry, Miss Frances M., Dept. of Educ., Baltimore, Md.
Betts, Emmett A., 323 Brown St., Iowa City, Iowa.
Betts, Mrs. Mary Tuite, 32 Verona Bldg., Walnut Hills, Cincinnati, Ohio.
Beveridge, J. H., Supt. of Instruction, Omaha, Neb.
Bick, Miss Anna, 2842A Victor St., St. Louis, Mo.
Bickford, C. W., Supt. of Schools, Lewiston, Me.

Biddle, Dr. Anna E., South Phil. H. S. for Girls, Philadelphia, Pa.
Birckhead, E. F., Supt. of Schools, Winchester, Ky.
Bird, Professor Charles, University of Minnesota, Minneapolis, Minn.
Birdsong, Miss Nellie W., Maryland State Normal School, Towson, Md.
Bishop, Fred G., Supt. of Schools, Two Rivers, Wis.
Bixler, H. H., Board of Education, Atlanta, Ga.
Blackburn, J. Albert, New Jersey College for Women, New Brunswick, N. J.
Blanton, Miss Annie Webb, Box 1742, University, Austin, Texas.
Blue, Harold G., University of Chicago, Chicago, Ill.
Blumberg, A. Alvin, 2136 S. Seventh St., Philadelphia, Pa.
Bly, Professor John, St. Olaf College, Northfield, Minn.
Bogan, Miss Eleanore M., 96 Woodland Ave., Detroit, Mich.
Bohan, John E., West Virginia University, Morgantown, W. Va.
Bohannan, C. D., Agricultural Economist, Dept. of Commerce, Washington, D. C.
Bolenius, Miss Emma Miller, 46 South Queen St., Lancaster, Pa.
Bolton, Dean Frederick E., University of Washington, Seattle, Wash.
Bonser, Professor F. G., Teachers College, Columbia Univ., New York City.
Boothby, Ralph E., Western Reserve Academy, Hudson, Ohio.
Boraas, Julius, St. Olaf College, Northfield, Minn.
Bossing, Professor Nelson L., 683 East Pine St., Portland, Ore.
Bott, Professor E. A., University of Toronto, Toronto 5, Canada.
Bowen, Wayne F., Box 84, Compton, Calif.
Bowyer, Vernon, Skinner School, 1070 W. Jackson Boul., Chicago, Ill.
Boyden, Wallace C., Teachers College of the City of Boston, Boston, Mass.
Boyer, Charles, Supt. of Schools, Atlantic City, N. J.
Boyer, Philip A., 6320 Lawnton Ave., Philadelphia, Pa.
Bracken, J., 305 N. Meramec, Clayton, Mo.
Bradner, J. W., Supt. of Schools, Middlesboro, Ky.
Bragdon, Professor Helen D., University of Minnesota, Minneapolis, Minn.
Braillier, J. Merle, University of Pittsburgh, Pittsburgh, Pa.
Breckenridge, Miss Elizabeth, Prin., Louisville Normal School, Louisville, Ky.
Breed, Frederick S., 1224 E. 57th St., Chicago, Ill.
Brewer, Miss Anne T., 74 W. 5th St., Oswego, N. Y.
Brewer, Professor John M., Harvard University, Cambridge, Mass.
Bridgett, Miss Alice E., Colony Street School, R.F.D. 1, Wallingford, Conn.
Briggs, Dr. Howard L., Dir. of Vocational Educ., Bd. of Educ., Cleveland, Ohio.
Briggs, Dr. Thomas H., Teachers College, Columbia Univ., New York City.
Brim, Professor Orville G., Ohio State University, Columbus, Ohio.
Brinkley, Sterling G., Emory University, Ga.
Brinser, Ira S., Supt. of Schools, Newark, Del.
Bristow, W. H., Dept. of Public Instruction, Harrisburg, Pa.
Brooks, Professor Fowler D., Johns Hopkins Univ., Baltimore, Md.
Brooks, Professor John D., Wilson College, Chambersburg, Pa.
Brown, F. J., Dir. Ext. Teaching, Univ. of Rochester, Rochester, N. Y.
Brown, George Earl, Supt. of Schools, Greeley, Colo.
Brown, J. C., Supt. of Schools, Pelham, N. Y.
Brown, Miss M. Ethel, Supervisor Elementary Grades, Schenectady, N. Y.
Brown, Stella E., 4213 Belmar Ave., Baltimore, Md.
Brown, W. O., 406 Beveridge St., Carbondale, Ill.
Browne, Mrs. Hetty S., Dir. River Road Country Day School, San Antonio, Texas.
Brownell, S. M., Supt. of Schools, Grosse Point, Mich.
Brownell, Professor W. A., Geo. Peabody College for Teachers, Nashville, Tenn.
Bruce, Miss Clara H., Ahmednagar, Bombay Presidency, India.
Brueckner, Dr. Leo J., University of Minnesota, Minneapolis, Minn.
Brust, Miss Huldah, Primary Supervisor, Rockville, Md.
Bubeck, Allan F., Dept. of Educ., State Teachers College, Kutztown, Pa.
Buchanan, William D., 5511 Vernon Ave., St. Louis, Mo.

Buckingham, Dr. B. R., Ginn & Co., Boston, Mass.
Buckner, Professor Chester A., Univ. of Pittsburgh, Pittsburgh, Pa.
Bull, Miss Elizabeth, Bay Avenue School, Patchogue, N. Y.
Burdick, Raymond C., Supt. of Schools, Watertown, N. Y.
Burkhard, William E., 1631 W. Ruscomb, Philadelphia, Pa.
Burnham, Professor Ernest, Western State Teachers College, Kalamazoo, Mich.
Burns, Robert L., Prin., Cliffside Park High School, Grantwood, N. J.
Buros, Francis C., 509 W. 121st St., New York City.
Buros, Oscar K., 509 W. 121st St., New York City.
Burton, Thomas C., Staten Island Academy, New Brighton, New York.
Buswell, Dr. G. T., School of Educ., Univ. of Chicago, Chicago, Ill.
Butler, Leslie A., Supt. of Schools, Grand Rapids, Mich.
Butsch, R. L. C., School of Education, University of Chicago, Chicago, Ill.
Butterworth, Julian E., Cornell University, Ithaca, N. Y.
Byrne, Lee, 319 12th Ave., S. E., Aberdeen, S. D.

Callahan, E. F., 726 Third Ave., West Haven, Conn.
Calvin, Miss Henrietta W., Dir. Div. of Home Economics, Bd. of Education, Grant Bldg., Philadelphia, Pa.
Cameron, Norman W., Prin., State Teachers College, West Chester, Pa.
Cameron, Walter C., Prin., Lincoln Junior H. S., Framingham, Mass.
Camp, Dr. H. L., 707 Elam Ave., Greensboro, N. C.
Canine, Edwin N., Dir. Supervisal Teaching, State Normal, Terre Haute, Ind.
Carkin, S. B., Prin., Packard Commercial School, New York City.
Carmichael, A. M., State Teachers College, Mayville, N. D.
Carpenter, Professor W. W., University of Missouri, Columbia, Mo.
Carrothers, George E., University of Michigan, Ann Arbor, Mich.
Carson, C. C., Supervising Prin., 1426 Drexel Ave., Miami Beach, Fla.
Carter, Professor Ralph E., Extension Div., Indiana Univ., Indianapolis, Ind.
Cassel, Lloyd S., Supt. of Schools, Freehold, N. J.
Castle, L. E. Supt. of Schools, Stuart, Iowa.
Cattell, Dr. J. McKeen, Garrison, N. Y.
Cattell, Miss Psyche, 2 Avon St., Cambridge, Mass.
Cavan, Professor Jordan, Rockford College, Rockford, Ill.
Chace, S. Howard, Supt. of Schools, Beverly, Mass.
Chadsey, Charles E., Dean, School of Educ., University of Illinois, Urbana, Ill.
Chadwick, Raymond D., Dean, Duluth Junior College, Duluth, Minn.
Chambers, Professor M. M., Oregon State Agricultural College, Corvallis, Ore.
Chambers, Will G., Dean of Education, State College, Pa.
Champlin, Carroll D., Pennsylvania State College, State College, Pa.
Chandler, Paul G., Millersville State Normal School, Millersville, Pa.
Chandor, Miss Valentine L., 137 E. 62nd St., New York City.
Chapman, Ira T., Supt. of Schools, So. Broad St., Elizabeth, N. J.
Charters, Professor W. W., Ohio State University, Columbus, Ohio.
Chauncey, Professor Marlin R., Okla. Ag. & Mech. College, Stillwater, Okla.
Chew, Samuel L., Supt. Dist. No. 5, Carlisle and Race Sts., Philadelphia, Pa.
Chidester, Albert J., Berea College, Berea, Ky.
Chiles, E. E., Prin., Harrison School, 4163 Green Lea Place, St. Louis, Mo.
Clark, Professor Donald Lemen, Columbia University, New York City.
Clark, Professor John R., 425 W. 123d St., New York City.
Clement, Professor W. P., Texas Technological College, Lubbock, Texas.
Cline, E. D., School Administration Bldg., South Bend, Ind.
Coats, Miss Marion, Sarah Lawrence College, Bronxville, N. Y.
Cochran, Professor T. E., Georgetown College, Georgetown, Ky.
Coffey, Wilford L., Dean, College of the City of Detroit, Detroit, Mich.
Coffin, Miss Rebecca J., Lincoln School of Teachers College, New York City.

Coffman, Lotus D., Pres., Univ. of Minnesota, Minneapolis, Minn.
Cole, C. E., R.F.D. 1, Temple, Pa.
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1. **Purpose.** The purpose of the National Society is to promote the investigation and discussion of educational questions. To this end it holds an annual meeting and publishes a series of yearbooks.

2. **Eligibility to Membership.** Any person who is interested in receiving its publications may become a member by sending to the Secretary-Treasurer information concerning name, address, and class of membership desired (see Item 4) and a check for $3.50 or $3.00 (see Item 5).

Membership may not be had by libraries or by institutions.

3. **Period of Membership.** Applicants for membership may not date their entrance back of the current calendar year, and all memberships terminate automatically on December 31st, unless the dues for the ensuing year are paid as indicated in Item 6.

4. **Classes of Members.** Application may be made for either active or associate membership. Active members pay dues of $2.50 annually, receive a cloth-bound copy of each publication, are entitled to vote, to participate in discussion and (under certain conditions) to hold office. Associate members pay dues of $2.00 annually, receive a paper-bound copy of each publication, may attend the meetings of the Society, but may not vote, hold office, contribute to the yearbooks, or participate in discussion. The names of active members only are printed in the yearbooks. There were in 1929 about 1200 active and 1200 associate members.

5. **Entrance Fee.** New active and new associate members are required the first year to pay, in addition to the dues, an entrance fee of one dollar.

6. **Payment of Dues.** Statements of dues are rendered in October or November for the following calendar year. By vote of the Society at the 1919 meeting, "any member so notified whose dues remain unpaid on January 1st, thereby loses his membership and can be reinstated only by paying the entrance fee of one dollar required of new members."

School warrants and vouchers from institutions must be accompanied by definite information concerning the name and address and class of membership of the person for whom membership fee is being paid.

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7. **Distribution of Yearbooks to Members.** The yearbooks, ready prior to each February meeting, will be mailed from the office of the publishers, only to members whose dues for that year have been paid. Members who desire yearbooks prior to the current year must purchase them directly from the publishers (see Item 8).

8. **Commercial Sales.** The distribution of all yearbooks prior to the current year, and also of those of the current year not regularly mailed to members in exchange for their dues, is in the hands of the publishers, not of the secretary. For such commercial sales, communicate directly with the Public School Publishing Company, Bloomington, Illinois, which will gladly send a price list covering all the publications of this Society and of its predecessor, the National Herbart Society.

9. **Yearbooks.** The yearbooks are issued about one month before the February meeting. They comprise from 700 to 800 pages annually. Unusual effort has been made to make them, on the one hand, of immediate practical value, and on the other hand, representative of sound scholarship and scientific investigation. Many of them are the fruit of coöperative work by committees of the Society.

10. **Meetings.** The annual meetings, at which the yearbooks are discussed, are held in February at the same time and place as the meeting of the Department of Superintendence of the National Education Association.

Applications for membership will be handled promptly at any time on receipt of name and address, together with check for the appropriate amount ($3.50 for new active membership, $3.00 for new associate membership). Generally speaking, applications entitle the new member to the yearbook slated for discussion during the calendar year the application is made, but those received in December are regarded as pertaining to the next calendar year.

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